Applications of Trigonometry

CHAPTER OUTLINE

CHAPTER CONNECTIONS

When an airline pilot charts a course, it's not as simple as pointing the airplane in the right direction. Wind currents must be taken into consideration, and compensated for by additional thrust or a change of heading to equalize the force of the wind and keep the plane flying in the desired direction. The effect of these forces working together can be modeled using a carefully drawn *vector diagram,* and with the aid of trigonometry, a pilot can easily determine any adjustments in navigation needed. This application appears as Exercise 85 in Section 7.3

Check out these other real-world connections:

- Tracking Large Game in a Wildlife Preserve (Section 7.1, Exercise 50)
- Calculating Distances between Cities Using Satellite Information (Section 7.2, Exercise 37)
- Forces Required to Tow a Van out of a Ditch (Section 7.3, Exercise 81)
- Measuring Forces Used by Contestents in a Tough-Man Contest (Section 7.4, Exercise 37)

7.1 Oblique Triangles and the Law of Sines

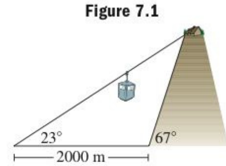
Learning Objectives

In Section 7.1 you will learn how to:

- ☐ **A.** Develop the law of sines, and use it to solve ASA and AAS triangles
- ☐ **B.** Solve SSA triangles (the ambiguous case) using the law of sines
- ☐ **C.** Use the law of sines to solve applications

Many applications of trigonometry involve *oblique triangles*, or triangles that do not have a 90° angle. For example, suppose a trolley carries passengers from ground level up to a mountain chateau, as shown in Figure 7.1. Assuming the cable could be held taut, what is its approximate length? Can we also determine the slant height of the mountain? To answer questions like these, we'll develop techniques that enable us to solve acute and obtuse triangles using fundamental trigonometric relationships.

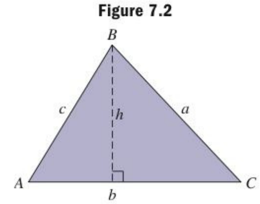
Figure 7.1



A. The Law of Sines and Unique Solutions

Consider the oblique triangle *ABC* pictured in Figure 7.2. Since it is not a right triangle, it seems the trigonometric ratios studied earlier cannot be applied. But if we draw the altitude *h* (from vertex *B*), two right triangles are formed that *share a common side*. By applying the sine ratio to angles *A* and *C*, we can develop a relationship that will help us solve the triangle.

Figure 7.2



WORTHY OF NOTE

As with right triangles, solving an oblique triangle involves determining the lengths of all three sides and the measures of all three angles.

For $\angle A$ we have $\sin A = \frac{h}{c}$ or $h = c\sin A$. For $\angle C$ we have $\sin C = \frac{h}{a}$ or $h = a\sin C$. Since both products are equal to *h*, the transitive property gives $c\sin A = a\sin C$, which leads to

$$c\sin A = a\sin C \qquad \text{since } h = h$$

$$\frac{c\sin A}{ac} = \frac{a\sin C}{ac} \qquad \text{divide by } ac$$

$$\frac{\sin A}{a} = \frac{\sin C}{c} \qquad \text{simplify}$$

Using the same triangle and the altitude drawn from *C* (Figure 7.3), we note a similar relationship involving angles *A* and *B*: $\sin A = \frac{h}{b}$ or $h = b\sin A$, and $\sin B = \frac{h}{a}$ or $h = a\sin B$. As before, we can then write $\frac{\sin A}{a} = \frac{\sin B}{b}$. If $\angle A$ is obtuse, the altitude *h* actually falls outside the triangle, as shown in Figure 7.4. In this case, consider that $\sin(180° - \alpha) = \sin\alpha$ from the difference formula for sines (Exercise 55, Section 6.3). In the figure we note $\sin(180° - \alpha) = \frac{h}{c} = \sin\alpha$, yielding $h = c\sin\alpha$

Figure 7.3

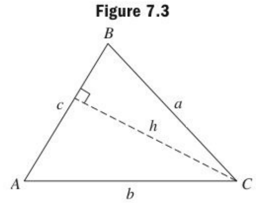
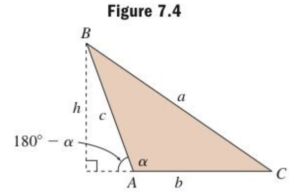


Figure 7.4



and the preceding relationship can now be stated using any pair of angles and corresponding sides. The result is called the **law of sines,** which is usually stated by combing the three possible proportions.

The Law of Sines

For any triangle *ABC*, the ratio of the sine of an angle to the side opposite that angle is constant:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

As a proportional relationship, the law requires that we have three parts in order to solve for the fourth. This suggests the following possibilities:

1. two angles and an included side (ASA)
2. two angles and a side opposite one of these angles (AAS)
3. two sides and a angle opposite one of these sides (SSA)
4. two sides and an included angle (SAS)
5. three sides (SSS)

Each of these possibilities is diagrammed in Figures 7.5 through 7.9.

Figure 7.5

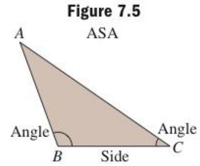


Figure 7.6

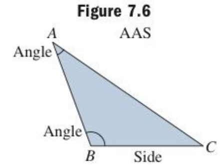


Figure 7.7

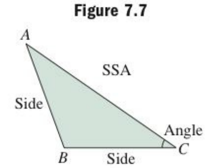


Figure 7.8

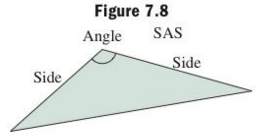
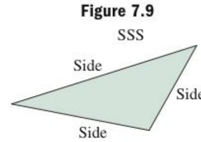


Figure 7.9



WORTHY OF NOTE

When working with triangles, keeping these basic properties in mind will prevent errors and assist in their solution:

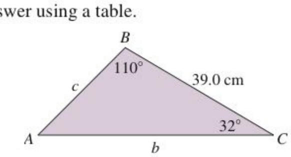
1. The angles must sum to 180°.
2. The combined length of any two sides must exceed the length of the third side.
3. Longer sides will be opposite larger angles.
4. This sine of an angle cannot be greater than 1.
5. For $y \in (0, 1)$, the equation $y = \sin\theta$ has two solutions in $(0°, 180°)$ that are supplements.

Since applying the law of sines requires we have a given side opposite a known angle, it cannot be used in the case of SAS or SSS triangles. These require the law of cosines, which we will develop in Section 7.2. In the case of ASA and AAS triangles, a unique triangle is formed since the measure of the third angle is fixed by the two angles given (they must sum to 180°) and the remaining sides must be of fixed length.

EXAMPLE 1 ► **Solving a Triangle Using the Law of Sines**

Solve the triangle shown, and state your answer using a table.

Solution ► This is *not* a right triangle, so the standard ratios cannot be used. Since $\angle B$ and $\angle C$ are given, we know $\angle A = 180° - (110° + 32°) = 38°$. With $\angle A$ and side a, we have



WORTHY OF NOTE

Although not a definitive check, always review the solution table to ensure the smallest side is opposite the smallest angle, the largest side is opposite the largest angle, and so on. If this is not the case, you should go back and check your work.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{law of sines applied to } \angle A \text{ and } \angle B$$

$$\frac{\sin 38°}{39} = \frac{\sin 110°}{b} \quad \text{substitute given values}$$

$$b \sin 38° = 39 \sin 110° \quad \text{multiply by } 39b$$

$$b = \frac{39 \sin 110°}{\sin 38°} \quad \text{divide by } \sin 38°$$

$$b \approx 59.5 \quad \text{result}$$

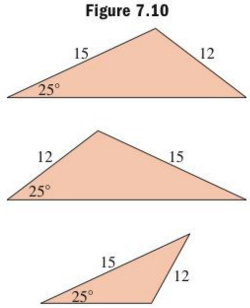
Repeating this procedure using $\frac{\sin A}{a} = \frac{\sin C}{c}$ shows side $c \approx 33.6$ cm. In table form we have

Angles	Sides (cm)
$A = 38°$	$\mathbf{a = 39.0}$
$\mathbf{B = 110°}$	$b \approx 59.5$
$\mathbf{C = 32°}$	$c \approx 33.6$

A. You've just learned to develop the law of sines and use it to solve ASA and AAS triangles

Now try Exercises 7 through 24 ►

Figure 7.10



B. Solving SSA Triangles—The Ambiguous Case

To understand the concept of unique and nonunique solutions regarding the law of sines, consider an instructor who asks a large group of students to draw a triangle with sides of 15 and 12 units, and a nonincluded 25° angle. Unavoidably, three different solutions will be offered (see Figure 7.10). For the SSA case, there is some doubt as to the number of solutions possible, or whether a solution even exists.

To further understand why, consider a triangle with side $c = 30$ cm, $\angle A = 30°$, and side a opposite the 30° angle (Figure 7.11—note the length of side b is yet to be determined). From our work with 30-60-90 triangles, we know if $a = 15$ cm, it is exactly the length needed to form a right triangle (Figure 7.12).

Figure 7.11

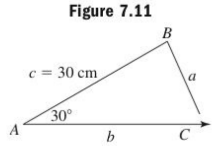
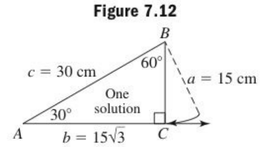


Figure 7.12



By varying the length of side a, we note three other possibilities. If side $a < 15$ cm, no triangle is possible since a is too short to contact side b (Figure 7.13), while if 15 cm $<$ side $a < 30$ cm, two triangles are possible since side a will then intersect side b at two points, C_1 and C_2 (Figure 7.14).

For future use, note that when two triangles are possible, angles C_1 and C_2 must be supplements since an-isosceles triangle is formed. Finally, if side $a > 30$ cm, it will

Figure 7.13

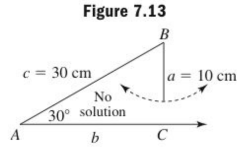
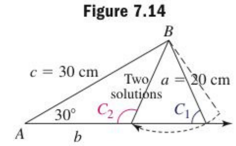


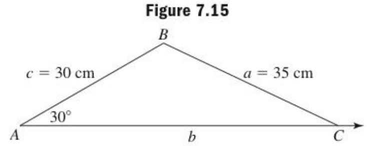
Figure 7.14



WORTHY OF NOTE

The case where three angles are known (AAA) is not considered since we then have a family of similar triangles, with infinitely many solutions.

Figure 7.15



intersect side b only once, forming the obtuse triangle shown in Figure 7.15, where we've assumed $a = 35$ cm. Since the final solution is in doubt until we do further work, the SSA case is called the **ambiguous case** of the law of sines.

EXAMPLE 2 ▶ Analyzing the Ambiguous Case of the Law of Sines

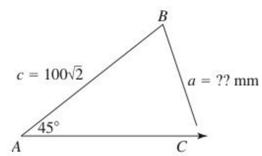
Given triangle ABC with $\angle A = 45°$ and side $c = 100\sqrt{2}$ mm,

a. What length for side a will produce a right triangle where $\angle C = 90°$?

b. How many triangles can be formed if side $a = 90$ mm?

c. If side $a = 120$ mm, how many triangles can be formed?

d. If side $a = 145$ mm, how many triangles can be formed?



Solution ▶

a. Recognizing the sides of a 45-45-90 triangle are in proportion according to $1x{:}1x{:}\ \sqrt{2}x$, side a must be 100 mm for a right triangle to be formed.

b. If $a = 90$ mm, it will be too short to contact side b and no triangle is possible.

c. As shown in Figure 7.16, if $a = 120$ mm, it will contact side b in two distinct places and two triangles are possible.

d. If $a = 145$ mm, it will contact side b only once, since it is longer than side c and will "miss" side b as it pivots around $\angle B$ (see Figure 7.17). One triangle is possible.

Figure 7.16

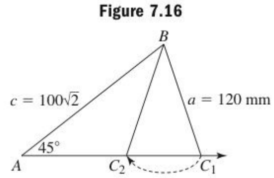
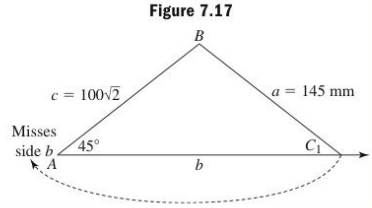


Figure 7.17



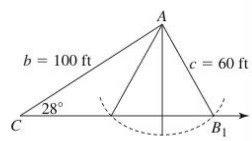
Now try Exercises 25 and 26 ▶

For a better understanding of the SSA (ambiguous) case, scaled drawings can initially be used along with a metric ruler and protractor. Begin with a horizontal line segment of undetermined length to represent the third (unknown) side, and use the protractor to draw the given angle on either the left or right side of this segment (we chose the left). Then use the metric ruler to draw an adjacent side of appropriate length, *choosing a scale that enables a complete diagram*. For instance, if the given sides are 3 ft and 5 ft, use 3 cm and 5 cm instead (1 cm = 1 ft). If the sides are 80 mi and 120 mi, use 8 cm and 12 cm (1 cm = 10 mi), and so on. Once the adjacent side is drawn, start at the free endpoint and draw a vertical segment to represent the remaining side. A careful sketch will often indicate whether none, one, or two triangles are possible (see the *Reinforcing Basic Concepts* feature on page 751).

EXAMPLE 3 ▶ Solving the Ambiguous Case of the Law of Sines

Solve the triangle with side $b = 100$ ft, side $c = 60$ ft, and $\angle C = 28.0°$.

Solution ▶ Two sides and an angle opposite are given (SSA), and we draw a diagram to help determine the possibilities. Draw the horizontal segment of some length and use a protractor to mark $\angle C = 28°$. Then draw a segment 10 cm long (to represent $b = 100$ ft) as the adjacent side of the angle, with a vertical segment 6 cm long from the free end of b (to represent $c = 60$ ft). It seems apparent that side c will intersect the horizontal side in two places (see figure), and two triangles are possible. We apply the law of sines to solve the first triangle, whose features we'll note with a subscript of 1.



$$\frac{\sin B_1}{b} = \frac{\sin C}{c} \quad \text{law of sines}$$

$$\frac{\sin B_1}{100} = \frac{\sin 28°}{60} \quad \text{substitute}$$

$$\sin B_1 = \frac{5}{3}\sin 28° \quad \text{solve for } \sin B_1$$

$$B_1 \approx 51.5° \quad \text{apply arcsine}$$

Since $\angle B_1 + \angle B_2 = 180°$, we know $\angle B_2 = 128.5°$. These values give 100.5° and 23.5° as the measures of $\angle A_1$ and $\angle A_2$, respectively. By once again applying the law of sines to each triangle, we find side $a_1 \approx 125.7$ ft and $a_2 \approx 51.0$ ft. See Figure 7.18.

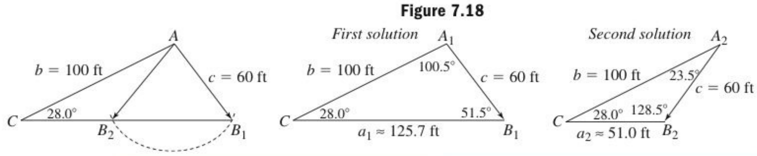
WORTHY OF NOTE

In Example 3, we found $\angle B_2$ using the property that states the angles in a triangle must sum to 180°. We could also view B_1 as a QI reference angle, which also gives a QII solution of $(180 - 51.5)° = 128.5°$.

Angles	Sides (ft)
$A_1 \approx 100.5°$	$a_1 \approx 125.7°$
$B_1 \approx 51.5°$	$b = 100$
$C = 28°$	$c = 60$

Angles	Sides (ft)
$A_2 \approx 23.5°$	$a_2 \approx 51.0$
$B_2 \approx 128.5°$	$b = 100$
$C = 28°$	$c = 60$

Figure 7.18



Now try Exercises 27 through 32 ▶

Admittedly, the scaled drawing approach has some drawbacks — it takes time to draw the diagrams and is of little use if the situation is a close call. It does, however, offer a deeper understanding of the subtleties involved in solving the SSA case. Instead of a scaled drawing, we can use a simple sketch *as a guide*, while keeping in mind the properties mentioned in the *Worthy of Note* on page 713.

EXAMPLE 4 ▶ Solving the Ambiguous Case of the Law of Sines

Solve the triangle with side $a = 220$ ft, side $b = 200$ ft, and $\angle A = 40°$.

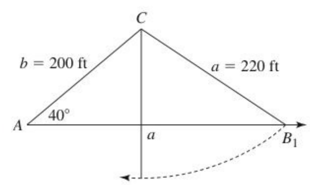
Solution ▶ The information given is again SSA, and we apply the law of sines with this in mind.

$$\frac{\sin A}{a} = \frac{\sin B}{b} \quad \text{law of sines}$$

$$\frac{\sin 40°}{220} = \frac{\sin B}{200} \quad \text{substitute}$$

$$\sin B = \frac{200 \sin 40°}{220} \quad \text{solve for } \sin B$$

$$B_1 \approx 35.7° \quad \text{apply arcsine}$$



This is the solution from Quadrant I. The QII solution is about $(180 - 35.7)° = 144.3°$. At this point our solution tables have this form:

Angles	Sides (ft)
$A = 40°$	$a = 220$
$B_1 \approx 35.7°$	$b = 200$
$C_1 =$	$c_1 =$

Angles	Sides (ft)
$A = 40°$	$a = 220$
$B_2 \approx 144.3°$	$b = 200$
$C_2 =$	$c_2 =$

It seems reasonable to once again find the remaining angles and finish by reapplying the law of sines, but observe that the sum of the two angles from the second solution *already exceeds 180°*: $40° + 144.3° = 188.3°$! This means no second solution is possible (side a is too long). We find that $C_1 \approx 104.3°$, and applying the law of sines gives a value of $c_1 \approx 331.7$ ft.

B. You've just learned how to solve SSA triangles (the ambiguous case) using the law of sines

Now try Exercises 33 through 44 ▶

C. Applications of the Law of Sines

As "ambiguous" as it is, the ambiguous case has a number of applications in engineering, astronomy, physics, and other areas. Here is an example from astronomy.

EXAMPLE 5 ▶ Solving an Application of the Ambiguous Case—Planetary Distance

The planet Venus can be seen from Earth with the naked eye, but as the diagram indicates, the position of Venus is uncertain (we are unable to tell if Venus is in the near position or the far position). Given the Earth is 93 million miles from the Sun and Venus is 67 million miles from the Sun, determine the closest and farthest possible distances that separate the planets in this alignment. Assume a viewing angle of $\theta \approx 18°$ and that the orbits of both planets are roughly circular.

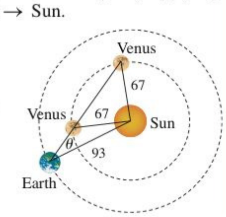
Solution ▶ A close look at the information and diagram shows a SSA case. Begin by applying the law of sines where $E \rightarrow$ Earth, $V \rightarrow$ Venus, and $S \rightarrow$ Sun.

$$\frac{\sin E}{e} = \frac{\sin V}{v} \quad \text{law of sines}$$

$$\frac{\sin 18°}{67} = \frac{\sin V}{93} \quad \text{substitute given values}$$

$$\sin V = \frac{93 \sin 18°}{67} \quad \text{solve for } \sin V$$

$$V \approx 25.4° \quad \text{apply arcsine}$$



This is the angle V_1 formed when Venus is farthest away. The angle V_2 at the closer distance is $180° - 25.4° = 154.6°$. At this point, our solution tables have this form:

Angles	Sides (10^6 mi)
$E = 18°$	$e = 67$
$V_1 \approx 25.4°$	$v = 93$
$S_1 =$	$s_1 =$

Angles	Sides (10^6 mi)
$E = 18°$	$e = 67$
$V_2 = 154.6°$	$v = 93$
$S_2 =$	$s_2 =$

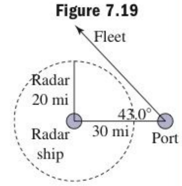
For S_1 and S_2 we have $S_1 \approx 180 - (18 + 25.4°) = 136.6°$ (larger angle) and $S_2 \approx 180 - (18 + 154.6°) = 7.4°$ (smaller angle). Re-applying the law of sines for s_1 shows the farther distance between the planets is about 149 million miles. Solving for s_2 shows that the closer distance is approximately 28 million miles.

Now try Exercises 47 and 48 ►

EXAMPLE 6 ► Solving an Application of the Ambiguous Case—Radar Detection

As shown in Figure 7.19, a radar ship is 30.0 mi off shore when a large fleet of ships leaves port at an angle of 43.0°.

Figure 7.19



a. If the maximum range of the ship's radar is 20.0 mi, will the departing fleet be detected?

b. If the maximum range of the ship's radar is 25.0 mi, how far from port is the fleet when it is first detected?

Solution ► **a.** This is again the SSA (ambiguous) case. Applying the law of sines gives

$$\frac{\sin 43°}{20} = \frac{\sin\theta}{30} \quad \text{law of sines}$$

$$\sin\theta = \frac{30\sin 43°}{20} \quad \text{solve for } \sin\theta$$

$$\sin\theta \approx 1.02299754 \quad \text{result}$$

No triangle is possible and the departing fleet will not be detected.

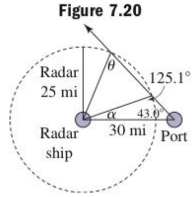
b. If the radar has a range of 25.0 mi, the radar beam will intersect the projected course of the fleet in two places.

$$\frac{\sin 43°}{25} = \frac{\sin\theta}{30} \quad \text{law of sines}$$

$$\sin\theta = \frac{30\sin 43°}{25} \quad \text{solve for } \sin\theta$$

$$\theta \approx 54.9° \quad \text{apply arcsine}$$

Figure 7.20



This is the acute angle θ related to the *farthest point* from port at which the fleet could be detected (see Figure 7.20). For the second triangle, we have $180° - 54.9° = 125.1°$ (the obtuse angle) giving a measure of $180° - (125.1° + 43°) = 11.9°$ for angle α. For d as the side opposite α we have

$$\frac{\sin 43°}{25} = \frac{\sin 11.9°}{d} \quad \text{law of sines}$$

$$d = \frac{25\sin 11.9°}{\sin 43°} \quad \text{solve for } d$$

$$\approx 7.6 \quad \text{simplify}$$

This shows the fleet is first detected about 7.6 mi from port.

Now try Exercises 49 and 50 ►

☑ **C.** You've just learned how to use the law of sines to solve applications

There are a number of additional, interesting applications in the exercise set **(see Exercises 51 through 70).**

7.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For the law of sines, if two sides and an angle opposite one side are given, this is referred to as the ________ case, since the solution is in doubt until further work.

2. Two inviolate properties of a triangle that can be used to help solve the ambiguous case are: (a) the angles must sum to ________ and (b) no sine ratio can exceed ________.

3. For positive k, the equation $\sin\theta = k$ has two solutions, one in Quadrant ________ and the other in Quadrant ________.

4. After a triangle is solved, you should always check to ensure that the ________ side is opposite the ________ angle.

5. In your own words, explain why the AAS case results in a unique solution while the SSA case does not. Give supporting diagrams.

6. Explain why no triangle is possible in each case:

a. $A = 34°, B = 73°, C = 52°,$ $a = 14', b = 22', c = 18'$

b. $A = 42°, B = 57°, C = 81°,$ $a = 7'', b = 9'', c = 22''$

▶ DEVELOPING YOUR SKILLS

Solve each of the following equations for the unknown part (if possible). Round sides to the nearest hundredth and degrees to the nearest tenth.

7. $\dfrac{\sin 32°}{15} = \dfrac{\sin 18.5°}{a}$ **8.** $\dfrac{\sin 52°}{b} = \dfrac{\sin 30°}{12}$

9. $\dfrac{\sin 63°}{21.9} = \dfrac{\sin C}{18.6}$ **10.** $\dfrac{\sin B}{3.14} = \dfrac{\sin 105°}{6.28}$

11. $\dfrac{\sin C}{48.5} = \dfrac{\sin 19°}{43.2}$ **12.** $\dfrac{\sin 38°}{125} = \dfrac{\sin B}{190}$

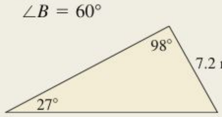
Solve each triangle using the law of sines. If the law of sines cannot be used, state why. Draw and label a triangle or label the triangle given before you begin.

13. side $a = 75$ cm, $\angle A = 38°$, $\angle B = 64°$

14. side $b = 385$ m, $\angle B = 47°$, $\angle A = 108°$

15. side $b = 10\sqrt{3}$ in., $\angle A = 30°$, $\angle B = 60°$

16.



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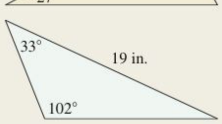

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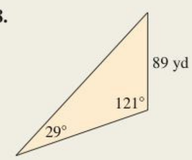
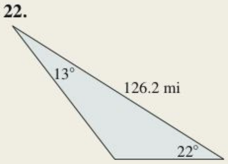


19. $\angle A = 45°$, $\angle B = 45°$, side $c = 15\sqrt{2}$ mi

20. $\angle A = 20.4°$, side $c = 12.9$ mi, $\angle B = 63.4°$

21. $\angle B = 103.4°$, side $a = 42.7$ km, $\angle C = 19.6°$

22.



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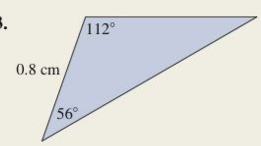

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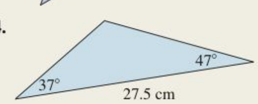


Answer each question and justify your response using a diagram, but do not solve.

25. Given $\triangle ABC$ with $\angle A = 30°$ and side $c = 20$ cm, (a) what length for side a will produce a right triangle? (b) How many triangles can be formed if side $a = 8$ cm? (c) If side $a = 12$ cm, how many triangles can be formed? (d) If side $a = 25$ cm, how many triangles can be formed?

26. Given $\triangle ABC$ with $\angle A = 60°$ and side $c = 6\sqrt{3}$ m, (a) what length for side a will produce a right triangle? (b) How many triangles can be formed if side $a = 8$ m? (c) If side $a = 10$ m, how many triangles can be formed? (d) If side $a = 15$ m, how many triangles can be formed?

Solve using the law of sines and a scaled drawing. If two triangles exist, solve both completely.

27. side $b = 385$ m, $\angle B = 67°$, side $a = 490$ m

28. side $a = 36.5$ yd, $\angle B = 67°$, side $b = 12.9$ yd

29. side $c = 25.8$ mi, $\angle A = 30°$, side $a = 12.9$ mi

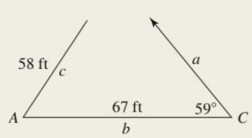
30. side $c = 10\sqrt{3}$ in., $\angle A = 60°$, side $a = 15$ in.

31. side $c = 58$ mi, $\angle C = 59°$, side $b = 67$ mi

32. side $b = 24.9$ km, $\angle B = 45°$, side $a = 32.8$ km

Use the law of sines to determine if no triangle, one triangle, or two triangles can be formed from the diagrams given (diagrams *may not be to scale*), then solve. If two solutions exist, solve both completely. Note the arrowhead marks the side of undetermined length.

33.



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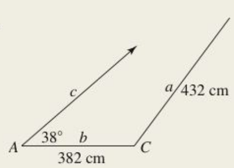

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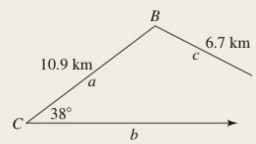

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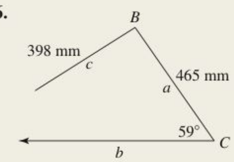

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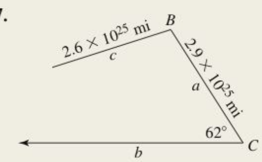

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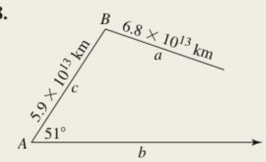


For Exercises 39 to 44, assume the law of sines is being applied to solve a triangle. Solve for the unknown angle (if possible), then determine if a second angle ($0° < \theta < 180°$) exists that also satisfies the proportion.

39. $\dfrac{\sin A}{12} = \dfrac{\sin 48°}{27}$

40. $\dfrac{\sin 60°}{32} = \dfrac{\sin B}{9}$

41. $\dfrac{\sin 57°}{35.6} = \dfrac{\sin C}{40.2}$

42. $\dfrac{\sin B}{5.2} = \dfrac{\sin 65°}{4.9}$

43. $\dfrac{\sin A}{280} = \dfrac{\sin 15°}{52}$

44. $\dfrac{\sin 29°}{121} = \dfrac{\sin B}{321}$

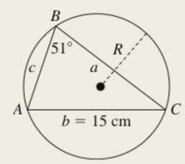
▶ WORKING WITH FORMULAS

45. Triple angle formula for sine: $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$

Most students are familiar with the double angle formula for sine: $\sin(2\theta) = 2\sin\theta\cos\theta$. The triple angle formula for sine is given here. Use the formula to find an exact value for sin 135°, then verify the result using a reference angle.

46. Radius of a circumscribed circle: $R = \dfrac{b}{2 \sin B}$

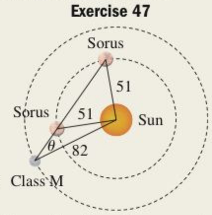
Given $\triangle ABC$ is circumscribed by a circle of radius R, the radius of the circle can be found using the formula shown, where side b is opposite angle B. Find the radius of the circle shown.



▶ APPLICATIONS

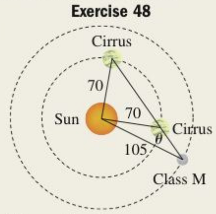
47. Planetary distances: In a solar system that parallels our own, the planet Sorus can be seen from a Class M planet with the naked eye, but as the diagram indicates, the position of Sorus is uncertain. Assume the orbits of both planets are roughly circular and that the viewing angle θ is about 20°. If the Class M planet is 82 million miles from its sun and Sorus is 51 million miles from this sun, determine the closest and farthest possible distances that separate the planets in this alignment.

Exercise 47

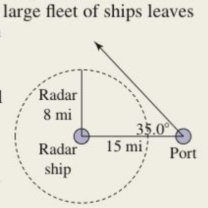


48. Planetary distances: In a solar system that parallels our own, the planet Cirrus can be seen from a Class M planet with the naked eye, but as the diagram indicates, the position of Cirrus is uncertain. Assume the orbits of both planets are roughly circular and that the viewing angle θ is about 15°. If the Class M planet is 105 million miles from its sun and Cirrus is 70 million miles from this sun, determine the closest and farthest possible distances that separate the planets in this alignment.

Exercise 48

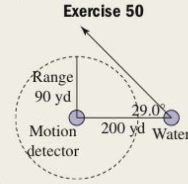


49. Radar detection: A radar ship is 15.0 mi off shore from a major port when a large fleet of ships leaves the port at the 35.0° angle shown. (a) If the maximum range of the ship's radar is 8.0 mi, will the departing fleet be detected? (b) If the maximum range of the ship's radar is 12 mi, how far from port is the fleet when it is first detected?



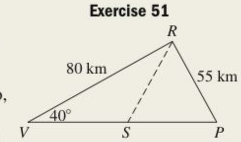
50. Motion detection: To notify environmentalists of the presence of big game, motion detectors are installed 200 yd from a watering hole. A pride of lions has just visited the hole and is leaving the area at the 29.0° angle shown. (a) If the maximum range of the motion detector is 90 yd, will the pride be detected? (b) If the maximum range of the motion detector is 120 yd, how far from the watering hole is the pride when first detected?

Exercise 50



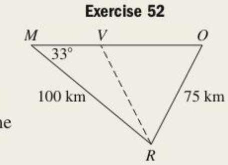
51. Distance between cities: The cities of ***V***an Gogh, ***R***embrandt, ***P***issarro, and ***S***eurat are situated as shown in the diagram. Assume that triangle *RSP* is isosceles and use the law of sines to find the distance between *V*an Gogh and *S*eurat, and between *V*an Gogh and *P*issarro.

Exercise 51



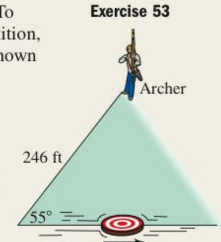
52. Distance between cities: The cities of ***M***ozart, ***R***ossini, ***O***ffenbach, and ***V***erdi are situated as shown in the diagram. Assume that triangle *ROV* is isosceles and use the law of sines to find the distance between *M*ozart and *V*erdi, and between *M*ozart and *O*ffenbach.

Exercise 52



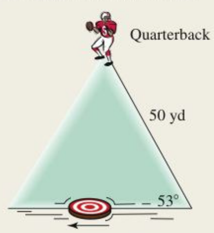
53. Distance to target: To practice for a competition, an archer stands as shown in the diagram and attempts to hit a moving target. (a) If the archer has a maximum effective range of about 180 ft, can the target be hit?

Exercise 53



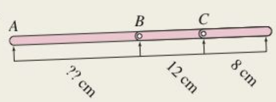
(b) What is the shortest range the archer can have and still hit the target? (c) If the archer's range is 215 ft and the target is moving at 10 ft/sec, how many seconds is the target within range?

54. Distance to target: As part of an All-Star competition, a quarterback stands as shown in the diagram and attempts to hit a moving target with a football. (a) If the quarterback has a maximum effective range of about 35 yd, can the target be hit? (b) What is the shortest range the quarterback can have and still hit the target? (c) If the quarterback's range is 45 yd and the target is moving at 5 yd/sec, how many seconds is the target within range?

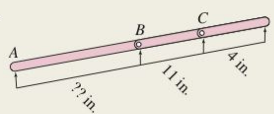


In Exercises 55 and 56, three rods are attached via pivot joints so the rods can be manipulated to form a triangle. How many triangles can be formed if angle *B* must measure 26°? If one triangle, solve it. If two, solve both. Diagrams are not drawn to scale.

55.

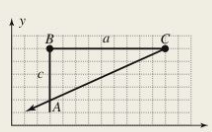


56.

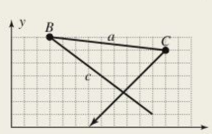


In the diagrams given, the measure of angle *C* and the length of sides *a* and *c* are fixed. Side *c* can be rotated at pivot point *B*. Solve any triangles that can be formed. (*Hint:* Begin by using the grid to find lengths *a* and *c*, then find angle *C*.)

57.

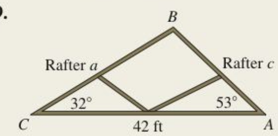


58.

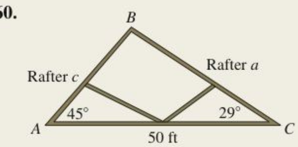


Length of a rafter: Determine the length of both roof rafters in the diagrams given.

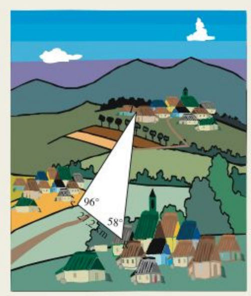
59.



60.

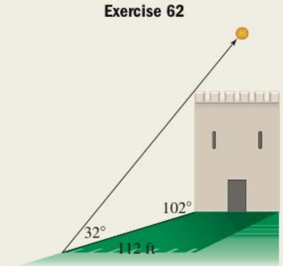


61. Map distance: A cartographer is using aerial photographs to prepare a map for publication. The distance from Sexton to Rhymes is known to be 27.2 km. Using a protractor, the map maker measures an angle of 96° from Sexton to Tarryson (a newly developed area) and an angle of 58° from Rhymes to Tarryson. Compute each unknown distance.

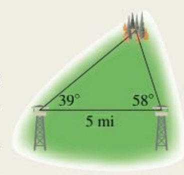


62. Height of a fortress: An ancient fortress is built on a steep hillside, with the base of the fortress walls making a 102° angle with the hill. At the moment the fortress casts a 112-ft shadow, the angle of elevation from the tip of the shadow to the top of the wall is 32°. What is the distance from the base of the fortress to the top of the tower?

Exercise 62

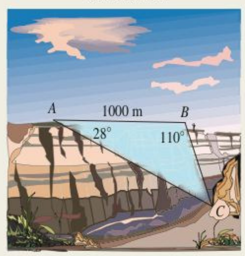


63. Distance to a fire: In Yellowstone Park, a fire is spotted by park rangers stationed in two towers that are known to be 5 mi apart. Using the line between them as a baseline, tower A reports the fire is at an angle of 39°, while tower B reports an angle of 58°. How far is the fire from the closer tower?



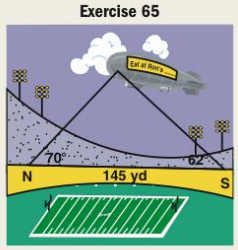
64. Width of a canyon: To find the distance across Waimea Canyon (on the island of Kauai), a surveyor marks a 1000-m baseline along the southern rim. Using a transit, she sights on a large rock formation on the north rim, and finds the angles indicated. How wide is the canyon from point *B* to point *C*?

Exercise 64



65. Height of a blimp: When the Good-Year Blimp is viewed from the field-level bleachers near the southern end-zone of a football stadium, the angle of elevation is 62°. From the field-level bleachers near the northern end-zone, the angle of elevation is 70°. Find the height of the blimp if the distance from the southern bleachers to the northern bleachers is 145 yd.

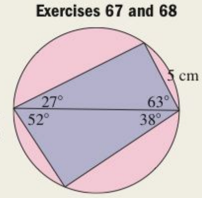
Exercise 65



66. Height of a blimp: The rock-n-roll group *Pink Floyd* just finished their most recent tour and has moored their touring blimp at a hangar near the airport in Indianapolis, Indiana. From an unknown distance away, the angle of elevation is measured at 26.5°. After moving 110 yd closer, the angle of elevation has become 48.3°. At what height is the blimp moored?

67. Circumscribed triangles: A triangle is circumscribed within the upper semicircle drawn in the figure. Use the law of sines to solve the triangle given the measures shown. What is the diameter of the circle? What do you notice about the triangle?

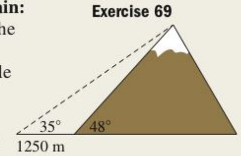
Exercises 67 and 68



68. Circumscribed triangles: A triangle is circumscribed within the lower semicircle shown. Use the law of sines to solve the triangle given the measures shown. How long is the longer chord? What do you notice about the triangle?

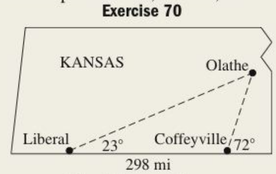
69. Height of a mountain: Approaching from the west, a group of hikers notes the angle of elevation to the summit of a steep mountain is 35° at a distance of 1250 meters. Arriving at the base of the mountain, they estimate this side of the mountain has an average slope of 48°. (a) Find the slant height of the mountain's west side. (b) Find the slant height of the east side of the mountain, if the east side has an average slope of 65°. (c) How tall is the mountain?

Exercise 69



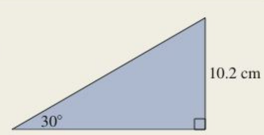
70. Distance on a map: Coffeyville and Liberal, Kansas, lie along the state's southern border and are roughly 298 miles apart. Olathe, Kansas, is very near the state's eastern border at an angle of 23° with Liberal and 72° with Coffeyville (using the southern border as one side of the angle). (a) Compute the distance between these cities. (b) What is the shortest (straight line) distance from Olathe to the southern border of Kansas?

Exercise 70

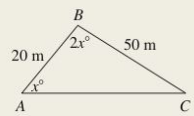


▶ EXTENDING THE CONCEPT

71. Solve the triangle shown in three ways—first by using the law of sines, second using right triangle trigonometry, and third using the standard 30-60-90 triangle. Was one method "easier" than the others? Use these connections to express the irrational number $\sqrt{3}$ as a quotient of two trigonometric functions of an angle. Can you find a similar expression for $\sqrt{2}$?



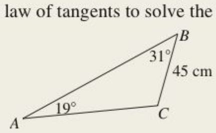
72. Use the law of sines and any needed identities to solve the triangles shown.



73. Similar to the law of sines, there is a *law of tangents.* The law says for any triangle

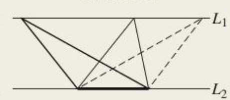
$$ABC, \frac{a+b}{a-b} = \frac{\tan\left[\frac{1}{2}(A+B)\right]}{\tan\left[\frac{1}{2}(A-B)\right]}.$$

Use the law of tangents to solve the triangle shown.



74. Lines L_1 and L_2 shown are parallel. The three triangles between these lines all share the same base (in bold). Explain why all three triangles must have the same area.

Exercise 74



75. A UFO is sighted on a direct line between the towns of Batesville and Cave City, sitting stationary in the sky. The towns are 13 mi apart as the crow flies. A student in Batesville calls a friend in Cave City and both take measurements of the angle of elevation: 35° from Batesville and 42° from Cave City. Suddenly the UFO zips across the sky at a level altitude heading directly for Cave City, then stops and hovers long enough for an additional measurement from Batesville: 24°. If the UFO was in motion for 1.2 sec, at what average speed (in mph) did it travel?

▶ MAINTAINING YOUR SKILLS

76. **(6.7)** Find all solutions to the equation $2 \sin x = \cos(2x)$

77. **(6.2)** Prove the given identity: $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

78. **(3.3)** Write an equation for the real polynomial with smallest degree, possible, having the solutions $x = 2$, $x = -1$, and $x = 1 + 2i$.

79. **(2.3)** Given the points $(-5, -3)$ and $(4, 2)$, find (a) the equation of the line containing these points and (b) the distance between these points.

7.2 The Law of Cosines; the Area of a Triangle

Learning Objectives

In Section 7.2 you will learn how to:

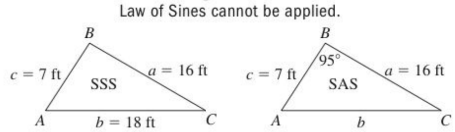
- ❑ **A.** Apply the law of cosines when two sides and an included angle are known (SAS)
- ❑ **B.** Apply the law of cosines when three sides are known (SSS)
- ❑ **C.** Solve applications using the law of cosines
- ❑ **D.** Use trigonometry to find the area of a triangle

The distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is traditionally developed by placing two arbitrary points on a rectangular coordinate system and using the Pythagorean theorem. The relationship known as the *law of cosines* is developed in much the same way, but this time by using *three* arbitrary points (the vertices of a triangle). After giving the location of one vertex in trigonometric form, we obtain a formula that enables us to solve SSS and SAS triangles, which cannot be solved using the law of sines alone.

A. The Law of Cosines and SAS Triangles

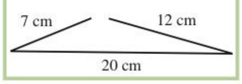
In situations where all three sides are known (but no angles), the law of sines cannot be applied. The same is true when two sides and the angle between them are known, since we must have an angle opposite one of the sides. In these two cases (Figure 7.21), side-side-side (**SSS**) and side-angle-side (**SAS**), we use the **law of cosines.**

Figure 7.21
Law of Sines cannot be applied.



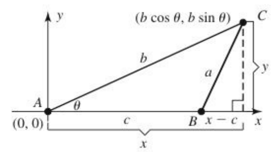
WORTHY OF NOTE

Keep in mind that the sum of any two sides of a triangle must be greater than the remaining side. For example, if $a = 7$, $B = 20$, and $C = 12$, no triangle is possible (see the figure).



To solve these cases, it's evident we need additional insight on the unknown angles. Consider a general triangle ABC on the rectangular coordinate system conveniently placed with vertex A at the origin, side c along the x-axis, and the vertex C at some point (x, y) in QI (Figure 7.22). Note $\cos\theta = \frac{x}{b}$ giving $x = b\cos\theta$, and $\sin\theta = \frac{y}{b}$ or $y = b\sin\theta$. This means we can write the point (x, y) as $(b\cos\theta, b\sin\theta)$ as shown, and use the Pythagorean theorem with side $x - c$ to find the length of side a of the exterior, right triangle. It follows that

Figure 7.22



$$\begin{aligned}
a^2 &= (x - c)^2 + y^2 && \text{Pythagorean theorem}\\
&= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{substitute } b\cos\theta \text{ for } x \text{ and } b\sin\theta \text{ for } y\\
&= b^2\cos^2\theta - 2bc\cos\theta + c^2 + b^2\sin^2\theta && \text{square binomial, square term}\\
&= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{rearrange terms}\\
&= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{factor out } b^2\\
&= b^2 + c^2 - 2bc\cos\theta && \text{substitute 1 for } \cos^2\theta + \sin^2\theta
\end{aligned}$$

We now have a formula relating all three sides and an included angle. Since the naming of the angles is purely arbitrary, the formula can be used in any of the three forms shown. For the derivation of the formula where $\angle B$ is acute, **see Exercise 61.**

The Law of Cosines

For any triangle ABC and corresponding sides a, b, and c,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

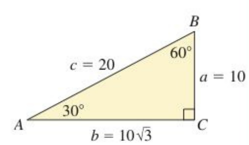
Note the relationship between the indicated angle and the squared term.

In words, the law of cosines says that the square of any side is equal to the sums of the squares of the other two sides, minus twice their product times the cosine of the included angle. It is interesting to note that if the included angle is 90°, the formula reduces to the Pythagorean theorem since $\cos 90° = 0$.

EXAMPLE 1 ▶ Verifying the Law of Cosines

For the triangle shown, verify:

a. $c^2 = a^2 + b^2 - 2ab\cos C$
b. $b^2 = a^2 + c^2 - 2ac\cos B$



Solution ▶ Note the included angle C is a right angle.

a.
$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos C\\
20^2 &= 10^2 + (10\sqrt{3})^2 - 2(10\sqrt{3})(10)\cos 90°\\
400 &= 100 + 300 - 0\\
&= 400 \checkmark
\end{aligned}$$

b.
$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos B\\
(10\sqrt{3})^2 &= 10^2 + 20^2 - 2(10)(20)\cos 60°\\
300 &= 100 + 400 - 400\left(\frac{1}{2}\right)\\
&= 500 - 200\\
&= 300 \checkmark
\end{aligned}$$

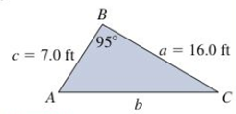
Now try Exercises 7 through 14 ▶

CAUTION ► When evaluating the law of cosines, a common error is to combine the coefficient of cos θ with the squared terms (the terms shown in blue): $a^2 = b^2 + c^2 - 2bc \cos A$. Be sure to use the correct order of operations when simplifying the expression.

Once additional information about the triangle is known, the law of sines can be used to complete the solution.

EXAMPLE 2 ► Solving a Triangle Using the Law of Cosines—SAS

Solve the triangle shown. Write the solution in table form.



Solution ► The given information is SAS. Apply the law of cosines with respect to side b and $\angle B$:

$b^2 = a^2 + c^2 - 2ac \cos B$ — law of cosines with respect to b

$b^2 = (16)^2 + (7)^2 - 2(16)(7)\cos 95°$ — substitute known values

≈ 324.522886 — simplify

$b \approx 18.0$ — $\sqrt{324.522886} \approx 18.0$

We now have side b opposite $\angle B$, and complete the solution using the law of sines, selecting the smaller angle to avoid the ambiguous case (we *could* apply the law of cosines again, if we chose).

$\frac{\sin C}{c} = \frac{\sin B}{b}$ — law of sines applied to $\angle C$ and $\angle B$

$\frac{\sin C}{7} \approx \frac{\sin 95°}{18}$ — substitute given values

$\sin C \approx 7 \cdot \frac{\sin 95°}{18}$ — solve for sin C

$C \approx \sin^{-1}\left(\frac{7 \sin 95°}{18}\right)$ — apply $\sin^{-1}$

$\approx 22.8°$ — result

For the remaining angle, $\angle C$: $180° - (95° + 22.8°) = 62.2°$. The finished solution is shown in the table (given information is in bold).

Angles	Sides (ft)
$A \approx 62.2°$	$\mathbf{a = 16.0}$
$\mathbf{B = 95.0°}$	$b \approx 18.0$
$C \approx 22.8°$	$\mathbf{c = 7.0}$

WORTHY OF NOTE

After using the law of cosines, we often use the law of sines to complete a solution. With a little foresight, we can avoid the ambiguous case—since the ambiguous case occurs only if θ *could be* obtuse (the largest angle of the triangle). After calculating the third side of a SAS triangle using the law of cosines, use the law of sines to find the smallest angle, since it cannot be obtuse. For SSS triangles, using the law of cosines to find the largest angle will ensure that when the second angle is found using the law of sines, it cannot be obtuse.

Now try Exercises 15 through 26 ►

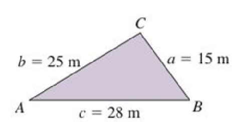
✓ A. You've just learned how to apply the law of cosines when two sides and an included angle are known (SAS)

B. The Law of Cosines and SSS Triangles

When three sides of a triangle are given, we use the law of cosines to find any one of the three angles. As a good practice, we first find the *largest* angle, or the angle opposite the largest side. This will ensure that the remaining two angles are acute, avoiding the ambiguous case if the law of sines is used to complete the solution.

EXAMPLE 3 ► Solving a Triangle Using the Law of Cosines—SSS

Solve the triangle shown. Write the solution in table form, with angles rounded to tenths of a degree.



Solution ▶ The information is given as SSS. Since side c is the longest side, we apply the law of cosines with respect to side c and $\angle C$:

$$c^2 = a^2 + b^2 - 2ab\cos C \quad \text{law of cosines with respect to } c$$
$$28^2 = (15)^2 + (25)^2 - 2(15)(25)\cos C \quad \text{substitute known values}$$
$$784 = 850 - 750\cos C \quad \text{simplify}$$
$$-66 = -750\cos C \quad \text{isolate variable term}$$
$$0.088 = \cos C \quad \text{divide}$$
$$\cos^{-1}0.088 = C \quad \text{solve for } C$$
$$85.0 \approx C \quad \text{result}$$

We now have side c opposite $\angle C$ and finish up using the law of sines.

$$\frac{\sin A}{a} = \frac{\sin C}{c} \quad \text{law of sines applied to } \angle A \text{ and } \angle C$$
$$\frac{\sin A}{15} \approx \frac{\sin 85°}{28} \quad \text{substitute given values}$$
$$\sin A \approx 15 \cdot \frac{\sin 85°}{28} \quad \text{solve for } \sin A$$
$$\approx 0.5336757311 \quad \text{simplify}$$
$$A \approx \sin^{-1}0.5336757311 \quad \text{solve for } A$$
$$\approx 32.3° \quad \text{result}$$

Since the remaining angle must be acute, we compute it directly. $\angle B$: $180° - (85° + 32.3°) = 62.7°$. The finished solution is shown in the table, with the information originally given shown in bold.

Angles	Sides (m)
$A \approx 32.3°$	$\mathbf{a = 15}$
$B \approx 62.7°$	$\mathbf{b = 25}$
$C \approx 85°$	$\mathbf{c = 28}$

B. You've just learned how to apply the law of cosines when three sides are known (SSS)

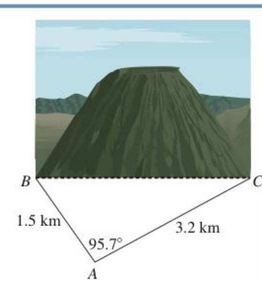
Now try Exercises 27 through 34 ▶

C. Applications Using the Law of Cosines

As with the law of sines, the law of cosines has a large number of applications from very diverse fields including geometry, navigation, surveying, and astronomy, as well as being put to use in solving recreational exercises **(see Exercises 37 through 40).**

EXAMPLE 4 ▶ Solving an Application of the Law of Cosines—Geological Surveys

A volcanologist needs to measure the distance across the base of an active volcano. Distance AB is measured at 1.5 km, while distance AC is 3.2 km. Using a theodolite (a sighting instrument used by surveyors), angle BAC is found to be 95.7°. What is the distance across the base?



Solution ▶ The information is given as SAS. To find the distance BC across the base of the volcano, we apply the law of cosines with respect to $\angle A$.

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{law of cosines with respect to } a$$
$$= (1.5)^2 + (3.2)^2 - 2(1.5)(3.2)\cos 95.7° \quad \text{substitute known values}$$
$$\approx 13.44347 \quad \text{simplify}$$
$$a \approx 3.7 \quad \text{solve for } a$$

The volcano is approximately 3.7 km wide at its base.

C. You've just learned how to solve applications using the law of cosines

Now try Exercises 41 through 52 ▶

A variety of additional applications can be found in the exercise set **(see Exercises 45 through 52).**

D. Trigonometry and the Area of a Triangle

While you're likely familiar with the most common formula for a triangle's area, $\mathcal{A} = \frac{1}{2}bh$, there are actually over 20 formulas for computing this area. Many involve basic trigonometric ideas, and we'll use some of these ideas to develop three additional formulas here.

For $\mathcal{A} = \frac{1}{2}bh$, recall that b represents the length of a designated base, and h represents the length of the altitude drawn to that base (see Figure 7.23). If the height h is unknown, but sides a and b with angle C between them are known, h can be found using $\sin C = \frac{h}{a}$, giving $h = a \sin C$. Figure 7.24 indicates the same result is obtained if C is obtuse, since $\sin(180° - C) = \sin C$. Substituting for in the formula $\mathcal{A} = \frac{1}{2}b$ gives $\mathcal{A} = \frac{1}{2}b$, or $\mathcal{A} = \frac{1}{2}ab \sin C$ in more common form. Since naming the angles in a triangle is arbitrary, the formulas $\mathcal{A} = \frac{1}{2}bc \sin A$ and $\mathcal{A} = \frac{1}{2}ac \sin B$ can likewise be obtained.

Figure 7.23

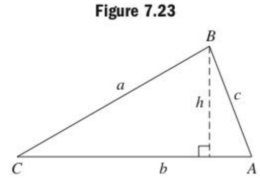
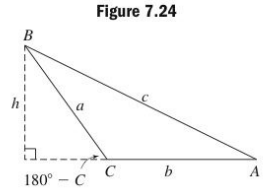


Figure 7.24



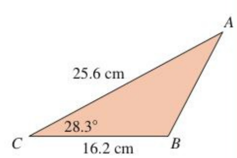
Area Given Two Sides and an Included Angle (SAS)

1. $\mathcal{A} = \frac{1}{2}ab \sin C$ **2.** $\mathcal{A} = \frac{1}{2}bc \sin A$ **3.** $\mathcal{A} = \frac{1}{2}ac \sin B$

In words, the formulas say the area of a triangle is equal to one-half the product of two sides times the sine of the angle between them.

EXAMPLE 5 ► **Finding the Area of a Nonright Triangle**

Find the area of $\triangle ABC$, if $a = 16.2$ cm, $b = 25.6$ cm, and $C = 28.3°$.



Solution ► Since sides a and b and angle C are given, we apply the first formula.

$$\mathcal{A} = \frac{1}{2}ab \sin C \quad \text{area formula}$$

$$= \frac{1}{2}(16.2)(25.6) \sin 28.3° \quad \text{substitute 15.2 for } a\text{, 25.6 for } b\text{, and 28.3° for } C$$

$$\approx 98.3 \text{ cm}^2 \quad \text{result}$$

The area of this triangle is approximately 98.3 cm^2.

Now try Exercises 53 and 54 ►

Using these formulas, a second formula type requiring two angles and one side (AAS or ASA) can be developed. Solving for b in $\mathcal{A} = \frac{1}{2}bc\sin A$ gives $b = \frac{2\mathcal{A}}{c\sin A}$. Likewise, solving for a in $\mathcal{A} = \frac{1}{2}ac\sin B$ yields $a =$. Substituting these for b and in $\mathcal{A} = \frac{1}{2}ab\sin C$ gives

$$\mathcal{A} = \frac{1}{2}\cdot a\cdot b\cdot \sin C \qquad \text{given formula}$$

$$2\mathcal{A} = \frac{2\mathcal{A}}{c\sin B}\cdot\frac{2\mathcal{A}}{c\sin A}\cdot\sin C \qquad \text{substitute } \frac{2\mathcal{A}}{c\sin B} \text{ for } a,\ \frac{2\mathcal{A}}{c\sin A} \text{ for } b\text{; multiply by 2}$$

$$c^2\sin A\cdot\sin B = 2\mathcal{A}\cdot\sin C \qquad \text{multiply by } c\sin A\cdot c\sin B\text{; divide by } 2\mathcal{A}$$

$$\frac{c^2\sin A\sin B}{2\sin C} = \mathcal{A} \qquad \text{solve for } \mathcal{A}$$

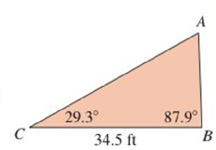
As with the previous formula, versions relying on side a or side b can also be found.

Area Given Two Angles and Any Side (AAS/ASA)

1. $\mathcal{A} = \frac{c^2\cdot\sin A\cdot\sin B}{2\sin C}$ **2.** $\mathcal{A} = \frac{a^2\cdot\sin B\cdot\sin C}{2\sin A}$ **3.** $\mathcal{A} = \frac{b^2\cdot\sin A\cdot\sin C}{2\sin B}$

EXAMPLE 6 ► Finding the Area of a Nonright Triangle

Find the area of $\triangle ABC$ if $a = 34.5$ ft, $B = 87.9°$, and $C = 29.3°$.



Solution ► Since side a is given, we apply the second version of the formula. First we find the measure of angle A, then make the appropriate substitutions:

$$A = 180° - (87.9 + 29.3)° = 62.8°$$

$$\mathcal{A} = \frac{a^2\sin B\sin C}{2\sin A} \qquad \text{area formula—side } a$$

$$= \frac{(34.5)^2\sin 87.9°\sin 29.3°}{2\sin 62.8°} \qquad \text{substitute 34.5 for } a,\ 87.9° \text{ for } B,\ 29.3° \text{ for } C\text{, and } 62.8° \text{ for } A$$

$$\approx 327.2\ \text{ft}^2 \qquad \text{simplify}$$

The area of this triangle is approximately 327.2 ft^2.

Now try Exercises 55 and 56 ►

Our final formula for a triangle's area is a useful addition to the other two, as it requires only the lengths of the three sides. The development of the formula requires only a Pythagorean identity and solving for the angle C in the law of cosines, as follows.

$$a^2 + b^2 - 2ab\cos C = c^2 \qquad \text{law of cosines}$$

$$a^2 + b^2 - c^2 = 2ab\cos C \qquad \text{add } 2ab\cos C\text{, subtract } c^2$$

$$\frac{a^2 + b^2 - c^2}{2ab} = \cos C \qquad \text{divide by } 2ab$$

Beginning with our first area formula, we then have

$$\mathcal{A} = \frac{1}{2}ab\sin C \quad \text{previous area formula}$$

$$= \frac{1}{2}ab\sqrt{1-\cos^2 C} \quad \sin^2 C + \cos^2 C = 1 \rightarrow \sin C = \sqrt{1-\cos^2 C}$$

$$= \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} \quad \text{substitute } \frac{a^2+b^2+c^2}{2ab} \text{ for } \cos C$$

and can find the area of any triangle given its three sides. While the formula certainly serves this purpose, it is not so easy to use. By working algebraically and using the perimeter of the triangle, we can derive a more elegant version.

$$\mathcal{A}^2 = \frac{1}{4}a^2b^2\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right] \quad \text{square both sides}$$

$$= \frac{1}{4}a^2b^2\left[1+\left(\frac{a^2+b^2-c^2}{2ab}\right)\right]\left[1-\left(\frac{a^2+b^2-c^2}{2ab}\right)\right] \quad \text{factor as a difference of squares}$$

$$= \frac{1}{4}a^2b^2\left[\frac{2ab+a^2+b^2-c^2}{2ab}\right]\left[\frac{2ab-a^2-b^2+c^2}{2ab}\right] \quad 1 = \frac{2ab}{2ab}\text{; combine terms}$$

$$= \frac{1}{4}\left[\frac{(a^2+2ab+b^2)-c^2}{2}\right]\left[\frac{-(a^2-2ab+b^2)+c^2}{2}\right] \quad \text{rewrite/regroup numerator; cancel } a^2b^2$$

$$= \frac{1}{16}[(a+b)^2-c^2][c^2-(a-b)^2] \quad \text{factor (binomial squares)}$$

$$= \frac{1}{16}(a+b+c)(a+b-c)(c+a-b)(c-a+b) \quad \text{factor (difference of squares)}$$

For the perimeter $p = a + b + c$, we note the following relationships:

$$a+b-c = p-2c \qquad c+a-b = p-2b \qquad c-a+b = p-2a$$

and making the appropriate substitutions gives

$$= \frac{1}{16}p(p-2c)(p-2b)(p-2a) \quad \text{substitute}$$

While this would provide a usable formula for the area in terms of the perimeter, we can refine it further using the *semi*perimeter $s = \frac{a+b+c}{2} = \frac{p}{2}$. Since $\frac{1}{16} = \left(\frac{1}{2}\right)^4$, we can write the expression as

$$= \frac{p}{2}\left(\frac{p-2c}{2}\right)\left(\frac{p-2b}{2}\right)\left(\frac{p-2a}{2}\right) \quad \text{rewrite expression}$$

$$= \frac{p}{2}\left(\frac{p}{2}-c\right)\left(\frac{p}{2}-b\right)\left(\frac{p}{2}-a\right) \quad \text{simplify}$$

$$= s(s-c)(s-b)(s-a) \quad \text{substitute } s \text{ for } \frac{p}{2}$$

Taking the square root of each side produces what is known as **Heron's formula.**

$$\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{Heron's formula}$$

Heron's Formula

Given $\triangle ABC$ with sides a, b, and c and semiperimeter $s = \frac{a + b + c}{2}$, the area of the triangle is

$$\mathcal{A} = \sqrt{s(s - a)(s - b)(s - c)}$$

EXAMPLE 7 ▶ Solving an Application of Heron's Formula—Construction Planning

A New York City developer wants to build condominiums on the triangular lot formed by Greenwich, Watts, and Canal Streets. How many square meters does the developer have to work with if the frontage along each street is approximately 34.1 m, 43.5 m, and 62.4 m, respectively?

Solution ▶ The perimeter of the lot is $p = 34.1 + 43.5 + 62.4 = 140$ m, so $s = 70$ m. By direct substitution we obtain

$$\begin{aligned}
\mathcal{A} &= \sqrt{s(s - a)(s - b)(s - c)} && \text{Heron's formula}\\
&= \sqrt{70(70 - 34.1)(70 - 43.5)(70 - 62.4)} && \text{substitute known values}\\
&= \sqrt{70(35.9)(26.5)(7.6)} && \text{simplify}\\
&= \sqrt{506{,}118.2} && \text{multiply}\\
&\approx 711.4 && \text{result}
\end{aligned}$$

The developer has about 711.4 m^2 of land to work with.

D. You've just learned how to use trigonometry to find the area of a triangle

Now try Exercises 57 and 58 ▶

For a derivation of Heron's formula that does not depend on trigonometry, see Appendix IV.

7.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

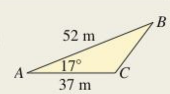
Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. When the information given is SSS or SAS, the law of ______ is used to solve the triangle.

2. Fill in the blank so that the law of cosines is complete: $c^2 = a^2 + b^2 -$ ______ $\cos C$

3. If the law of cosines is applied to a right triangle, the result is the same as the ______ theorem, since $\cos 90° = 0$.

4. Write out which version of the law of cosines you would use to begin solving the triangle shown: ______



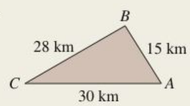
5. Solve the triangle in Exercise 4 using only the law of cosines, then by using the law of cosines followed by the law of sines. Which method was more efficient?

6. Begin with $a^2 = b^2 + c^2 - 2bc \cos A$ and write $\cos A$ in terms of a, b, and c (solve for $\cos A$). Why must $b^2 + c^2 - a^2 < 2bc$ hold in order for a solution to exist?

▶ DEVELOPING YOUR SKILLS

Determine whether the law of cosines can be used to begin the solution process for each triangle.

7.



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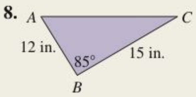

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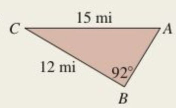

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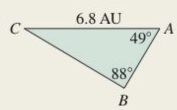
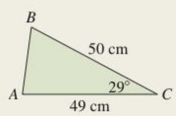


11.



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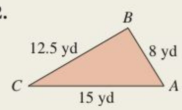
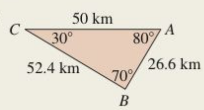


For each triangle, verify all three forms of the law of cosines.

13.



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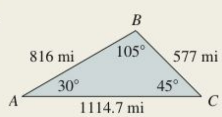


Solve each of the following equations for the unknown part.

15. $4^2 = 5^2 + 6^2 - 2(5)(6)\cos B$

16. $12.9^2 = 15.2^2 + 9.8^2 - 2(15.2)(9.8)\cos C$

17. $a^2 = 9^2 + 7^2 - 2(9)(7)\cos 52°$

18. $b^2 = 3.9^2 + 9.5^2 - 2(3.9)(9.5)\cos 30°$

19. $10^2 = 12^2 + 15^2 - 2(12)(15)\cos A$

20. $202^2 = 182^2 + 98^2 - 2(182)(98)\cos B$

Solve each triangle using the law of cosines.

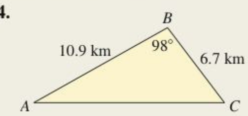
21. side $a = 75$ cm
$\angle C = 38°$
side $b = 32$ cm

22. side $b = 385$ m
$\angle C = 67°$
side $a = 490$ m

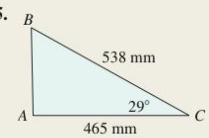
23. side $c = 25.8$ mi
$\angle B = 30°$
side $a = 12.9$ mi

Solve using the law of cosines (if possible). Label each triangle appropriately before you begin.

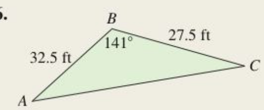
24.



25.



26.

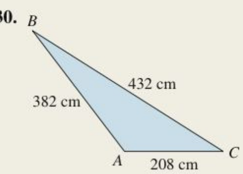


27. side $c = 10\sqrt{3}$ in.
side $b = 6\sqrt{3}$ in.
side $a = 15\sqrt{3}$ in.

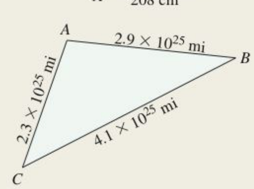
28. side $a = 282$ ft
side $b = 129$ ft
side $c = 300$ ft

29. side $a = 32.8$ km
side $b = 24.9$ km
side $c = 12.4$ km

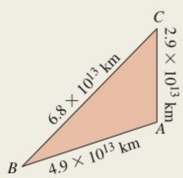
30.



31.



32.

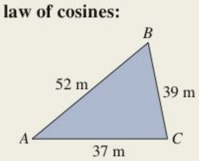


33. side $a = 12\sqrt{3}$ yd
side $b = 12.9$ yd
side $c = 9.2$ yd

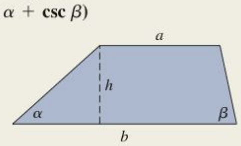
34. side $a = 36.5$ AU
side $b = 12.9$ AU
side $c = 22$ AU

▶ WORKING WITH FORMULAS

35. Alternative form for the law of cosines:
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
By solving the law of cosines for the cosine of the angle, the formula can be written as shown. Derive this formula (solve for $\cos \theta$), beginning from $a^2 = b^2 + c^2 - 2bc \cos A$, then use this form to begin the solution of the triangle given.

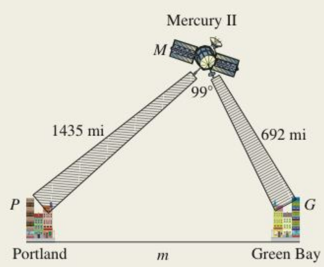


36. The Perimeter of a Trapezoid:
$$P = a + b + h(\csc \alpha + \csc \beta)$$
The perimeter of a trapezoid can be found using the formula shown, where a and b represent the lengths of the parallel sides, h is the height of the trapezoid, and α and β are the base angles. Find the perimeter of Trapezoid Park (to the nearest foot) if $a = 5000$ ft, $b = 7500$ ft, and $h = 2000$ ft, with base angles $\alpha = 42°$ and $\beta = 78°$.

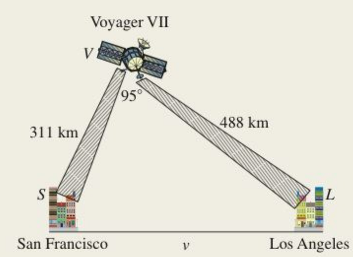


▶ APPLICATIONS

37. Distance between cities: The satellite Mercury II measures its distance from Portland and from Green Bay using radio waves as shown. Using an on-board sighting device, the satellite determines that $\angle M$ is 99°. How many miles is it from Portland to Green Bay?

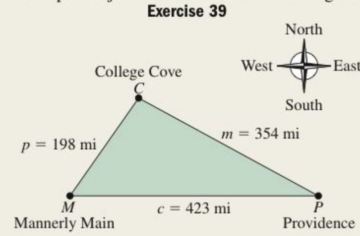


38. Distance between cities: Voyager VII measures its distance from Los Angeles and from San Francisco using radio waves as shown. Using an on-board sighting device, the satellite determines $\angle V$ is 95°. How many kilometers separate Los Angeles and San Francisco?



39. Trip planning: A business executive is going to fly the corporate jet from Providence to College Cove.

Exercise 39



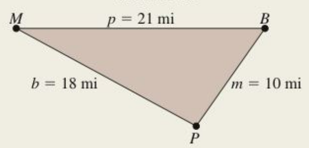
WORTHY OF NOTE

In navigation, there are two basic methods for defining a course. *Headings* are understood to be the amount of rotation from due north in the clockwise direction ($0 \le \theta < 360°$). *Bearings* give the number of degrees East or West from a due North or due South orientation, hence the angle indicated is always less than 90°. For instance, the bearing N 25° W and a heading of 335° would indicate the same direction.

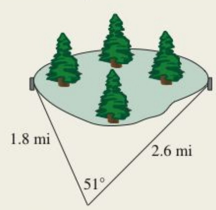
She calculates the distances shown using a map, with Mannerly Main for reference since it is due east of Providence. What is the measure of angle *P*? What heading should she set for this trip?

40. Trip planning: A troop of Scouts is planning a hike from ***M***ontgomery to ***P***attonville. They calculate the distances shown using a map, using ***B***radleyton for reference since it is due east of ***M***ontgomery. What is the measure of angle *M*? What heading should they set for this trip?

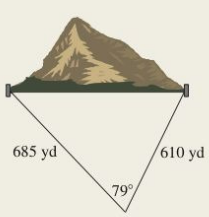
Exercise 40



41. Runway length: Surveyors are measuring a large, marshy area outside of the city as part of a feasibility study for the construction of a new airport. Using a theodolite and the markers shown gives the information indicated. If the main runway must be at least 11,000 ft long, and environmental concerns are satisfied, can the airport be constructed at this site (recall that 1 mi = 5280 ft)?



42. Tunnel length: An engineering firm decides to bid on a proposed tunnel through Harvest Mountain. In order to find the tunnel's length, the measurements shown are taken. (a) How long will the tunnel be? (b) Due to previous tunneling experience, the firm estimates a cost of $5000 per foot for boring through this type of rock and constructing the tunnel according to required specifications. If management insists on a 25% profit, what will be their minimum bid to the nearest hundred?

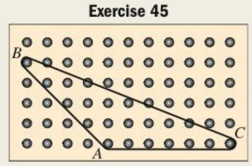


43. Aerial distance: Two planes leave Los Angeles International Airport at the same time. One travels due west (at heading 270°) with a cruising speed of 450 mph, going to Tokyo, Japan, with a group that seeks tranquility at the foot of Mount Fuji. The other travels at heading 225° with a cruising speed of 425 mph, going to Brisbane, Australia, with a group seeking adventure in the Great Outback. Approximate the distance between the planes after 5 hr of flight.

44. Nautical distance: Two ships leave Honolulu Harbor at the same time. One travels 15 knots (nautical miles per hour) at heading 150°, and is going to the Marquesas Islands (*Crosby, Stills, and Nash*). The other travels 12 knots at heading 200°, and is going to the Samoan Islands (*Samoa, le galu a tu*). How far apart are the two ships after 10 hr?

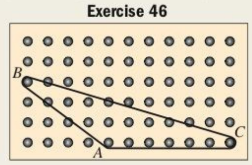
45. Geoboard geometry: A rubber band is placed on a geoboard (a board with all pegs 1 cm apart) as shown. Approximate the perimeter of the triangle formed by the rubber band *and* the angle formed at each vertex. (*Hint:* Use a standard triangle to find $\angle A$ and length $\overline{AB}$.)

Exercise 45



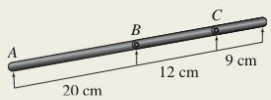
46. Geoboard geometry: A rubber band is placed on a geoboard as shown. Approximate the perimeter of the triangle formed by the rubber band *and* the angle formed at each vertex. (*Hint:* Use a Pythagorean triple, then find angle *A*.)

Exercise 46

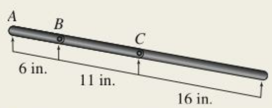


In Exercises 47 and 48, three rods are attached via pivot joints so the rods can be manipulated to form a triangle. Find the three angles of the triangle formed.

47.

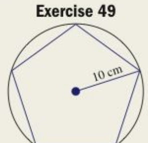


48.



49. Pentagon perimeter: Find the perimeter of a regular *pentagon* that is circumscribed by a circle with radius $r = 10$ cm.

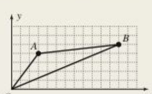
Exercise 49



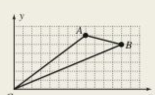
50. Hexagon perimeter: Find the perimeter of a regular *hexagon* that is circumscribed by a circle with radius $r = 15$ cm.

Solve the following triangles. Round sides and angles to the nearest tenth. (*Hint:* Use Pythagorean triples.)

51.

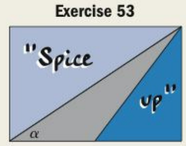


52.



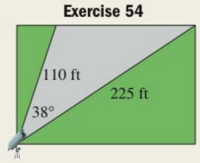
53. Billboard design: Creative Designs iNc. has designed a flashy, new billboard for one of its clients. Using a rectangular highway billboard measuring 20 ft by 30 ft, the primary advertising area is a triangle formed using the diagonal of the billboard as one side, and one-half the base as another (see figure). Use the dimensions given to find the angle α formed at the corner, then compute the area of the triangle using two sides and this included angle.

Exercise 53



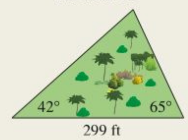
54. Area caught by surveillance camera: A stationary surveillance camera is set up to monitor activity in the parking lot of a shopping mall. If the camera has a 38° field of vision, how many square feet of the parking lot can it tape using the dimensions given?

Exercise 54

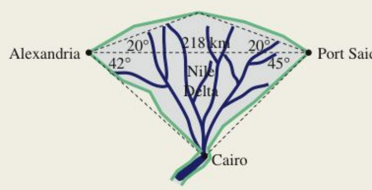


55. Pricing for undeveloped lots: Undeveloped land in a popular resort area is selling for \$3,000,000/acre. Given the dimensions of the lot shown, (a) find what percent of a full acre is being purchased (to the nearest whole percent), and (b) compute the cost of the lot. Recall that 1 acre $= 43{,}560$ ft^2.

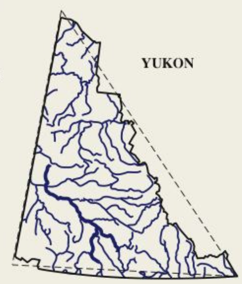
Exercise 55



56. Area of the Nile River Delta: The Nile River Delta is one of the world's largest. The delta begins slightly up river from the Egyptian capitol (Cairo) and stretches along the Mediterranean from Alexandria in the west to Port Said in the east (over 240 km). Approximate the area of this rich agricultural region using the two triangles shown.



57. Area of the Yukon Territory: The Yukon Territory in northwest Canada is roughly triangular in shape with sides of 1289 km, 1063 km, and 922 km. What is the approximate area covered by this territory?

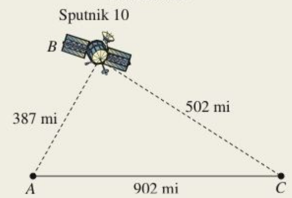


58. Alternate method for computing area: Referring to Exercise 53, since the dimensions of the billboard are known, all three sides of the triangle can actually be determined. Find the length of the sides rounded to the nearest whole, then use Heron's formula to find the area of the triangle. How close was your answer to that in Exercise 53?

▶ EXTENDING THE CONCEPT

59. No matter how hard I try, I cannot solve the triangle shown. Why?

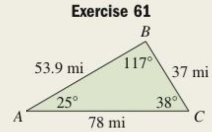
Exercise 59



60. In Figure 7.22 (page 725), note that if the x-coordinate of vertex B is greater than the x-coordinate of vertex C, $\angle B$ becomes acute, and $\angle C$ obtuse. How does this change the relationship between x and c? Verify the law of cosines remains unchanged.

61. For the triangle shown, verify that $c = b\cos A + a\cos B$, then use two different forms of the law of cosines to show this relationship holds for *any* triangle ABC.

Exercise 61



62. Most students are familiar with this double-angle formula for cosine: $\cos(2\theta) = \cos^2\theta - \sin^2\theta$. The *triple* angle formula for cosine is $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$. Use the formula to find an exact value for cos 135°. Show that you get the same result as when using a reference angle.

▶ MAINTAINING YOUR SKILLS

63. **(4.4)** Write the expression as a single term in simplest form: $2\log_2 4 + 2\log_2 3 - 2\log_2 6$

64. **(5.4)** State exact forms for each of the following: $\sin\left(\frac{\pi}{6}\right)$, $\cos\left(\frac{7\pi}{6}\right)$, and $\tan\left(\frac{\pi}{3}\right)$.

65. **(5.7)** Use fundamental identities to find the values of all six trig functions that satisfy the conditions. $\sin x = -\frac{5}{13}$ and $\cos x > 0$.

66. **(3.2)** Use synthetic division to show $f(-2) > 0$ for $f(x) = -x^4 - x^3 + 7x^2 + x - 6$.

7.3 Vectors and Vector Diagrams

Learning Objectives

In Section 7.3 you will learn how to:

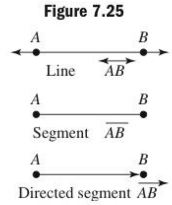
- ❑ **A.** Represent a vector quantity geometrically
- ❑ **B.** Represent a vector quantity graphically
- ❑ **C.** Perform defined operations on vectors
- ❑ **D.** Represent a vector quantity algebraically and find unit vectors
- ❑ **E.** Use vector diagrams to solve applications

The study of vectors is closely connected to the study of force, motion, velocity, and other related phenomena. Vectors enable us to quantify certain characteristics of these phenomena and to physically represent their magnitude and direction with a simple model. To quantify something means we assign it a relative numeric value for purposes of study and comparison. While very uncomplicated, this model turns out to be a powerful mathematical tool.

A. The Notation and Geometry of Vectors

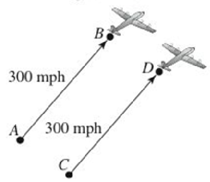
Measurements involving time, area, volume, energy, and temperature are called **scalar measurements** or **scalar quantities** because each can be adequately described by their magnitude alone and the appropriate unit or "scale." The related real number is simply called a **scalar.** Concepts that require more than a single quantity to describe their attributes are called **vector quantities.** Examples might include force, velocity, and displacement, which require knowing a magnitude *and* direction to describe them completely.

Figure 7.25



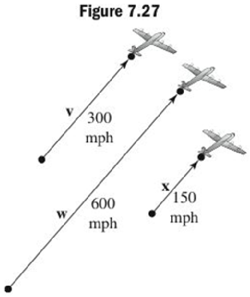
To begin our study, consider two identical airplanes flying at 300 mph, on a parallel course and in the same direction. Although we don't know how far apart they are, what direction they're flying, or if one is "ahead" of the other, we can still model, "300 mph on a parallel course," using **directed line segments** (Figure 7.25). Drawing these segments parallel with the arrowheads pointing the same way models the direction of flight, while drawing segments the *same length* indicates the velocities are equal. The directed segment used to represent a vector quantity is simply called a **vector.** In this case the length of the vector models the **magnitude** of the velocity, while the arrowhead indicates the **direction** of travel. The origin of the segment is called the **initial point,** with the arrowhead pointing to the **terminal point.** Both are labeled using capital letters as shown in Figure 7.26 and we call this a *geometric representation* of the vectors.

Figure 7.26



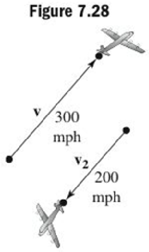
Vectors can be named using the initial and terminal points that define them (initial point first) as in $\overrightarrow{AB}$ and $\overrightarrow{CD}$, or using a bold, small case letter with the favorites being **v** (first letter of the word vector) and **u.** Other small case, bold letters can be used and subscripted vector names ($\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$) are also common. Two **vectors are equal** if they have the same magnitude and direction. For $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{CD}$, we can say $\mathbf{u} = \mathbf{v}$ or $\overrightarrow{AB} = \overrightarrow{CD}$ since both airplanes are flying at the same speed and in the same direction (Figure 7.26).

Figure 7.27



Based on these conventions, it seems reasonable to represent an airplane flying at 600 mph with a vector that is twice as long as **u** and **v,** and one flying at 150 mph with a vector that is half as long. If all planes are flying in the same direction on a parallel course, we can represent them geometrically as shown in Figure 7.27, and state that $\mathbf{w} = 2\mathbf{v}$, $\mathbf{x} = \frac{1}{2}\mathbf{v}$, and $\mathbf{w} = 4\mathbf{x}$. The multiplication of a vector by a constant is called **scalar multiplication,** since the product changes only the scale or size of the vector and not its direction.

Figure 7.28

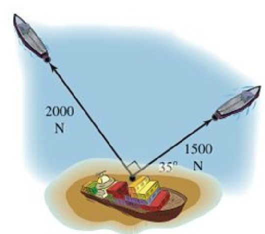


Finally, consider the airplane represented by vector $\mathbf{v}_2$, flying at 200 mph on a parallel course *but in the opposite direction* (see Figure 7.28). In this case, the directed segment will be $\frac{200}{300} = \frac{2}{3}$ as long as **v** and point in the opposite or "negative" direction. In perspective we can now state: $\mathbf{v}_2 = -\frac{2}{3}\mathbf{v}$, $\mathbf{v}_2 = -\frac{1}{3}\mathbf{w}$, $\mathbf{v}_2 = -\frac{4}{3}\mathbf{x}$, or any equivalent form of these equations.

EXAMPLE 1 ► **Using Geometric Vectors to Model Forces Acting on a Point**

Two tugboats are attempting to free a barge that is stuck on a sand bar. One is pulling with a force of 2000 newtons (N) in a certain direction, the other is pulling with a force of 1500 N in a direction that is *perpendicular to the first.* Represent the situation geometrically using vectors.

Solution ► We could once again draw a vector of arbitrary length and let it represent the 2000-N force applied by the first tugboat. For better perspective, we can actually use a ruler and choose a convenient length, say 6 cm. We then represent the pulling force of the second tug with a vector that is $\frac{1500}{2000} = \frac{3}{4}$ as long (4.5 cm), drawn at a 90° angle with relation to the first. Note that many correct solutions are possible, depending on the direction of the first vector drawn.

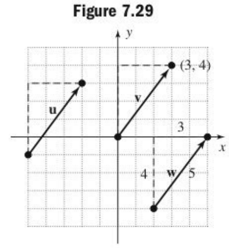


A. You've just learned how to represent a vector quantity geometrically

Now try Exercises 7 through 12 ►

B. Vectors and the Rectangular Coordinate System

Figure 7.29



Representing vectors geometrically (with a directed line segment) is fine for simple comparisons, but many applications involve numerous vectors acting on a single point or changes in a vector quantity over time. For these situations, a graphical representation in the coordinate plane helps to analyze this interaction. The only question is *where* to place the vector on the grid, and the answer is—it really doesn't matter. Consider the three vectors shown in Figure 7.29. From the initial point of each, counting four units in the vertical direction, then three units in the horizontal direction, puts us at the terminal point. This shows the vectors are all 5 units long (since a 3-4-5 triangle is formed) and are all parallel (since slopes are equal: $\frac{\Delta y}{\Delta x} = \frac{4}{3}$). In other words, they are **equivalent vectors.**

Since a vector's location is unimportant, we can replace any given vector with a unique and equivalent vector whose initial point is (0, 0), called the **position vector.**

WORTHY OF NOTE

For vector **u**, the initial and terminal points are $(-5, -1)$ and $(-2, 3)$, respectively, yielding the position vector $\langle -2 - (-5), 3 - (-1)\rangle = \langle 3, 4\rangle$ as before.

Position Vectors

For a vector **v** with initial point (x_1, y_1) and terminal point (x_2, y_2), the position vector for **v** is

$$\mathbf{v} = \langle x_2 - x_1, y_2 - y_1\rangle,$$

an equivalent vector with initial point (0, 0) and terminal point $(x_2 - x_1, y_2, -y_1)$.

For instance, the initial and terminal points of vector **w** in Figure 7.29 are $(2, -4)$ and $(5, 0)$, respectively, with $(5 - 2, 0 - (-4)) = (3, 4)$. Since $(3, 4)$ is also the terminal point of **v** (whose initial point is at the origin), **v** is the position vector for **u** and **w.** This observation also indicates that every geometric vector in the *xy*-plane corresponds to a unique ordered pair of real numbers (a, b), with a as the **horizontal component** and b as the **vertical component** of the vector. As indicated, we denote the vector in **component form** as $\langle \boldsymbol{a}, \boldsymbol{b}\rangle$, using the new notation to prevent confusing vector $\langle a, b\rangle$ with the ordered pair (a, b). Finally, while each of the vectors in Figure 7.29 has a component form of $\langle 3, 4\rangle$, the horizontal and vertical components can be read directly only from $\mathbf{v} = \langle 3, 4\rangle$, giving it a distinct advantage.

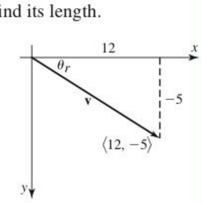
EXAMPLE 2 ► Verifying the Components of a Position Vector

Vector $\mathbf{v} = \langle 12, -5\rangle$ has initial point $(-4, 3)$.

a. Find the coordinates of the terminal point.

b. Verify the position vector for **v** is also $\langle 12, -5\rangle$ and find its length.

Solution ►



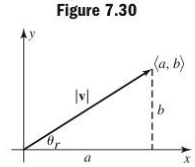
a. Since **v** has a horizontal component of 12 and a vertical component of −5, we add 12 to the *x*-coordinate and −5 to the *y*-coordinate of the initial point. This gives a terminal point of $(12 + (-4), -5 + 3) = (8, -2)$.

b. To verify we use the initial and terminal points to compute $\langle x_2 - x_1, y_2, -y_1\rangle$, giving a position vector of $\langle 8 - (-4), -2 - 3\rangle = \langle 12, -5\rangle$. To find its length we can use either the Pythagorean theorem or simply note that a 5-12-13 Pythagorean triple is formed. Vector **v** has a length of 13 units.

Now try Exercises 13 through 20 ►

For the remainder of this section, vector $\mathbf{v} = \langle a, b\rangle$ will refer to the unique position vector for all those equivalent to **v.** Upon considering the graph of $\langle a, b\rangle$ (shown in QI for convenience in Figure 7.30), several things are immediately evident. The length or **magnitude** of the vector, which is denoted $|\mathbf{v}|$, can be determined using the Pythagorean theorem: $|\mathbf{v}| = \sqrt{a^2 + b^2}$. In addition, basic trigonometry shows the horizontal component can be found using $\cos\theta = \frac{a}{|\mathbf{v}|}$ or $a = |\mathbf{v}|\cos\theta$, with the vertical component being $\sin\theta = \frac{b}{|\mathbf{v}|}$ or $b = |\mathbf{v}|\sin\theta$. Finally, we note the angle θ can be determined using $\tan\theta = \left(\frac{b}{a}\right)$, or $\theta_r = \tan^{-1}\left(\frac{b}{a}\right)$ and the quadrant of $\boldsymbol{v}$.

Figure 7.30



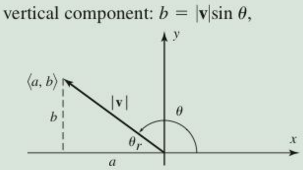
Vector Components in Trig Form

For a position vector $\mathbf{v} = \langle a, b\rangle$ and angle θ, we have

horizontal component: $a = |\mathbf{v}|\cos\theta$ vertical component: $b = |\mathbf{v}|\sin\theta$,

where $\theta_r = \tan^{-1}\left(\frac{b}{a}\right)$ and

$|\mathbf{v}| = \sqrt{a^2 + b^2}$



The ability to model characteristics of a vector using these equations is a huge benefit to solving applications, since we must often work out solutions using only the partial information given.

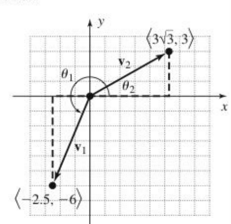
EXAMPLE 3 ► **Finding the Magnitude and Direction Angle of a Vector**

For $\mathbf{v}_1 = \langle -2.5, -6\rangle$ and $\mathbf{v}_2 = \langle 3\sqrt{3}, 3\rangle$,

a. Graph each vector and name the quadrant where it is located.

b. Find their magnitudes.

c. Find the angle θ for each vector (round to tenths of a degree as needed).



Solution ► **a.** The graphs of $\mathbf{v}_1$ and $\mathbf{v}_2$ are shown in the figure. Using the signs of each coordinate, we note that $\mathbf{v}_1$ is in QIII, and $\mathbf{v}_2$ is in QI.

b. $|\mathbf{v}_1| = \sqrt{(-2.5)^2 + (-6)^2} = \sqrt{6.25 + 36} = \sqrt{42.25} = 6.5$

$|\mathbf{v}_2| = \sqrt{(3\sqrt{3})^2 + (3)^2} = \sqrt{27 + 9} = \sqrt{36} = 6$

c. For $\mathbf{v}_1$: $\theta_r = \tan^{-1}\left(\frac{-6}{-2.5}\right) = \tan^{-1}(2.4) \approx 67.4°$

In QIII, $\theta \approx 247.4°$.

For $\mathbf{v}_2$: $\theta_r = \tan^{-1}\left(\frac{3}{3\sqrt{3}}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = 30°$

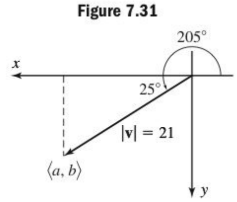
In QI, $\theta = 30°$.

Now try Exercises 21 through 24 ►

EXAMPLE 4 ► Finding the Horizontal and Vertical Components of a Vector

The vector $\mathbf{v} = \langle a, b \rangle$ is in QIII, has a magnitude of $|\mathbf{v}| = 21$, and forms an angle of 25° with the negative x-axis (Figure 7.31). Find the horizontal and vertical components of the vector, rounded to tenths.

Figure 7.31



Solution ► Begin by graphing the vector and setting up the equations for its components. For $\theta_r = 25°$, $\theta = 205°$.

For the horizontal component:

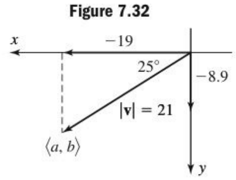
$$a = |\mathbf{v}|\cos\theta$$
$$= 21\cos 205°$$
$$\approx -19$$

For the vertical component:

$$b = |\mathbf{v}|\sin\theta$$
$$= 21\sin 205°$$
$$\approx -8.9$$

With **v** in QIII, its component form is approximately $\langle -19, -8.9 \rangle$. As a check, we apply the Pythagorean theorem: $\sqrt{(-19)^2 + (-8.9)^2} \approx 21$ ✓. See Figure 7.32.

Figure 7.32



Now try Exercises 25 through 30 ►

☑ **B.** You've just learned how to represent a vector quantity graphically

C. Operations on Vectors and Vector Properties

The operations defined for vectors have a close knit graphical representation. Consider a local park having a large pond with pathways around both sides, so that a park visitor can enjoy the view from either side. Suppose $\mathbf{v} = \langle 8, 2 \rangle$ is the position vector representing a person who decides to turn to the right at the pond, while $\mathbf{u} = \langle 2, 6 \rangle$ represents a person who decides to first turn left. At (8, 2) the first person changes direction and walks to (10, 8) on the other side of the pond, while the second person arrives at (2, 6) and turns to head for (10, 8) as well. This is shown graphically in Figure 7.33 and demonstrates that (1) a parallelogram is formed (opposite sides equal and parallel), (2) the path taken is unimportant relative to the destination, and (3) the coordinates of the destination represent the *sum of corresponding coordinates* from the terminal points of **u** and **v:** $(2, 6) + (8, 2) = (2 + 8, 6 + 2) = (10, 8)$. In other words, the result of adding **u** and **v** gives the new position vector $\mathbf{u} + \mathbf{v} = \mathbf{w}$, called the **resultant** or the **resultant vector.** Note the resultant vector is a diagonal of the parallelogram formed. Geometrically or graphically, the addition of vectors can be viewed as a "tail-to-tip" combination of one with another, by shifting one vector (without changing its direction) so that its tail (initial point) is at the tip (terminal point) of the other vector. This is illustrated in Figures 7.34 through 7.36.

Figure 7.33

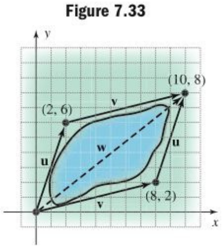


Figure 7.34
Given vectors u and v

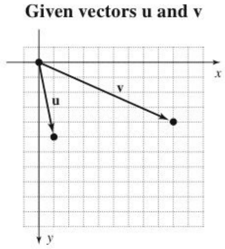


Figure 7.35
Shift vector v

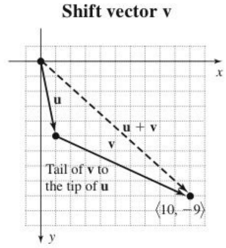
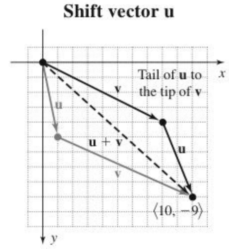


Figure 7.36
Shift vector u

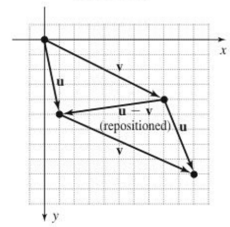


WORTHY OF NOTE

The geometry of vector subtraction is a key part of resolving a vector into orthogonal components that are nonquadrantal. Applications of this concept are wide ranging, and include thrust and drag forces, tension and stress limits in a cable, and others.

The subtraction of vectors can be understood as either as $\mathbf{u} - \mathbf{v}$ or $\mathbf{u} + (-\mathbf{v})$. Since the location of a vector is unimportant relative to the information it carries, vector subtraction can be interpreted as the *tip-to-tip diagonal* of the parallelogram from vector addition. In Figures 7.34 to 7.36, assume $\mathbf{u} = \langle 1, -5\rangle$ and $\mathbf{v} = \langle 9, -4\rangle$. Then $\mathbf{u} - \mathbf{v} = \langle 1, -5\rangle - \langle 9, -4\rangle = \langle 1 - 9, -5 + 4\rangle$ giving the position vector $\langle -8, -1\rangle$. By repositioning this vector with its tail at the tip of **v,** we note the new vector points directly at **u,** forming the diagonal (see Figure 7.37). Scalar multiplication of vectors also has a graphical representation that corresponds to the geometric description given earlier.

Figure 7.37



Operations on Vectors

Given vectors $\mathbf{u} = \langle a, b\rangle$, $\mathbf{v} = \langle c, d\rangle$, and a scalar k,

1. $\mathbf{u} + \mathbf{v} = \langle a + c, b + d\rangle$
2. $\mathbf{u} - \mathbf{v} = \langle a - c, b - d\rangle$
3. $k\mathbf{u} = \langle ka, kb\rangle$ for $k \in \mathbb{R}$

If $k > 0$, the new vector points in the same direction as **u.**
If $k < 0$, the new vector points in the opposite direction as **u.**

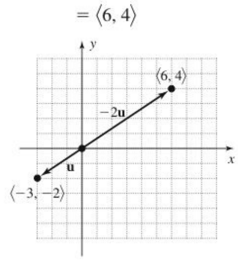
EXAMPLE 5 ► Representing Operations on Vectors Graphically

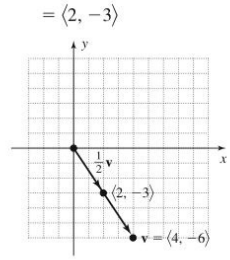
Given $\mathbf{u} = \langle -3, -2\rangle$ and $\mathbf{v} = \langle 4, -6\rangle$ compute each of the following and represent the result graphically:

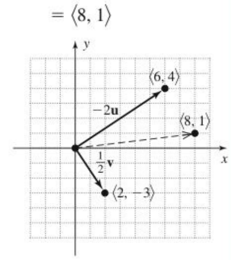
a. $-2\mathbf{u}$ **b.** $\frac{1}{2}\mathbf{v}$ **c.** $-2\mathbf{u} + \frac{1}{2}\mathbf{v}$

Note the relationship between part (c) and parts (a) and (b).

Solution ► **a.** $-2\mathbf{u} = -2\langle -3, -2\rangle = \langle 6, 4\rangle$ **b.** $\frac{1}{2}\mathbf{v} = \frac{1}{2}\langle 4, -6\rangle = \langle 2, -3\rangle$ **c.** $-2\mathbf{u} + \frac{1}{2}\mathbf{v} = \langle 6, 4\rangle + \langle 2, -3\rangle = \langle 8, 1\rangle$







Now try Exercises 31 through 48 ►

The properties that guide operations on vectors closely resemble the familiar properties of real numbers. Note we define the zero vector $\mathbf{0} = \langle 0, 0\rangle$ as one having no magnitude or direction.

Properties of Vectors

For vector quantities **u**, **v**, and **w** and real numbers c and k,

1. $1\mathbf{u} = \mathbf{u}$	2. $0\mathbf{u} = \mathbf{0} = k\mathbf{0}$
3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	4. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$
5. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	6. $(ck)\mathbf{u} = c(k\mathbf{u}) = k(c\mathbf{u})$
7. $\mathbf{u} + \mathbf{0} = \mathbf{u}$	8. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
9. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$	10. $(c + k)\mathbf{u} = c\mathbf{u} + k\mathbf{u}$

Proof of Property 3

For $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, we have

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \langle a, b\rangle + \langle c, d\rangle && \text{sum of } \mathbf{u} \text{ and } \mathbf{v} \\ &= \langle a + c, b + d\rangle && \text{vector addition} \\ &= \langle c + a, d + b\rangle && \text{commutative property} \\ &= \langle c, d\rangle + \langle a, b\rangle && \text{vector addition} \\ &= \mathbf{v} + \mathbf{u} && \text{result} \end{aligned}$$

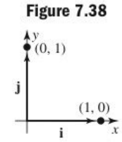
C. You've just learned how to perform defined operations on vectors

Proofs of the other properties are similarly derived (see **Exercises 89 through 97**).

D. Algebraic Vectors, Unit Vectors, and i, j Form

While the bold, small case **v** and the $\langle a, b\rangle$ notation for vectors has served us well, we now introduce an alternative form that is somewhat better suited to the **algebra of vectors,** and is used extensively in some of the physical sciences. Consider the vector $\langle 1, 0\rangle$, a vector 1 unit in length extending along the x-axis. It is called the **horizontal unit vector** and given the special designation **i** (not to be confused with the imaginary unit $i = \sqrt{-1}$). Likewise, the vector $\langle 0, 1\rangle$ is called the **vertical unit vector** and given the designation **j** (see Figure 7.38). Using scalar multiplication, the unit vector along the negative x-axis is $-\mathbf{i}$ and along the negative y-axis is $-\mathbf{j}$. Similarly, the vector $4\mathbf{i}$ represents a position vector 4 units long along the x-axis, and $-5\mathbf{j}$ represents a position vector 5 units long along the negative y-axis. Using these conventions, any nonquadrantal vector $\langle a, b\rangle$ can be written as a **linear combination** of **i** and **j**, with a and b expressed as multiples of **i** and **j**, respectively: $a\mathbf{i} + b\mathbf{j}$. These ideas can easily be generalized and applied to any vector.

Figure 7.38



WORTHY OF NOTE

Earlier we stated, "Two vectors were equal if they have the same magnitude and direction." Note that this means two vectors are equal if *their components are equal*.

Algebraic Vectors and i, j Form

For the unit vectors $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$, any arbitrary vector $\mathbf{v} = \langle a, b\rangle$ can be written as a linear combination of **i** and **j**:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$$

Graphically, **v** is being expressed as the resultant of a vector sum.

EXAMPLE 6 ► **Finding the Horizontal and Vertical Components of Algebraic Vectors**

Vector **u** is in QII, has a magnitude of 15, and makes an angle of 20° with the negative x-axis.

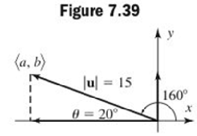
a. Graph the vector.

b. Find the horizontal and vertical components (round to one decimal place) then write **u** in component form.

c. Write **u** in terms of **i** and **j**.

Solution ► **a.** The vector is graphed in Figure 7.39.

Figure 7.39



b. **Horizontal Component** | **Vertical Component**

$a = |\mathbf{v}|\cos\theta$ | $b = |\mathbf{v}|\sin\theta$

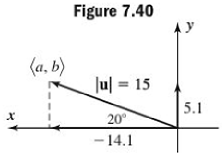
$= 15\cos 160°$ | $= 15\sin 160°$

≈ -14.1 | ≈ 5.1

With the vector in QII, $\mathbf{u} = \langle -14.1, 5.1\rangle$ in component form.

c. In terms of **i** and **j** we have $\mathbf{u} = -14.1\mathbf{i} + 5.1\mathbf{j}$. See Figure 7.40

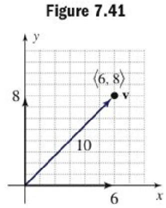
Figure 7.40



Now try Exercises 49 through 62 ►

Some applications require that we find a nonhorizontal, nonvertical vector one unit in length, having the same direction as a given vector **v**. To understand how this is done, consider vector $\mathbf{v} = \langle 6, 8\rangle$. Using the Pythagorean theorem we find $|\mathbf{v}| = 10$, and can form a 6-8-10 triangle using the horizontal and vertical components (Figure 7.41). Knowing that similar triangles have sides that are proportional, we can find a unit vector in the same direction as **v** by dividing all three sides by 10, giving a triangle with sides $\frac{3}{5}$, $\frac{4}{5}$, and 1. The new vector "**u**" (along the hypotenuse) indeed points in the same direction since we have merely shortened **v**, and is a unit vector since $\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 \rightarrow \frac{9}{25} + \frac{16}{25} = 1$. In retrospect, we have divided the components of vector **v** by its magnitude $|\mathbf{v}|$ (or multiplied components by the reciprocal of $|\mathbf{v}|$) to obtain the desired unit vector: $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 6, 8\rangle}{10} = \left\langle \frac{6}{10}, \frac{8}{10}\right\rangle = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$. In general we have the following:

Figure 7.41



Unit Vectors

For any nonzero vector $\mathbf{v} = \langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$, the vector

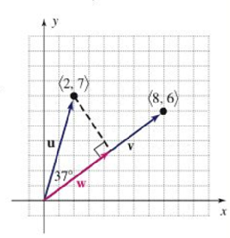
$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{a}{\sqrt{a^2 + b^2}}\mathbf{i} + \frac{b}{\sqrt{a^2 + b^2}}\mathbf{j}$$

is a unit vector in the same direction as **v**.

You are asked to verify this relationship in **Exercise 100.** In summary, for vector $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j}$, we find $|\mathbf{v}| = \sqrt{6^2 + 8^2} = 10$, so the unit vector pointing in the same direction is $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$. **See Exercises 63 through 74.**

EXAMPLE 7 ► **Using Unit Vectors to Find Coincident Vectors**

Vectors **u** and **v** form the 37° angle illustrated in the figure. Find the vector **w** (in red), which points in the same direction as **v** (is coincident with **v**) and forms the base of the right triangle shown.



Solution ► Using the Pythagorean theorem we find $|\mathbf{u}| \approx 7.3$ and $|\mathbf{v}| = 10$. Using the cosine of 37° the magnitude of **w** is then $|\mathbf{w}| \approx 7.3\cos 37°$ or about 5.8. To ensure that **w** will point in the same direction as **v**, we simply multiply the 5.8 magnitude by the unit vector for **v**: $|\mathbf{w}|\frac{\mathbf{v}}{|\mathbf{v}|} \approx (5.8)\frac{\langle 8, 6\rangle}{10} = (5.8)\langle 0.8, 0.6\rangle$, and we find that $\mathbf{w} \approx \langle 4.6, 3.5\rangle$. As a check we use the Pythagorean theorem: $\sqrt{4.6^2 + 3.5^2} = \sqrt{33.41} \approx 5.8$.

WORTHY OF NOTE

In this context **w** is called the projection of **u** on **v**, an idea applied more extensively in Section 7.4

☑ **D. You've just learned how to represent a vector quantity algebraically and find unit vectors**

Now try Exercises 75 through 78 ►

E. Vector Diagrams and Vector Applications

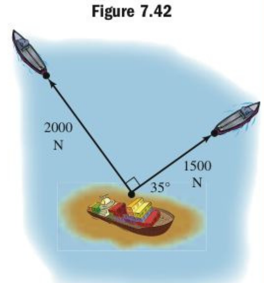
Applications of vectors are virtually unlimited, with many of these in the applied sciences. Here we'll look at two applications that are an extension of our work in this section. In Section 7.4 we'll see how vectors can be applied in a number of other creative and useful ways.

In Example 1, two tugboats were pulling on a barge to dislodge it from a sand bar, with the pulling force of each represented by a vector. Using our knowledge of vector components, vector addition, and **resultant forces** (a force exerted along the resultant), we can now determine the direction and magnitude of the resultant force if we know the angle formed by one of the vector forces and the barge.

EXAMPLE 8 ▶ Solving an Application of Vectors—Force Vectors Acting on a Barge

Two tugboats are attempting to free a barge that is stuck on a sand bar, and are exerting the forces shown in Figure 7.42. Find the magnitude and direction of the resultant force.

Figure 7.42



Solution ▶ Begin by orienting the diagram on a coordinate grid (see Figure 7.43). Since the angle between the vectors is 90°, we know the acute angle formed by the first tugboat and the x-axis is 55°. With this information, we can write each vector in "**i**, **j**" form and add the vectors to find the resultant.

For vector $\mathbf{v}_1$ (in QII):

Horizontal Component	Vertical Component
$a = \|\mathbf{v}_1\|\cos\theta$	$b = \|\mathbf{v}_1\|\sin\theta$
$= 2000\cos 125°$	$= 2000\sin 125°$
≈ -1147	≈ 1638

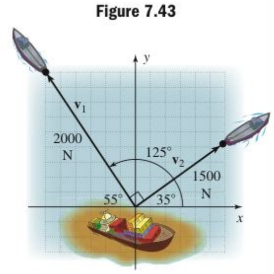
$$\mathbf{v}_1 \approx -1147\mathbf{i} + 1638\mathbf{j}.$$

For vector $\mathbf{v}_2$ (in QI):

Horizontal Component	Vertical Component
$a = \|\mathbf{v}_2\|\cos\theta$	$b = \|\mathbf{v}_2\|\sin\theta$
$= 1500\cos 35°$	$= 1500\sin 35°$
≈ 1229	≈ 860

$$\mathbf{v}_2 \approx 1229\mathbf{i} + 860\mathbf{j}.$$

Figure 7.43



This gives a resultant of $\mathbf{v}_1 + \mathbf{v}_2 \approx (-1147\mathbf{i} + 1638\mathbf{j}) + (1229\mathbf{i} + 860\mathbf{j}) = 82\mathbf{i} + 2498\mathbf{j}$, with magnitude $|\mathbf{v}_1 + \mathbf{v}_2| \approx \sqrt{82^2 + 2498^2} \approx 2499$ N. To find the direction of the force, we have $\theta_r = \tan^{-1}\left(\frac{2498}{82}\right)$, or about 88°.

Now try Exercises 81 and 82 ▶

It's worth noting that a single tugboat pulling at 88° with a force of 2499 N would have the same effect as the two tugs in the original diagram. In other words, the resultant vector $82\mathbf{i} + 2498\mathbf{j}$ truly represents the "result" of the two forces.

Knowing that the location of a vector is unimportant enables us to model and solve a great number of seemingly unrelated applications. Although the final example concerns aviation, **headings,** and crosswinds, the solution process has a striking similarity to the "tugboat" example just discussed. In navigation, headings involve a single angle, which is understood to be the amount of rotation from due north in the clockwise direction. Several headings are illustrated in Figures 7.44 through 7.47.

Figure 7.44

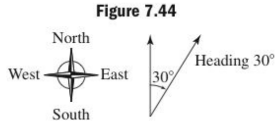


Figure 7.45

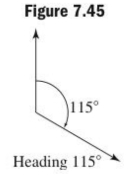


Figure 7.46

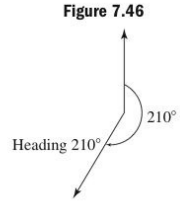
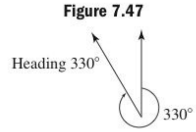


Figure 7.47



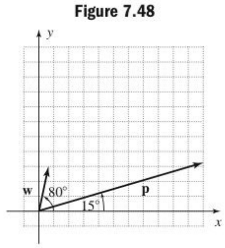
In order to keep an airplane on course, the captain must consider the direction and speed of any wind currents, since the plane's true course (relative to the ground) will be affected. Both the plane and the wind can be represented by vectors, with the plane's true course being the resultant vector.

EXAMPLE 9 ► **Solving an Application of Vectors—Airplane Navigation**

An airplane is flying at 240 mph, heading 75°, when it suddenly encounters a strong, 60 mph wind blowing *from* the southwest, heading 10°. What is the actual course and speed of the plane (relative to the ground) as it flies through this wind?

Solution ► Begin by drawing a vector **p** to represent the speed and direction of the airplane (Figure 7.48). Since the heading is 75°, the angle between the vector and the x-axis must be 15°. For convenience (and because location is unimportant) we draw it as a position vector. Note the vector **w** representing the wind will be $\frac{60}{240} = \frac{1}{4}$ as long, and can also be drawn as a position vector—with an acute 80° angle. To find the resultant, we first find the components of each vector, then add. For vector **w** (in QI):

Figure 7.48



Horizontal Component	Vertical Component
$a = \lvert\mathbf{w}\rvert\cos\theta$	$b = \lvert\mathbf{w}\rvert\sin\theta$
$= 60\cos 80°$	$= 60\sin 80°$
≈ 10.4	≈ 59.1

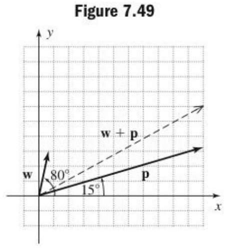
$$\mathbf{w} \approx 10.4\mathbf{i} + 59.1\mathbf{j}.$$

For vector **p** (in QI):

Horizontal Component	Vertical Component
$a = \lvert\mathbf{p}\rvert\cos\theta$	$b = \lvert\mathbf{p}\rvert\sin\theta$
$= 240\cos 15°$	$= 240\sin 15°$
$= 231.8$	≈ 62.1

$$\mathbf{p} \approx 231.8\mathbf{i} + 62.1\mathbf{j}.$$

Figure 7.49



The resultant is $\mathbf{w} + \mathbf{p} \approx (10.4\mathbf{i} + 59.1\mathbf{j}) + (231.8\mathbf{i} + 62.1\mathbf{j}) = 242.2\mathbf{i} + 121.2\mathbf{j}$, with magnitude $\lvert\mathbf{w} + \mathbf{p}\rvert \approx \sqrt{(242.2)^2 + (121.2)^2} \approx 270.8$ mph (see Figure 7.49). To find the heading of the plane relative to the ground we use $\theta_r = \tan^{-1}\left(\frac{121.2}{242.2}\right)$, which shows $\theta_r \approx 26.6°$. The plane is flying on a course heading of $90° - 26.6° = 63.4°$ at a speed of about 270.8 mph relative to the ground. Note the airplane has actually "increased speed" due to the wind.

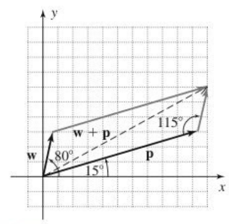
Now try Exercises 83 through 86 ►

WORTHY OF NOTE

Be aware that using the rounded values of intermediate calculations may cause slight variations in the final result. In Example 9, if we calculate $\mathbf{w} + \mathbf{p} = (60 \cos 80° + 240 \cos 15°)\mathbf{i} + (60 \sin 80° + 240 \sin 15°)\mathbf{j}$, then find $|\mathbf{w} + \mathbf{p}|$, the result is actually closer to 270.9 mph.

Applications like those in Examples 8 and 9 can also be solved using what is called the **parallelogram method,** which takes its name from the tail-to-tip vector addition noted earlier (See Figure 7.50). The resultant will be a diagonal of the parallelogram, whose magnitude can be found using the law of cosines. For Example 9, we note the parallelogram has two acute angles of $(80 - 15)° = 65°$, and since the adjacent angles must sum to 180°, the obtuse angles must be 115°. Using the law of cosines,

Figure 7.50



$$|\mathbf{w} + \mathbf{p}|^2 = \mathbf{p}^2 + \mathbf{w}^2 - 2\mathbf{pw} \cos 115° \quad \text{law of cosines}$$
$$= 240^2 + 60^2 - 2(240)(60) \cos 115° \quad \text{substitute 240 for } \mathbf{p}\text{, 60 for } \mathbf{w}$$
$$= 73371.40594 \quad \text{compute result}$$
$$|\mathbf{w} + \mathbf{p}| \approx 270.9 \quad \text{take square roots}$$

E. You've just learned how to use vector diagrams to solve applications

Note this answer is slightly more accurate, since there was no rounding until the final stage.

TECHNOLOGY HIGHLIGHT

Vector Components Given the Magnitude and the Angle θ

The TABLE feature of a graphing calculator can help us find the horizontal and vertical components of any vector with ease. Consider the vector **v** shown in Figure 7.51, which has a magnitude of 9.5 with $\theta = 15°$. Knowing this magnitude is used in both computations, first store 9.5 in storage location A: 9.5 STO→ ALPHA MATH. Next, enter the expressions for the horizontal and vertical components as Y_1 and Y_2 on the Y= screen (see Figure 7.52). Note that storing the magnitude 9.5 in memory will prevent our having to alter Y_1 and Y_2 as we apply these ideas to other values of θ. As an additional check, note that Y_3 recomputes the magnitude of the vector using the components generated in Y_1 and Y_2. To access the function variables we press: VARS ▶ ENTER and select the desired function. Although our primary interest is the components for $\theta = 15°$, we use the TBLSET screen to begin at TblStart $= 0°$, ΔTbl $= 5$, and have it count **AUTO**matically, so we can make additional observations. Pressing 2nd GRAPH (**TABLE**) brings up the screen shown in Figure 7.53. As expected, at $\theta = 0°$ the horizontal component is the same as the magnitude and the vertical component is zero. At $\theta = 15°$ we have the components of the vector pictured in Figure 7.51, approximately $\langle 9.18, 2.46\rangle$. If the angle were increased to $\theta = 30°$, a 30-60-90 triangle could be formed and one component should be $\sqrt{3}$ times the other. Sure enough, $\sqrt{3}(4.75) \approx 8.2272$.

Figure 7.51

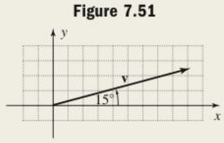


Figure 7.52

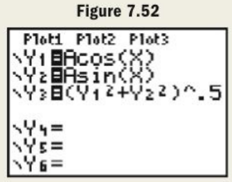
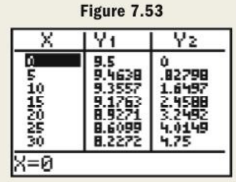


Figure 7.53



X	Y1	Y2
0	9.5	0
5	9.4638	.82798
10	9.3557	1.6497
15	9.1763	2.4588
20	8.9271	3.2492
25	8.6099	4.0149
30	8.2272	4.75

X=0

Exercise 1: If $\theta = 45°$, what would you know about the lengths of the horizontal and vertical components? Scroll down to $\theta = 45°$ to verify.

Exercise 2: If $\theta = 60°$, what would you know about the lengths of the horizontal and vertical components? Scroll down to $\theta = 60°$ to verify.

Exercise 3: We used column Y_3 as a double check on the magnitude of **v** for any given θ. What would this value be for $\theta = 45°$ and $\theta = 60°$? Press the right arrow ▶ to verify. What do you notice?

7.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Measurements that can be described using a single number are called ______ quantities.
2. ______ quantities require more than a single number to describe their attributes. Examples are force, velocity, and displacement.
3. To represent a vector quantity geometrically we use a ______ ______ segment.
4. Two vectors are equal if they have the same ______ and ______.
5. Discuss/Explain the geometric interpretation of vector addition. Give several examples and illustrations.
6. Describe the process of finding a resultant vector given the magnitude and direction of two arbitrary vectors **u** and **v**. Follow-up with an example.

▶ DEVELOPING YOUR SKILLS

Draw the comparative geometric vectors indicated.

7. Three oceanic research vessels are traveling on a parallel course in the same direction, mapping the ocean floor. One ship is traveling at 12 knots (nautical miles per hour), one at 9 knots, and the third at 6 knots.
8. As part of family reunion activities, the Williams Clan is at a bowling alley and using three lanes. Being amateurs they all roll the ball straight on, aiming for the 1 pin. Grand Dad in Lane 1 rolls his ball at 50 ft/sec. Papa in Lane 2 lets it rip at 60 ft/sec, while Junior in Lane 3 can muster only 30 ft/sec.
9. Vector $\mathbf{v}_1$ is a geometric vector representing a boat traveling at 20 knots. Vectors $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are geometric vectors representing boats traveling at 10 knots, 15 knots, and 25 knots, respectively. Draw these vectors given that $\mathbf{v}_2$ and $\mathbf{v}_3$ are traveling the same direction and parallel to $\mathbf{v}_1$, while $\mathbf{v}_4$ is traveling in the opposite direction and parallel to $\mathbf{v}_1$.
10. Vector $\mathbf{F}_1$ is a geometric vector representing a force of 50 N. Vectors $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ are geometric vectors representing forces of 25 N, 35 N, and 65 N, respectively. Draw these vectors given that $\mathbf{F}_2$ and $\mathbf{F}_3$ are applied in the same direction and parallel to $\mathbf{F}_1$, while $\mathbf{F}_4$ is applied in the opposite direction and parallel to $\mathbf{F}_1$.

Represent each situation described using geometric vectors.

11. Two tractors are pulling at a stump in an effort to clear land for more crops. The Massey-Ferguson is pulling with a force of 250 N, while the John Deere is pulling with a force of 210 N. The chains attached to the stump and each tractor form a 25° angle.
12. In an effort to get their mule up and plowing again, Jackson and Rupert are pulling on ropes attached to the mule's harness. Jackson pulls with 200 lb of force, while Rupert, who is really upset, pulls with 220 lb of force. The angle between their ropes is 16°.

Draw the vector **v** indicated, then graph the equivalent position vector.

13. initial point $(-3, 2)$; terminal point $(4, 5)$

14. initial point $(-4, -4)$; terminal point $(2, 3)$

15. initial point $(5, -3)$; terminal point $(-1, 2)$

16. initial point $(1, 4)$; terminal point $(-2, 2)$

For each vector $\mathbf{v} = \langle a, b\rangle$ and initial point (x, y) given, find the coordinates of the terminal point and the magnitude $|\mathbf{v}|$ of the vector.

17. $\mathbf{v} = \langle 7, 2\rangle$; initial point $(-2, -3)$

18. $\mathbf{v} = \langle -6, 1\rangle$; initial point $(5, -2)$

19. $\mathbf{v} = \langle -3, -5\rangle$; initial point $(2, 6)$

20. $\mathbf{v} = \langle 8, -2\rangle$; initial point $(-3, -5)$

For each position vector given, (a) graph the vector and name the quadrant, (b) compute its magnitude, and (c) find the acute angle θ formed by the vector and the nearest x-axis.

21. $\langle 8, 3\rangle$ **22.** $\langle -7, 6\rangle$

23. $\langle -2, -5\rangle$ **24.** $\langle 8, -6\rangle$

For Exercises 25 through 30, the magnitude of a vector is given, along with the quadrant of the terminal point and the angle it makes with the nearest x-axis. Find the horizontal and vertical components of each vector and write the result in component form.

25. $|\mathbf{v}| = 12$; $\theta = 25°$; QII

26. $|\mathbf{u}| = 25$; $\theta = 32°$; QIII

27. $|\mathbf{w}| = 140.5$; $\theta = 41°$; QIV

28. $|\mathbf{p}| = 15$; $\theta = 65°$; QI

29. $|\mathbf{q}| = 10$; $\theta = 15°$; QIII

30. $|\mathbf{r}| = 4.75$; $\theta = 62°$; QII

For each pair of vectors **u** and **v** given, compute (a) through (d) and illustrate the indicated operations graphically.

a. $\mathbf{u} + \mathbf{v}$ **b.** $\mathbf{u} - \mathbf{v}$

c. $2\mathbf{u} + 1.5\mathbf{v}$ **d.** $\mathbf{u} - 2\mathbf{v}$

31. $\mathbf{u} = \langle 2, 3\rangle$; $\mathbf{v} = \langle -3, 6\rangle$

32. $\mathbf{u} = \langle -3, -4\rangle$; $\mathbf{v} = \langle 0, 5\rangle$

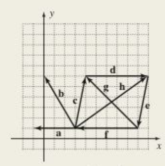
33. $\mathbf{u} = \langle 7, -2\rangle$; $\mathbf{v} = \langle 1, 6\rangle$

34. $\mathbf{u} = \langle -5, -3\rangle$; $\mathbf{v} = \langle 6, -4\rangle$

35. $\mathbf{u} = \langle -4, 2\rangle$; $\mathbf{v} = \langle 1, 4\rangle$

36. $\mathbf{u} = \langle 7, 3\rangle$; $\mathbf{v} = \langle -7, 3\rangle$

Use the graphs of vectors **a, b, c, d, e, f, g,** and **h** given to determine if the following statements are true or false.



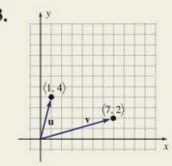
37. $\mathbf{a} + \mathbf{c} = \mathbf{b}$ **38.** $\mathbf{f} - \mathbf{e} = \mathbf{g}$

39. $\mathbf{c} + \mathbf{f} = \mathbf{h}$ **40.** $\mathbf{b} + \mathbf{h} = \mathbf{c}$

41. $\mathbf{d} - \mathbf{e} = \mathbf{h}$ **42.** $\mathbf{d} + \mathbf{f} = \mathbf{0}$

For the vectors **u** and **v** shown, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ and represent each result graphically.

43.



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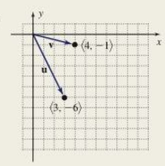

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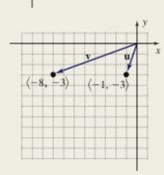

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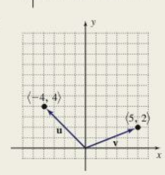

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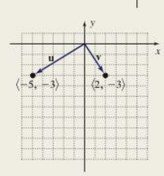
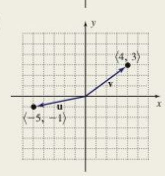


48.



Graph each vector and write it as a linear combination of **i** and **j.** Then compute its magnitude.

49. $\mathbf{u} = \langle 8, 15\rangle$ **50.** $\mathbf{v} = \langle -5, 12\rangle$

51. $\mathbf{p} = \langle -3.2, -5.7\rangle$ **52.** $\mathbf{q} = \langle 7.5, -3.4\rangle$

For each vector here, θ_r represents the acute angle formed by the vector and the x-axis. (a) Graph each vector, (b) find the horizontal and vertical components and write the vector in component form, and (c) write the vector in **i, j** form. Round to the nearest tenth.

53. **v** in QIII, $|\mathbf{v}| = 12$, $\theta_r = 16°$

54. **u** in QII, $|\mathbf{u}| = 10.5$, $\theta_r = 25°$

55. **w** in QI, $|\mathbf{w}| = 9.5$, $\theta_r = 74.5°$

56. **v** in QIV, $|\mathbf{v}| = 20$, $\theta_r = 32.6°$

For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ given, compute the vector sums (a) through (d) and find the magnitude and direction of each resultant.

a. $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{p}$ **b.** $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{q}$

c. $2\mathbf{v}_1 + 1.5\mathbf{v}_2 = \mathbf{r}$ **d.** $\mathbf{v}_1 - 2\mathbf{v}_2 = \mathbf{s}$

57. $\mathbf{v}_1 = 2\mathbf{i} - 3\mathbf{j}$; $\mathbf{v}_2 = -4\mathbf{i} + 5\mathbf{j}$

58. $\mathbf{v}_1 = 7.8\mathbf{i} + 4.2\mathbf{j}$; $\mathbf{v}_2 = 5\mathbf{j}$

59. $\mathbf{v}_1 = 5\sqrt{2}\mathbf{i} + 7\mathbf{j}$; $\mathbf{v}_2 = -3\sqrt{2}\mathbf{i} - 5\mathbf{j}$

60. $\mathbf{v}_1 = 6.8\mathbf{i} - 9\mathbf{j}$; $\mathbf{v}_2 = -4\mathbf{i} + 9\mathbf{j}$

61. $\mathbf{v}_1 = 12\mathbf{i} + 4\mathbf{j}$; $\mathbf{v}_2 = -4\mathbf{i}$

62. $\mathbf{v}_1 = 2\sqrt{3}\mathbf{i} - 6\mathbf{j}$; $\mathbf{v}_2 = -4\sqrt{3}\mathbf{i} + 2\mathbf{j}$

Find a unit vector pointing in the same direction as the vector given. Verify that a unit vector was found.

63. $\mathbf{u} = \langle 7, 24\rangle$ **64.** $\mathbf{v} = \langle -15, 36\rangle$

65. $\mathbf{p} = \langle -20, 21\rangle$ **66.** $\mathbf{q} = \langle 12, -35\rangle$

67. $20\mathbf{i} - 21\mathbf{j}$ **68.** $-4\mathbf{i} - 7.5\mathbf{j}$

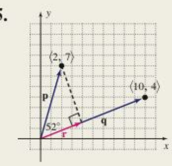
69. $3.5\mathbf{i} + 12\mathbf{j}$ **70.** $-9.6\mathbf{i} + 18\mathbf{j}$

71. $\mathbf{v}_1 = \langle 13, 3\rangle$ **72.** $\mathbf{v}_2 = \langle -4, 7\rangle$

73. $6\mathbf{i} + 11\mathbf{j}$ **74.** $-2.5\mathbf{i} + 7.2\mathbf{j}$

Vectors **p** and **q** form the angle indicated in each diagram. Find the vector **r** that points in the same direction as **q** and forms the base of the right triangle shown.

75.



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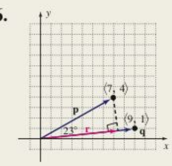

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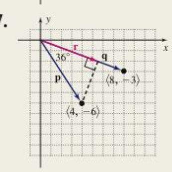

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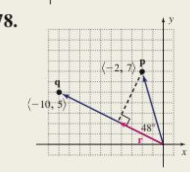


▶ WORKING WITH FORMULAS

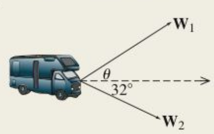
The magnitude of a vector in three dimensions: $|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$

79. The magnitude of a vector in three dimensional space is given by the formula shown, where the components of the position vector **v** are $\langle a, b, c\rangle$. Find the magnitude of **v** if $\mathbf{v} = \langle 5, 9, 10\rangle$.

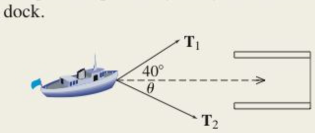
80. Find a cardboard box of any size and carefully measure its length, width, and height. Then use the given formula to find the magnitude of the box's diagonal. Verify your calculation by direct measurement.

▶ APPLICATIONS

81. Tow forces: A large van has careened off of the road into a ditch, and two tow trucks are attempting to winch it out. The cable from the first winch exerts a force of 900 lb, while the cable from the second exerts a force of 700 lb. Determine the angle θ for the first tow truck that will bring the van directly out of the ditch and along the line indicated.



82. Tow forces: Two tugboats are pulling a large ship into dry dock. The first is pulling with a force of 1250 N and the second with a force of 1750 N. Determine the angle θ for the second tugboat that will keep the ship moving straight forward and into the dock.



83. Projectile components: An arrow is shot into the air at an angle of 37° with an initial velocity of 100 ft/sec. Compute the horizontal and vertical components of the representative vector.

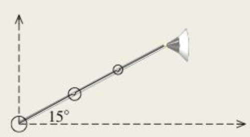
84. Projectile components: A football is punted (kicked) into the air at an angle of 42° with an initial velocity of 20 m/sec. Compute the horizontal and vertical components of the representative vector.

85. Headings and cross-winds: An airplane is flying at 250 mph on a heading of 75°. There is a strong, 35 mph wind blowing from the southwest on a heading of 10°. What is the true course and speed of the plane (relative to the ground)?

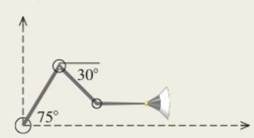
86. Headings and currents: A cruise ship is traveling at 16 knots on a heading of 300°. There is a strong water current flowing at 6 knots from the northwest on a heading of 120°. What is the true course and speed of the cruise ship?

The lights used in a dentist's office are multijointed so they can be configured in multiple ways to accommodate various needs. As a simple model, consider such a light that has the three joints, as illustrated. The first segment has a length of 45 cm, the second is 40 cm in length, and the third is 35 cm.

87. If the joints of the light are positioned so a straight line is formed and the angle made with the horizontal is 15°, determine the approximate coordinates of the joint nearest the light.



88. If the first segment is rotated 75° above horizontal, the second segment −30° (below the horizontal), and the third segment is parallel to the horizontal, determine the approximate coordinates of the joint nearest the light.



► EXTENDING THE CONCEPT

For the arbitrary vectors $\mathbf{u} = \langle a, b\rangle$, $\mathbf{v} = \langle c, d\rangle$, and $\mathbf{w} = \langle e, f\rangle$ and the scalars c and k, prove the following vector properties using the properties of real numbers.

89. $1\mathbf{u} = \mathbf{u}$ **90.** $0\mathbf{u} = \mathbf{0} = k\mathbf{0}$

91. $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

92. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

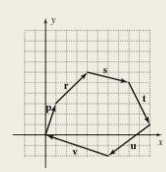
93. $(ck)\mathbf{u} = c(k\mathbf{u}) = k(c\mathbf{u})$

94. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ **95.** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

96. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ **97.** $(c + k)\mathbf{u} = c\mathbf{u} + k\mathbf{u}$

98. Consider an airplane flying at 200 mph at a heading of 45°. Compute the groundspeed of the plane under the following conditions. A strong, 40-mph wind is blowing (a) in the same direction; (b) in the direction of due north (0°); (c) in the direction heading 315°; (d) in the direction heading 270°; and (e) in the direction heading 225°. What did you notice about the groundspeed for (a) and (b)? Explain why the plane's speed is greater than 200 mph for (a) and (b), but less than 200 mph for the others.

99. Show that the sum of the vectors given, which form the sides of a closed polygon, is the zero vector. Assume all vectors have integer coordinates and each tick mark is 1 unit.



100. Verify that for $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ and $|\mathbf{v}| = \sqrt{a^2 + b^2}$, $\frac{\mathbf{v}}{|\mathbf{v}|} = 1$.

(*Hint:* Create the vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ and find its magnitude.)

101. Referring to Exercises 87 and 88, suppose the dentist needed the pivot joint at the light (the furthest joint from the wall) to be at (80, 20) for a certain patient or procedure. Find at least one set of "joint angles" that will make this possible.

▶ MAINTAINING YOUR SKILLS

102. **(6.1)** Derive the other two common versions of the Pythagorean identities, given $\sin^2 x + \cos^2 x = 1$.

103. **(2.5)** Evaluate each expression for $x = 3$ (if possible):
a. $y = \ln(2x - 7)$ **b.** $y = \dfrac{5}{x - 3}$
c. $y = \sqrt{\dfrac{1}{3}x - 5}$

104. **(6.5)** Evaluate the expression $\csc\left[\tan^{-1}\left(\dfrac{55}{48}\right)\right]$ by drawing a representative triangle.

105. **(3.4)** Graph the function $g(x) = x^3 - 7x$ and find its zeroes.

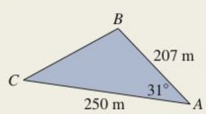
MID-CHAPTER CHECK

1. Beginning with $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$, solve for $\sin B$.

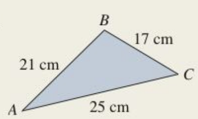
2. Given $b^2 = a^2 + c^2 - 2ac\cos B$, solve for $\cos B$.

Solve the triangles shown below using any appropriate method.

3.



4.

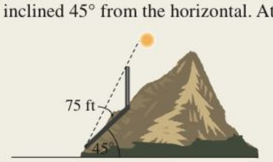


Solve the triangles described below using the law of sines. If more than one triangle exists, solve both.

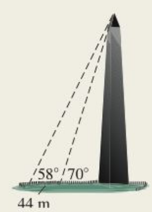
5. $A = 44°$, $a = 2.1$ km, $c = 2.8$ km

6. $C = 27°$, $a = 70$ yd, $c = 100$ yd

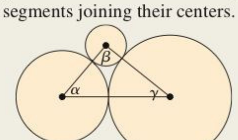
7. A large highway sign is erected on a steep hillside that is inclined 45° from the horizontal. At 9:00 A.M. the sign casts a 75 ft shadow. Find the height of the sign if the angle of elevation (measured from a horizontal line) from the tip of the shadow to the top of the sign is 65°.



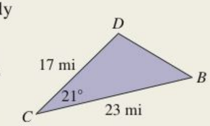
8. Modeled after an Egyptian obelisk, the Washington Monument (Washington, D.C.) is one of the tallest masonry buildings in the world. Find the height of the monument given the measurements shown (see the figure).



9. The circles shown here have radii of 4 cm, 9 cm, and 12 cm, and are tangent to each other. Find the angles formed by the line segments joining their centers.



10. On her delivery route, Judy drives 23 miles to ***C***olumbus, then 17 mi to ***D***rake, then back home to ***B***alboa. Use the diagram given to find the distance from ***D***rake to ***B***alboa.

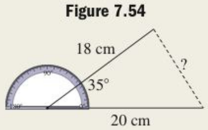


REINFORCING BASIC CONCEPTS

Scaled Drawings and the Laws of Sine and Cosine

In mathematics, there are few things as satisfying as the tactile verification of a concept or computation. In this *Reinforcing Basic Concepts,* we'll use scaled drawings to verify the relationships stated by the law of sines and the law of cosines. First, gather a blank sheet of paper, a ruler marked in centimeters/millimeters, and a protractor. When working with scale models, always measure and mark as carefully as possible. The greater the care, the better the results. For the first illustration (see Figure 7.54), we'll draw a 20-cm horizontal line segment near the bottom of the paper, then use the left endpoint to mark off a 35° angle. Draw the second side a length of 18 cm. Our first goal is to compute the length of the side needed to complete the triangle, then verify our computation by measurement. Since the current "triangle" is SAS, we use the law of cosines. Label the 35° as $\angle A$, the top vertex as $\angle B$, and the right endpoint as $\angle C$.

Figure 7.54



$a^2 = b^2 + c^2 - 2bc\cos A$ law of cosines with respect to *a*

$= (20)^2 + (18)^2 - 2(20)(18)\cos 35$ substitute known values

$\approx 724 - 589.8$ simplify (round to 10)

$= 134.2$ combine terms

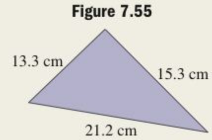
$a \approx 11.6$ solve for *a*

The computed length of side *a* is 11.6 cm, and if you took great care in drawing your diagram, you'll find the missing side is indeed very close to this length.

Exercise 1: Finish solving the triangle above using the law of sines. Once you've computed $\angle B$ and $\angle C$, measure these angles from the diagram using your protractor. How close was the computed measure to the actual measure?

For the second illustration (see Figure 7.55), draw *any arbitrary triangle* on a separate blank sheet, noting that the larger the triangle, the easier it is to measure the angles. After you've drawn it, measure the length of each side to the nearest millimeter (our triangle turned out to be 21.2 cm × 13.3 cm × 15.3 cm). Now use the law of cosines to find one angle, then the law of sines to solve the triangle. The computations for our triangle gave angles of 95.4°, 45.9°, and 38.7°. What angles did your computations give? Finally, use your protractor to measure the angles of the triangle you drew. With careful drawings, the measured results are often remarkably accurate!

Figure 7.55



Exercise 2: Using sides of 18 cm and 15 cm, draw a 35° angle, a 50° angle, and a 70° angle, then complete each triangle by connecting the endpoints. Use the law of cosines to compute the length of this third side, then actually measure each one. Was the actual length close to the computed length?

7.4 Vector Applications and the Dot Product

Learning Objectives

In Section 7.4 you will learn how to:

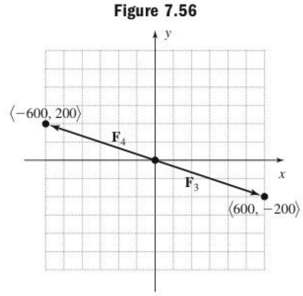
- ❑ **A.** Use vectors to investigate forces in equilibrium
- ❑ **B.** Find the components of one vector along another
- ❑ **C.** Solve applications involving work
- ❑ **D.** Compute dot products and the angle between two vectors
- ❑ **E.** Find the projection of one vector along another and resolve a vector into orthogonal components
- ❑ **F.** Use vectors to develop an equation for nonvertical projectile motion, and solve related applications

In Section 7.3 we introduced the concept of a vector, with its geometric, graphical, and algebraic representations. We also looked at operations on vectors and employed vector diagrams to solve basic applications. In this section we introduce additional ideas that enable us to solve a variety of new applications, while laying a strong foundation for future studies.

A. Vectors and Equilibrium

Much like the intuitive meaning of the word, vector forces are in **equilibrium** when they "counterbalance" each other. The simplest example is two vector forces of equal magnitude acting on the same point but in opposite directions. Similar to a tug-of-war with both sides equally matched, no one wins. If vector $\mathbf{F}_1$ has a magnitude of 500 lb in the positive direction, $\mathbf{F}_1 = \langle 500, 0\rangle$ would need vector $\mathbf{F}_2 = \langle -500, 0\rangle$ to counter it. If the forces are nonquadrantal, we intuitively sense the components must still sum to zero, and that $\mathbf{F}_3 = \langle 600, -200\rangle$ would need $\mathbf{F}_4 = \langle -600, 200\rangle$ for equilibrium to occur (see Figure 7.56). In other words, two vectors are in equilibrium when their sum is

Figure 7.56



$\mathbf{F}_3 + \mathbf{F}_4 = \langle -600, 200\rangle + \langle 600, -200\rangle$
$= \langle 0, 0\rangle = \mathbf{0}$

the zero vector **0**. If the forces have unequal magnitudes or do not pull in opposite directions, recall a resultant vector $\mathbf{F} = \mathbf{F}_a + \mathbf{F}_b$ can be found that represents the combined force. Equilibrium will then occur by adding the vector -1 (**F**) and this vector is sometimes called the **equilibriant.**

These ideas can be extended to include any number of vector forces acting on the same point. In general, we have the following:

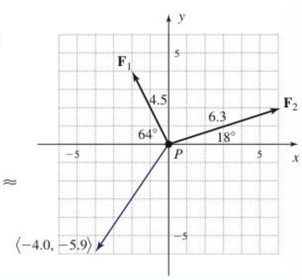
Vectors and Equilibrium

Given vectors $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting on a point P,

1. The resultant vector is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n$.
2. Equilibrium for these forces requires the vector $-1\mathbf{F}$, where $\mathbf{F} + (-1)\mathbf{F} = 0$

EXAMPLE 1 ► Finding the Equilibriant for Vector Forces

Two force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the point P as shown. Find a force $\mathbf{F}_3$ so equilibrium will occur, and sketch it on the grid.



Solution ► Begin by finding the horizontal and vertical components of each vector. For $\mathbf{F}_1$ we have $\langle -4.5\cos 64°, 4.5\sin 64°\rangle \approx \langle -2.0, 4.0\rangle$, and for $\mathbf{F}_2$ we have $\langle 6.3\cos 18°, 6.3\sin 18°\rangle \approx \langle 6.0, 1.9\rangle$. The resultant vector is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \langle 4.0, 5.9\rangle$, meaning equilibrium will occur by applying the force $-1\mathbf{F} = \langle -4.0, -5.9\rangle$ (see figure).

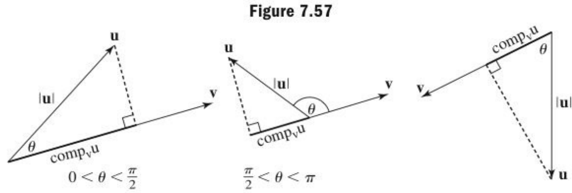
A. You've just learned how to use vectors to investigate forces in equilibrium

Now try Exercises 7 through 20 ►

B. The Component of u along v: comp$_\mathbf{v}$u

As in Example 1, many simple applications involve position vectors where the angle and horizontal/vertical components are known or can easily be found. In these situations, the components are often quadrantal, that is, they lie along the x- and y-axes and meet at a right angle. Many other applications require us to find components of a vector that are nonquadrantal, with one of the components parallel to, or lying along a second vector. Given vectors **u** and **v**, as shown in Figure 7.57, we symbolize the component of **u** that lies along **v** as comp$_\mathbf{v}$u, noting its value is simply $|\mathbf{u}|\cos\theta$ since $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\text{comp}_\mathbf{v}\mathbf{u}}{|\mathbf{u}|}$. As the diagrams further indicate, $\text{comp}_\mathbf{v}\mathbf{u} = |\mathbf{u}|\cos\theta$ regardless

Figure 7.57



of how the vectors are oriented. Note that even when the components of a vector do not lie along the x- or y-axes, they are still **orthogonal** (meet at a 90° angle).

It is important to note that comp$_\mathbf{v}$u is a *scalar quantity* (not a vector), giving only the magnitude of this component (the **vector projection** of **u** along **v** is studied later in this section). From these developments we make the following observations regarding the angle θ at which vectors **u** and **v** meet:

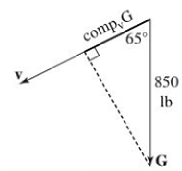
Vectors and the Component of u Along v

Given vectors **u** and **v**, which meet at an angle θ,

1. $\text{comp}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta$.
2. If $0 < \theta < 90°$, $\text{comp}_{\mathbf{v}}\mathbf{u} > 0$; if $90° < \theta < 180°$, $\text{comp}_{\mathbf{v}}\mathbf{u} < 0$.
3. If $\theta = 0$, **u** and **v** have the same direction and $\text{comp}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|$.
4. If $\theta = 90°$, **u** and **v** are orthogonal and $\text{comp}_{\mathbf{v}}\mathbf{u} = 0$.
5. If $\theta = 180°$, **u** and **v** have opposite directions and $\text{comp}_{\mathbf{v}}\mathbf{u} = -|\mathbf{u}|$.

EXAMPLE 2 ► **Finding the Component of Vector G Along Vector v**

Given the vectors **G** and **v** with $|\mathbf{G}| = 850$ lb as shown in the figure, find $\text{comp}_{\mathbf{v}}\mathbf{G}$.



Solution ► Using $|\mathbf{G}|\cos\theta = \text{comp}_{\mathbf{v}}\mathbf{G}$ we have $850\cos 65° \approx 359$ lb. The component of **G** along **v** is about 359 pounds.

Now try Exercises 21 through 26 ►

One interesting application of equilibrium and $\text{comp}_{\mathbf{v}}\mathbf{u}$ involves the force of gravity acting on an object placed on a ramp or an inclined plane. The greater the incline, the greater the tendency of the object to slide down the plane (for this study, we assume there is no friction between the object and the plane). While the force of gravity continues to pull straight downward (represented by the vector **G** in Figure 7.58), **G** is now the resultant of a force acting parallel to the plane along vector **v** (causing the object to slide) and a force acting perpendicular to the plane along vector **p** (causing the object to press against the plane). If we knew the component of **G** along **v** (indicated by the shorter, bold segment), we would know the force required to keep the object stationary as the two forces must be opposites. Note that **G** forms a right angle with the base of the inclined plane (see Figure 7.59), meaning that α and β must be complementary angles. Also note that since the location of a vector is unimportant, vector **p** has been repositioned for clarity.

Figure 7.58

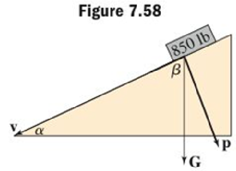
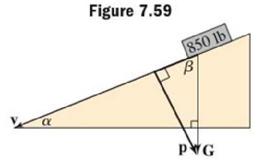


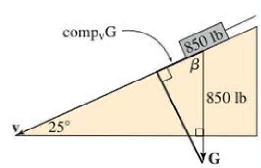
Figure 7.59



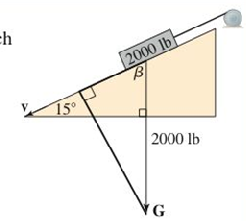
EXAMPLE 3A ► **Finding Components of Force for an Object on a Ramp**

A 850-lb object is sitting on a ramp that is inclined at 25°. Find the force needed to hold the object stationary (in equilibrium).

Solution ► Given $\alpha = 25°$, we know $\beta = 65°$. This means the component of **G** along the inclined plane is $\text{comp}_{\mathbf{v}}\mathbf{G} = 850\cos 65°$ or about 359 lb. A force of 359 lb is required to keep the object from sliding down the incline (compare to Example 2).



EXAMPLE 3B ► A winch is being used to haul a 2000-lb block of granite up a ramp that is inclined at 15°. If the winch has a maximum tow rating of 500 lb, will it be successful?



Solution ► We again need the component of **G** along the inclined plane: $\text{comp}_{\mathbf{v}}\mathbf{G} = 2000 \cos 75 \approx 518$ lb. Since the capacity of the winch is exceeded, the attempt will likely not be successful.

B. You've just learned how to find the components of one vector along another

Now try Exercises 27 through 30 ►

C. Vector Applications Involving Work

Figure 7.60

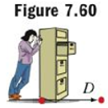
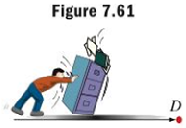


Figure 7.61



In common, everyday usage, **work** is understood to involve the exertion of energy or force to move an object a certain distance. For example, digging a ditch is hard work and involves moving dirt (exerting a force) from the trench to the bankside (over a certain distance). In an office, moving a filing cabinet likewise involves work. If the filing cabinet is heavier, or the distance it needs to be moved is greater, more work is required to move it (Figures 7.60 and 7.61).

To determine how much work was done by each person, we need to quantify the concept. Consider a constant force **F**, applied to move an object a distance D *in the same direction as the force*. In this case, work is defined as the product of the force applied and the distance the object is moved: Work = Force × Distance or $W = |\mathbf{F}|D$. If the force is given in pounds and the distance in feet, the amount of work is measured in a unit called **foot-pounds** (ft-lb). If the force is in newtons and the distance in meters, the amount of work is measured in **newton-meters** (N-m).

EXAMPLE 4 ► Solving Applications of Vectors—Work and Force Parallel to the Direction of Movement

While rearranging the office, Carrie must apply a force of 55.8 N to relocate a filing cabinet 4.5 m, while Bernard applies a 77.5 N force to move a second cabinet 3.2 m. Who did the most work?

Solution ► For Carrie:
$$W = |\mathbf{F}|D = (55.8)(4.5) = 251.1 \text{ N-m}$$
For Bernard:
$$W = |\mathbf{F}|D = (77.5)(3.2) = 248 \text{ N-m}$$

Carrie did $251.1 - 248 = 3.1$ N-m more work than Bernard.

Now try Exercises 31 and 32 ►

In many applications of work, the force **F** is not applied parallel to the direction of movement, as illustrated in Figures 7.62 and 7.63.

In calculating the amount of work done, the general concept of force × distance is preserved, *but only the component of force in the direction of movement is used*. In

Figure 7.62

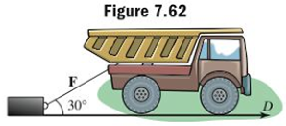
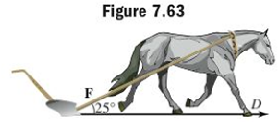


Figure 7.63



terms of the component forces discussed earlier, if **F** is a constant force applied at angle θ to the direction of movement, the amount of work done is *the component of force along D times the distance the object is moved.*

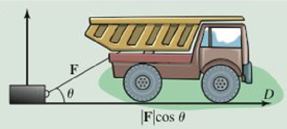
WORTHY OF NOTE

In the formula $W = |\mathbf{F}|\cos\theta \times D$, observe that if $\theta = 0$, we have the old formula for work when the force is applied in the direction of movement $W = \mathbf{F}D$. If $\theta \neq 0$, $\cos\theta \neq 1$ and the "effective force" on the object becomes $|\mathbf{F}|\cos\theta$.

Force Vectors and Work W

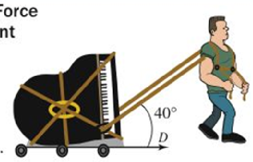
Given a force **F** applied in the direction of movement at the acute angle θ to an object, and D the distance it is moved,

$$W = |\mathbf{F}|\cos\theta \times D$$



EXAMPLE 5 ► Solving an Application of Vectors—Work and Force Applied at Angle θ to the Direction of Movement

To help move heavy pieces of furniture across the floor, movers sometime employ a body harness similar to that used for a plow horse. A mover applies a constant 200-lb force to drag a piano 100 ft down a long hallway and into another room. If the straps make a 40° angle with the direction of movement, find the amount of work performed.



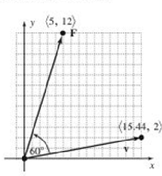
Solution ► The component of force in the direction of movement is 200 cos 40° or about 153 lb. The amount of work done is $W \approx 153(100) = 15{,}300$ ft-lb.

Now try Exercises 35 through 40 ►

These ideas can be generalized to include work problems where the component of force in the direction of motion is along a *nonhorizontal* vector **v**. Consider Example 6.

EXAMPLE 6 ► Solving an Application of Vectors—Forces Along a Nonhorizontal Vector

The force vector $\mathbf{F} = \langle 5, 12\rangle$ moves an object along the vector $\mathbf{v} = \langle 15.44, 2\rangle$ as shown. Find the amount of work required to move the object along the entire length of **v**. Assume force is in pounds and distance in feet.



Solution ► To begin, we first determine the angle between the vectors.
In this case we have $\theta = \tan^{-1}\left(\frac{12}{5}\right) - \tan^{-1}\left(\frac{2}{15.44}\right) \approx 60°$.

For $|\mathbf{F}| = 13$ (5-12-13 triangle), the component of force in the direction of motion is $\text{comp}_{\mathbf{v}}\mathbf{F} = 13\cos 60° = 6.5$. With $|\mathbf{v}| = \sqrt{(15.44)^2 + (2)^2} \approx 15.57$, the work required is $W = \text{comp}_{\mathbf{v}}\mathbf{F} \times |\mathbf{v}|$ or $(6.5)(15.57) \approx 101.2$ ft-lb.

☑ **C.** You've just learned how to solve applications involving work

Now try Exercises 41 through 44 ►

D. Dot Products and the Angle Between Two Vectors

When the component of force in the direction of motion lies along a *nonhorizontal* vector (as in Example 6), the work performed can actually be computed more efficiently using an operation called the **dot product.** For any two vectors **u** and **v**, the dot product $\mathbf{u} \cdot \mathbf{v}$ is equivalent to $\text{comp}_{\mathbf{v}}\mathbf{u} \times |\mathbf{v}|$, yet is much easier to compute (for the proof of $\mathbf{u} \cdot \mathbf{v} = \text{comp}_{\mathbf{v}}\mathbf{u} \times |\mathbf{v}|$, see Appendix IV). The operation is defined as follows:

The Dot Product u · v

Given vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, $\mathbf{u} \cdot \mathbf{v} = \langle a, b\rangle \cdot \langle c, d\rangle = ac + bd$. In words, it is the *real number* found by taking the sum of corresponding component products.

EXAMPLE 7 ► Using the Dot Product to Determine Force Along a Nonhorizontal Vector

Verify the answer to Example 6 using the dot product $\mathbf{u} \cdot \mathbf{v}$.

Solution ► For $\mathbf{u} = \langle 5, 12\rangle$ and $\mathbf{v} = \langle 15.44, 2\rangle$, we have $\mathbf{u} \cdot \mathbf{v} = \langle 5, 12\rangle \cdot \langle 15.44, 2\rangle$ giving $5(15.44) + 12(2) = 101.2$. The result is 101.2, as in Example 6.

Now try Exercises 45 through 48 ►

Note that dot products can also be used in the simpler case where the direction of motion is along a horizontal distance (Examples 4 and 5). While the dot product offers a powerful and efficient way to compute the work performed, it has many other applications; for example, to find the angle between two vectors. Consider that for any two vectors **u** and **v**, $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\cos\theta \times |\mathbf{v}|$, leading directly to $\cos\theta = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}$ (solve for $\cos\theta$).

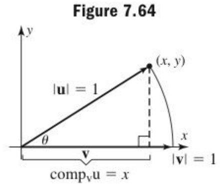
In summary,

The Angle θ Between Two Vectors

Given the nonzero vectors **u** and **v**:

$$\cos\theta = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \quad \text{and} \quad \theta = \cos^{-1}\left(\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right)$$

Figure 7.64



In the special case where **u** *and* **v** *are unit vectors*, this simplifies to $\cos\theta = \mathbf{u} \cdot \mathbf{v}$ since $|\mathbf{u}| = |\mathbf{v}| = 1$. This relationship is shown in Figure 7.64. The dot product $\mathbf{u} \cdot \mathbf{v}$ gives $\text{comp}_{\mathbf{v}}\mathbf{u} \times |\mathbf{v}|$, but $|\mathbf{v}| = 1$ and the component of **u** along **v** is simply the adjacent side of a right triangle whose hypotenuse is 1. Hence $\mathbf{u} \cdot \mathbf{v} = \cos\theta$.

EXAMPLE 8 ► Determining the Angle Between Two Vectors

Find the angle between the vectors given.

a. $\mathbf{u} = \langle -3, 4\rangle$; $\mathbf{v} = \langle 5, 12\rangle$ **b.** $\mathbf{v}_1 = 2\mathbf{i} - 3\mathbf{j}$; $\mathbf{v}_2 = 6\mathbf{i} + 4\mathbf{j}$

Solution ►

a.
$$\begin{aligned}\cos\theta &= \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\\ &= \left\langle \frac{-3}{5}, \frac{4}{5}\right\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle\\ &= \frac{-15}{65} + \frac{48}{65}\\ &= \frac{33}{65}\\ \theta &= \cos^{-1}\left(\frac{33}{65}\right)\\ &\approx 59.5°\end{aligned}$$

b.
$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}_1}{|\mathbf{v}_1|} \cdot \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\\ &= \left\langle \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}}\right\rangle \cdot \left\langle \frac{6}{\sqrt{52}}, \frac{4}{\sqrt{52}}\right\rangle\\ &= \frac{12}{\sqrt{676}} + \frac{-12}{\sqrt{676}}\\ &= \frac{0}{26} = 0\\ \theta &= \cos^{-1}0\\ &= 90°\end{aligned}$$

Now try Exercises 49 through 66 ►

Note we have implicitly shown that if $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u}$ is orthogonal to $\mathbf{v}$. As with other vector operations, recognizing certain properties of the dot product will enable us to work with them more efficiently.

Properties of the Dot Product

Given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and a constant k,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$	**2.** $\mathbf{u} \cdot \mathbf{u} = \lvert\mathbf{u}\rvert^2$
3. $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$	**4.** $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$
5. $\mathbf{0} \cdot \mathbf{u} = \mathbf{u} \cdot 0 = 0$	**6.** $\dfrac{\mathbf{u}}{\lvert\mathbf{u}\rvert} \cdot \dfrac{\mathbf{v}}{\lvert\mathbf{v}\rvert} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$

Property 6 offers an alternative to unit vectors when finding $\cos\theta$—the dot product of the vectors can be computed first, and the result divided by the product of their magnitudes: $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert}$. Proofs of the first two properties are given here. Proofs of the others have a similar development **(see Exercises 79 through 82).** For any two nonzero vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$:

Property 1:
$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \langle a, b\rangle \cdot \langle c, d\rangle \\ &= ac + bd \\ &= ca + db \\ &= \langle c, d\rangle \cdot \langle a, b\rangle \\ &= \mathbf{v} \cdot \mathbf{u} \end{aligned}$$

Property 2:
$$\begin{aligned} \mathbf{u} \cdot \mathbf{u} &= \langle a, b\rangle \cdot \langle a, b\rangle \\ &= a^2 + b^2 \\ &= \lvert\mathbf{u}\rvert^2 \\ &(\text{since } \lvert\mathbf{u}\rvert = \sqrt{a^2 + b^2}) \end{aligned}$$

D. You've just learned how to compute dot products and the angle between two vectors

Using $\text{comp}_{\mathbf{v}}\mathbf{u} = \lvert\mathbf{u}\rvert\cos\theta$ and $\mathbf{u} \cdot \mathbf{v} = \text{comp}_{\mathbf{v}}\mathbf{u} \times \lvert\mathbf{v}\rvert$, we can also state the following relationships, which give us some flexibility on how we approach applications of the dot product.

For any two vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$:

(1)	$\mathbf{u} \cdot \mathbf{v} = ac + bd$	standard computation of the dot product
(2)	$\mathbf{u} \cdot \mathbf{v} = \lvert\mathbf{u}\rvert\cos\theta \times \lvert\mathbf{v}\rvert$	alternative computation of the dot product
(3)	$\mathbf{u} \cdot \mathbf{v} = \text{comp}_{\mathbf{v}}\mathbf{u} \times \lvert\mathbf{v}\rvert$	replace $\lvert\mathbf{u}\rvert\cos\theta$ in (2) with $\text{comp}_{\mathbf{v}}\mathbf{u}$
(4)	$\dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert} = \cos\theta$	divide (2) by scalars $\lvert\mathbf{u}\rvert$ and $\lvert\mathbf{v}\rvert$
(5)	$\dfrac{\mathbf{u} \cdot \mathbf{v}}{\lvert\mathbf{v}\rvert} = \text{comp}_{\mathbf{v}}\mathbf{u}$	divide (3) by $\lvert\mathbf{v}\rvert$

E. Vector Projections and Orthogonal Components

In work problems and other simple applications, it is enough to find and apply $\text{comp}_{\mathbf{v}}\mathbf{u}$ (Figure 7.65). However, applications involving thrust and drag forces, tension and stress limits in a cable, electronic circuits, and cartoon animations often require that we also find the *vector form of* $\text{comp}_{\mathbf{v}}\mathbf{u}$. This is called the **projection** of **u** along **v** or $\mathbf{proj_v u}$, and is a vector in the same direction of **v** with magnitude $\text{comp}_{\mathbf{v}}\mathbf{u}$ (Figures 7.66 and 7.67).

Figure 7.65

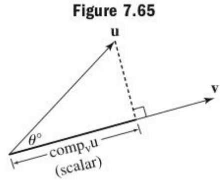


Figure 7.66

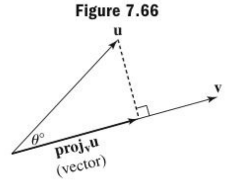
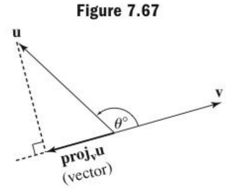


Figure 7.67



By its design, the unit vector $\frac{\mathbf{v}}{|\mathbf{v}|}$ has a length of one and points in the same direction as $\mathbf{v}$, so $\mathbf{proj_v u}$ can be computed as $\text{comp}_{\mathbf{v}}\mathbf{u} \times \frac{\mathbf{v}}{|\mathbf{v}|}$ (see Example 7, Section 7.3). Using equation (5) above and the properties shown earlier, an alternative formula for $\mathbf{proj_v u}$ can be found that is usually easier to simplify:

$$\begin{aligned}\mathbf{proj_v u} &= \text{comp}_{\mathbf{v}}\mathbf{u} \times \frac{\mathbf{v}}{|\mathbf{v}|} && \text{definition of a projection}\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \times \frac{\mathbf{v}}{|\mathbf{v}|} && \text{substitute } \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|} \text{ for comp}_{\mathbf{v}}\mathbf{u}\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2} \times \mathbf{v} && \text{rewrite factors}\end{aligned}$$

Vector Projections

Given vectors $\mathbf{u}$ and $\mathbf{v}$, the projection of $\mathbf{u}$ along $\mathbf{v}$ is the *vector*

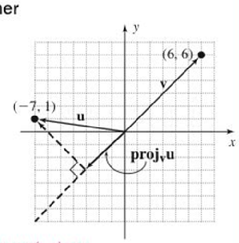
$$\mathbf{proj_v u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$$

EXAMPLE 9A ▶ Finding the Projection of One Vector Along Another

Given $\mathbf{u} = \langle -7, 1\rangle$ and $\mathbf{v} = \langle 6, 6\rangle$, find $\mathbf{proj_v u}$.

Solution ▶ To begin, find $\mathbf{u}\cdot\mathbf{v}$ and $|\mathbf{v}|$.

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= \langle -7, 1\rangle \cdot \langle 6, 6\rangle & |\mathbf{v}| &= \sqrt{6^2 + 6^2}\\ &= -42 + 6 & &= \sqrt{72}\\ &= -36 & &= 6\sqrt{2}\end{aligned}$$



$$\begin{aligned}\mathbf{proj_v u} &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} && \text{projection of } \mathbf{u} \text{ along } \mathbf{v}\\ &= \left(\frac{-36}{72}\right)\langle 6, 6\rangle && \text{substitute } -36 \text{ for } \mathbf{u}\cdot\mathbf{v},\ \sqrt{72} \text{ for } |\mathbf{v}|, \text{ and } \langle 6, 6\rangle \text{ for } \mathbf{v}\\ &= \langle -3, -3\rangle && \text{result}\end{aligned}$$

WORTHY OF NOTE

Note that $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ is the shorter diagonal of the parallelogram formed by the vectors $\mathbf{u}$ and $\mathbf{u}_1 = \mathbf{proj_v u}$. This can also be seen in the graph supplied for Example 9B.

A useful consequence of computing $\mathbf{proj_v u}$ is we can then **resolve** the vector $\mathbf{u}$ into **orthogonal components** *that need not be quadrantal.* One component will be parallel to $\mathbf{v}$ and the other perpendicular to $\mathbf{v}$ (the dashed line in the diagram in Example 9A). In general terms, this means we can write $\mathbf{u}$ as the vector sum $\mathbf{u}_1 + \mathbf{u}_2$, where $\mathbf{u}_1 = \mathbf{proj_v u}$ and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$ (note $\mathbf{u}_1 \| \mathbf{v}$).

Resolving a Vector into Orthogonal Components

Given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{proj_v u}$, $\mathbf{u}$ can be resolved into the orthogonal components $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1 = \mathbf{proj_v u}$, and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$.

EXAMPLE 9B ▶ Resolving a Vector into Orthogonal Components

Given $\mathbf{u} = \langle 2, 8\rangle$ and $\mathbf{v} = \langle 8, 6\rangle$, resolve $\mathbf{u}$ into orthogonal components $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1 \| \mathbf{v}$ and $\mathbf{u}_2 \perp \mathbf{v}$. Also verify $\mathbf{u}_1 \perp \mathbf{u}_2$.

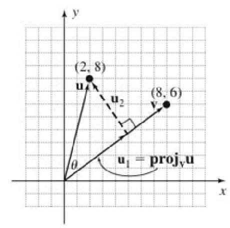
Solution ► Once again, begin by finding $\mathbf{u}\cdot\mathbf{v}$ and $|\mathbf{v}|$.

$$\mathbf{u}\cdot\mathbf{v} = \langle 2, 8\rangle\cdot\langle 8, 6\rangle = 16 + 48 = 64 \qquad |\mathbf{v}| = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$$

$$\mathbf{proj_v u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v} \quad \text{projection of } \mathbf{u} \text{ along } \mathbf{v}$$

$$= \left(\frac{64}{100}\right)\langle 8, 6\rangle \quad \text{substitute 64 for } \mathbf{u}\cdot\mathbf{v}\text{, 10 for } |\mathbf{v}|\text{, and } \langle 8, 6\rangle \text{ for } \mathbf{v}$$

$$= \langle 5.12, 3.84\rangle \quad \text{result}$$



For $\mathbf{proj_v u} = \mathbf{u}_1 = \langle 5.12, 3.84\rangle$, we have $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1 = \langle 2, 8\rangle - \langle 5.12, 3.84\rangle = \langle -3.12, 4.16\rangle$. To verify $\mathbf{u}_1 \perp \mathbf{u}_2$, we need only show $\mathbf{u}_1\cdot\mathbf{u}_2 = 0$:

$$\mathbf{u}_1\cdot\mathbf{u}_2 = \langle 5.12, 3.84\rangle\cdot\langle -3.12, 4.16\rangle = (5.12)(-3.12) + (3.84)(4.16) = 0✓$$

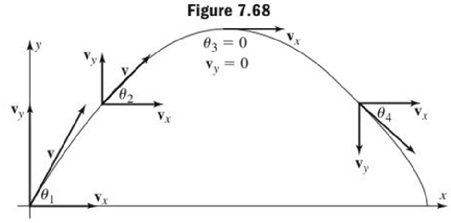
E. You've just learned how to find the projection of one vector along another and resolve a vector into orthogonal components

Now try Exercises 67 through 72 ►

F. Vectors and the Height of a Projectile

Our final application of vectors involves **projectile motion.** A projectile is any object that is thrown or projected upward, with no source of propulsion to sustain its motion. In this case, the only force acting on the projectile is gravity (air resistance is neglected), so the maximum height and the range of the projectile depend solely on its initial velocity and the angle θ at which it is projected. In a college algebra course, the equation $y = v_0t - 16t^2$ is developed to model the height in feet (at time t) of a projectile thrown vertically upward with initial velocity of v_0 feet per second. Here, we'll modify the equation slightly to take into account that the object is now moving horizontally as well as vertically. As you can see in Figure 7.68, the vector $\mathbf{v}$ representing the initial velocity, as well as the velocity vector at other times, can easily be decomposed into horizontal and vertical components. This will enable us to find a more general relationship for the position of the projectile. For now, we'll let $\mathbf{v}_y$ represent the component of velocity in the vertical (y) direction, and $\mathbf{v}_x$ represent the component of velocity in the horizontal (x) direction. Since gravity acts only in the vertical (and negative) direction, the horizontal component of the velocity remains constant at $\mathbf{v}_x = |\mathbf{v}|\cos\theta$. Using $D = RT$, the x-coordinate of the projectile at time t is $x = (|\mathbf{v}|\cos\theta)t$. For the vertical component $\mathbf{v}_y$ we use the projectile equation developed earlier, *substituting* $|\mathbf{v}|\sin\theta$ *for* $\mathbf{v}_0$, since the angle of projection is no longer 90°. This gives the y-coordinate at time t as $y = v_0t - 16t^2 = (|\mathbf{v}|\sin\theta)t - 16t^2$.

Figure 7.68



Projectile Motion

Given an object is projected upward from the origin with initial velocity $|\mathbf{v}|$ at angle $\theta°$.
The x-coordinate of its position at time t is $x = (|\mathbf{v}|\cos\theta)t$.
The y-coordinate of its position at time t is $y = (|\mathbf{v}|\sin\theta)t - 16t^2$.

EXAMPLE 10 ▶ Solving an Application of Vectors—Projectile Motion

An arrow is shot upward with an initial velocity of 150 ft/sec at an angle of 50°.

a. Find the position of the arrow after 2 sec.

b. How many seconds does it take to reach a height of 190 ft?

Solution ▶ **a.** Using the preceding equations yields these coordinates for its position at $t = 2$:

$$x = (|\mathbf{v}|\cos\theta)t \qquad y = (|\mathbf{v}|\sin\theta)t - 16t^2$$
$$= (150\cos 50°)(2) \qquad = (150\sin 50°)(2) - 16(2)^2$$
$$\approx 193 \qquad \approx 166$$

The arrow has traveled a horizontal distance of about 193 ft and is 166 ft high.

b. To find the time required to reach 190 ft in height, set the equation for the y coordinate equal to 190, which yields a quadratic equation in t:

$$y = (|\mathbf{v}|\sin\theta)t - 16t^2 \quad \text{equation for } y$$
$$190 = (150\sin 50°)t - 16t^2 \quad \text{substitute 150 for } |\mathbf{v}| \text{ and } 50° \text{ for } \theta$$
$$0 \approx -16(t)^2 + 115t - 190 \quad 150\sin 50 \approx 115$$

Using the quadratic formula we find that $t \approx 2.6$ sec and $t \approx 4.6$ sec are solutions. This makes sense, since the arrow reaches a given height once on the way up and again on the way down, as long as it hasn't reached its maximum height.

Now try Exercises 73 through 78 ▶

☑ **F. You've just learned how to use vectors to develop an equation for nonvertical, projectile motion and solve related applications**

For more on projectile motion, see the *Calculator Exploration and Discovery* feature at the end of this chapter.

7.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Vector forces are in ________ when they counterbalance each other. Such vectors have a sum of ________.

2. The component of a vector **u** along another vector **v** is written notationally as ________, and is computed as ________.

3. Two vectors that meet at a right angle are said to be ________.

4. The component of **u** along **v** is a ________ quantity. The projection of **u** along **v** is a ________.

5. Explain/Discuss exactly what information the dot product of two vectors gives us. Illustrate with a few examples.

6. Compare and contrast the projectile equations $y = v_0 t - 16t^2$ and $y = (v_0\sin\theta)t - 16t^2$. Discuss similarities/differences using illustrative examples.

▶ DEVELOPING YOUR SKILLS

The force vectors given are acting on a common point P. Find an additional force vector so that equilibrium takes place.

7. $\mathbf{F}_1 = \langle -8, -3\rangle$; $\mathbf{F}_2 = \langle 2, -5\rangle$

8. $\mathbf{F}_1 = \langle -2, 7\rangle$; $\mathbf{F}_2 = \langle 5, 3\rangle$

9. $\mathbf{F}_1 = \langle -2, -7\rangle$; $\mathbf{F}_2 = \langle 2, -7\rangle$; $\mathbf{F}_3 = \langle 5, 4\rangle$

10. $\mathbf{F}_1 = \langle -3, 10\rangle$; $\mathbf{F}_2 = \langle -10, 3\rangle$; $\mathbf{F}_3 = \langle -9, -2\rangle$

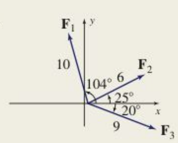
11. $\mathbf{F}_1 = 5\mathbf{i} - 2\mathbf{j}$; $\mathbf{F}_2 = \mathbf{i} + 10\mathbf{j}$

12. $\mathbf{F}_1 = -7\mathbf{i} + 6\mathbf{j}$; $\mathbf{F}_2 = -8\mathbf{i} - 3\mathbf{j}$

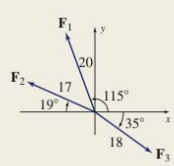
13. $\mathbf{F}_1 = 2.5\mathbf{i} + 4.7\mathbf{j}$; $\mathbf{F}_2 = -0.3\mathbf{i} + 6.9\mathbf{j}$; $\mathbf{F}_3 = -12\mathbf{j}$

14. $\mathbf{F}_1 = 3\sqrt{2}\mathbf{i} - 2\sqrt{3}\mathbf{j}$; $\mathbf{F}_2 = -2\mathbf{i} + 7\mathbf{j}$; $\mathbf{F}_3 = 5\mathbf{i} + 2\sqrt{3}\mathbf{j}$

15.



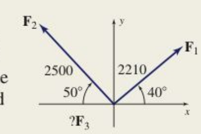
16.



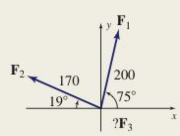
17. The force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ are simultaneously acting on a point P. Find a third vector $\mathbf{F}_3$ so that equilibrium takes place if $\mathbf{F}_1 = \langle 19, 10\rangle$ and $\mathbf{F}_2 = \langle 5, 17\rangle$.

18. The force vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ are simultaneously acting on a point P. Find a fourth vector $\mathbf{F}_4$ so that equilibrium takes place if $\mathbf{F}_1 = \langle -12, 2\rangle$, $\mathbf{F}_2 = \langle -6, 17\rangle$, and $\mathbf{F}_3 = \langle 3, 15\rangle$.

19. A new "Survivor" game involves a three-team tug-of-war. Teams 1 and 2 are pulling with the magnitude and at the angles indicated in the diagram. If the teams are currently in a stalemate, find the magnitude and angle of the rope held by team 3.

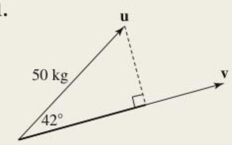


20. Three cowhands have roped a wild stallion and are attempting to hold him steady. The first and second cowhands are pulling with the magnitude and at the angles indicated in the diagram. If the stallion is held fast by the three cowhands, find the magnitude and angle of the rope from the third cowhand.

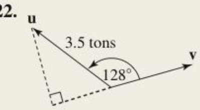


Find the component of u along v (compute $\text{comp}_\mathbf{v}\mathbf{u}$) for the vectors u and v given.

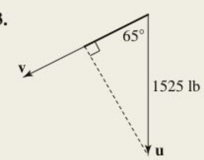
21.



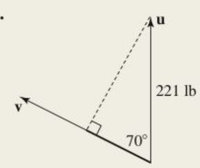
22.



23.



24.



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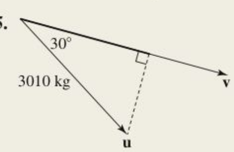

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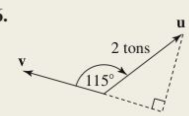
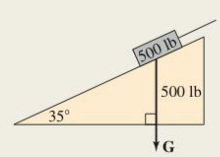
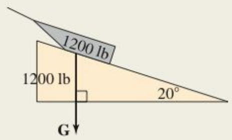


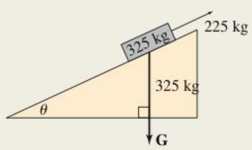
27. Static equilibrium: A 500-lb crate is sitting on a ramp that is inclined at 35°. Find the force needed to hold the object stationary.



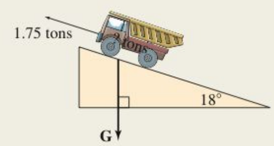
28. Static equilibrium: A 1200-lb skiff is being pulled from a lake, using a boat ramp inclined at 20°. Find the minimum force needed to dock the skiff.



29. Static equilibrium: A 325-kg carton is sitting on a ramp, held stationary by 225 kg of tension in a restraining rope. Find the ramps's angle of incline.



30. Static equilibrium: A heavy dump truck is being winched up a ramp with an 18° incline. Approximate the weight of the truck if the winch is working at its maximum capacity of 1.75 tons and the truck is barely moving.



31. While rearranging the patio furniture, Rick has to push the weighted base of the umbrella stand 15 m. If he uses a constant force of 75 N, how much work did he do?

32. Vinny's car just broke down in the middle of the road. Luckily, a buddy is with him and offers to steer if Vinny will get out and push. If he pushes with a constant force of 185 N to move the car 30 m, how much work did he do?

▶ WORKING WITH FORMULAS

The range of a projectile: $R = \dfrac{v^2 \sin\theta \cos\theta}{16}$

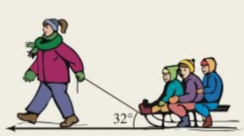
33. The range of a projected object (total horizontal distance traveled) is given by the formula shown, where v is the initial velocity and θ is the angle at which it is projected. If an arrow leaves the bow traveling 175 ft/sec at an angle of 45°, what horizontal distance will it travel?

34. A collegiate javelin thrower releases the javelin at a 40° angle, with an initial velocity of about 95 ft/sec. If the NCAA record is 280 ft, will this throw break the record? What is the smallest angle of release that will break this record? If the javelin were released at the optimum 45°, by how many feet would the record be broken?

▶ APPLICATIONS

35. Plowing a field: An old-time farmer is plowing his field with a mule. How much work does the mule do in plowing one length of a field 300 ft long, if it pulls the plow with a constant force of 250 lb and the straps make a 30° angle with the horizontal.

36. Pulling a sled: To enjoy a beautiful snowy day, a mother is pulling her three children on a sled along a level street. How much work (play) is done if the street is 100 ft long and she pulls with a constant force of 55 lb with the tow-rope making an angle of 32° with the street?



37. Tough-man contest: As part of a "tough-man" contest, participants are required to pull a bus along a level street for 100 ft. If one contestant did 45,000 ft-lb of work to accomplish the task and the straps used made an angle of 5° with the street, find the tension in the strap during the pull.

38. Moving supplies: An arctic explorer is hauling supplies from the supply hut to her tent, a distance of 150 ft, in a sled she is dragging behind her. If 9000 ft-lb of work was done and the straps used made an angle of 25° with the snow-covered ground, find the tension in the strap during the task.

39. Wheelbarrow rides: To break up the monotony of a long, hot, boring Saturday, a father decides to (carefully) give his kids a ride in a wheelbarrow. He applies a force of 30 N to move the "load" 100 m, then stops to rest. Find the amount of work done if the wheelbarrow makes an angle of 20° with level ground while in motion.

40. Mowing the lawn: A home owner applies a force of 40 N to push her lawn mower back and forth across the back yard. Find the amount of work done if the yard is 50 m long, requires 24 passes to get the lawn mowed, and the mower arm makes an angle of 39° with the level ground.

Force vectors: For the force vector **F** and vector **v** given, find the amount of work required to move an object along the entire length of **v**. Assume force is in pounds and distance in feet.

41. $\mathbf{F} = \langle 15, 10\rangle$; $\mathbf{v} = \langle 50, 5\rangle$

42. $\mathbf{F} = \langle -5, 12\rangle$; $\mathbf{v} = \langle -25, 10\rangle$

43. $\mathbf{F} = \langle 8, 2\rangle$; $\mathbf{v} = \langle 15, -1\rangle$

44. $\mathbf{F} = \langle 15, -3\rangle$; $\mathbf{v} = \langle 24, -20\rangle$

45. Use the dot product to verify the solution to Exercise 41.

46. Use the dot product to verify the solution to Exercise 42.

47. Use the dot product to verify the solution to Exercise 43.

48. Use the dot product to verify the solution to Exercise 44.

For each pair of vectors given, (a) compute the dot product $\mathbf{p} \cdot \mathbf{q}$ and (b) find the angle between the vectors to the nearest tenth of a degree.

49. $\mathbf{p} = \langle 5, 2\rangle$; $\mathbf{q} = \langle 3, 7\rangle$

50. $\mathbf{p} = \langle -3, 6\rangle$; $\mathbf{q} = \langle 2, -5\rangle$

51. $\mathbf{p} = -2\mathbf{i} + 3\mathbf{j}$; $\mathbf{q} = -6\mathbf{i} - 4\mathbf{j}$

52. $\mathbf{p} = -4\mathbf{i} + 3\mathbf{j}$; $\mathbf{q} = -6\mathbf{i} - 8\mathbf{j}$

53. $\mathbf{p} = 7\sqrt{2}\mathbf{i} - 3\mathbf{j}$; $\mathbf{q} = 2\sqrt{2}\mathbf{i} + 9\mathbf{j}$

54. $\mathbf{p} = \sqrt{2}\mathbf{i} - 3\mathbf{j}$; $\mathbf{q} = 3\sqrt{2}\mathbf{i} + 5\mathbf{j}$

Determine if the pair of vectors given are orthogonal.

55. $\mathbf{u} = \langle 7, -2\rangle$; $\mathbf{v} = \langle 4, 14\rangle$

56. $\mathbf{u} = \langle -3.5, 2.1\rangle$; $\mathbf{v} = \langle -6, -10\rangle$

57. $\mathbf{u} = \langle -6, -3\rangle$; $\mathbf{v} = \langle -8, 15\rangle$

58. $\mathbf{u} = \langle -5, 4\rangle$; $\mathbf{v} = \langle -9, -11\rangle$

59. $\mathbf{u} = -2\mathbf{i} - 6\mathbf{j}$; $\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$

60. $\mathbf{u} = 3\sqrt{2}\mathbf{i} - 2\mathbf{j}$; $\mathbf{v} = 2\sqrt{2}\mathbf{i} + 6\mathbf{j}$

Find $\text{comp}_{\mathbf{v}}\mathbf{u}$ for the vectors **u** and **v** given.

61. $\mathbf{u} = \langle 3, 5\rangle$; $\mathbf{v} = \langle 7, 1\rangle$

62. $\mathbf{u} = \langle 3, 5\rangle$; $\mathbf{v} = \langle -7, 1\rangle$

63. $\mathbf{u} = -7\mathbf{i} + 4\mathbf{j}$; $\mathbf{v} = -10\mathbf{j}$

64. $\mathbf{u} = 8\mathbf{i}$; $\mathbf{v} = 10\mathbf{i} + 3\mathbf{j}$

65. $\mathbf{u} = 7\sqrt{2}\mathbf{i} - 3\mathbf{j}$; $\mathbf{v} = 6\mathbf{i} + 5\sqrt{3}\mathbf{j}$

66. $\mathbf{u} = -3\sqrt{2}\mathbf{i} + 6\mathbf{j}$; $\mathbf{v} = 2\mathbf{i} + 5\sqrt{5}\mathbf{j}$

For each pair of vectors given, (a) find the projection of **u** along **v** (compute $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$) and (b) resolve **u** into vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1 \parallel \mathbf{v}$ and $\mathbf{u}_2 \perp \mathbf{v}$.

67. $\mathbf{u} = \langle 2, 6\rangle$; $\mathbf{v} = \langle 8, 3\rangle$

68. $\mathbf{u} = \langle -3, 8\rangle$; $\mathbf{v} = \langle -12, 3\rangle$

69. $\mathbf{u} = \langle -2, -8\rangle$; $\mathbf{v} = \langle -6, 1\rangle$

70. $\mathbf{u} = \langle -4.2, 3\rangle$; $\mathbf{v} = \langle -5, -8.3\rangle$

71. $\mathbf{u} = 10\mathbf{i} + 5\mathbf{j}$; $\mathbf{v} = 12\mathbf{i} + 2\mathbf{j}$

72. $\mathbf{u} = -3\mathbf{i} - 9\mathbf{j}$; $\mathbf{v} = 5\mathbf{i} - 3\mathbf{j}$

Projectile motion: A projectile is launched from a catapult with the initial velocity v_0 and angle θ indicated. Find (a) the position of the object after 3 sec and (b) the time required to reach a height of 250 ft.

73. $v_0 = 250$ ft/sec; $\theta = 60°$

74. $v_0 = 300$ ft/sec; $\theta = 55°$

75. $v_0 = 200$ ft/sec; $\theta = 45°$

76. $v_0 = 500$ ft/sec; $\theta = 70°$

77. At the circus, a "human cannon ball" is shot from a large cannon with an initial velocity of 90 ft/sec at an angle of 65° from the horizontal. How high is the acrobat after 1.2 sec? How long until the acrobat is again at this same height?

78. A center fielder runs down a long hit by an opposing batter and whirls to throw the ball to the infield to keep the hitter to a double. If the initial velocity of the throw is 130 ft/sec and the ball is released at an angle of 30° with level ground, how high is the ball after 1.5 sec? How long until the ball again reaches this same height?

▶ EXTENDING THE CONCEPT

For the arbitrary vectors $\mathbf{u} = \langle a, b \rangle$, $\mathbf{v} = \langle c, d \rangle$, and $\mathbf{w} = \langle e, f \rangle$ and the scalar k, prove the following vector properties using the properties of real numbers.

79. $\mathbf{w} \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{w} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{v}$

80. $k(\mathbf{u} \cdot \mathbf{v}) = k\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot k\mathbf{v}$

81. $\mathbf{0} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{0} = 0$ **82.** $\frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$

83. As alternative to $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ for finding the angle between two vectors, the equation $\tan\theta = \frac{m_2 - m_1}{1 + m_2m_1}$ can be used, where m_1 and m_2 represent the slopes of the vectors. Find the angle between the vectors $1\mathbf{i} + 5\mathbf{j}$ and $5\mathbf{i} + 2\mathbf{j}$ using each equation and comment on which you found more efficient. Then see if you can find a geometric connection between the two equations.

84. Use the equations for the horizontal and vertical components of the projected object's position to obtain the equation of trajectory $y = (\tan\theta)x - \frac{16}{v^2\cos^2\theta}x^2$. This is a quadratic equation in x. What can you say about its graph? Include comments about the concavity, x-intercepts, maximum height, and so on.

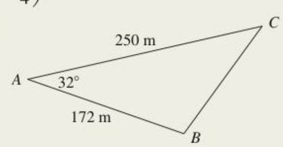
▶ MAINTAINING YOUR SKILLS

85. **(4.4)** Solve for t: $2.9e^{-0.25t} + 7.6 = 438$

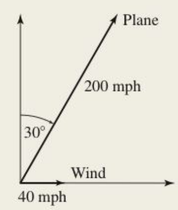
86. **(5.5)** Graph the function using a reference rectangle and the *rule of fourths*:

$y = 3\cos\left(2\theta - \frac{\pi}{4}\right)$

87. **(7.2)** Solve the triangle shown, then compute its perimeter and area.



88. **(7.3)** A plane is flying 200 mph at heading 30°, with a 40 mph wind blowing from due west. Find the true course and speed of the plane.



7.5 Complex Numbers in Trigonometric Form

Learning Objectives

In Section 7.5 you will learn how to:

- ❑ **A.** Graph a complex number
- ❑ **B.** Write a complex number in trigonometric form
- ❑ **C.** Convert from trigonometric form to rectangular form
- ❑ **D.** Interpret products and quotients geometrically
- ❑ **E.** Compute products and quotients in trigonometric form
- ❑ **F.** Solve applications involving complex numbers (optional)

Once the set of complex numbers became recognized and defined, the related basic operations matured very quickly. With little modification—sums, differences, products, quotients, and powers all lent themselves fairly well to the algebraic techniques used for real numbers. But roots of complex numbers did not yield so easily and additional tools and techniques were needed. Writing complex numbers in trigonometric form enables us to find complex roots (Section 7.6) and in some cases, makes computing products, quotients, and powers more efficient.

A. Graphing Complex Numbers

In previous sections we defined a vector quantity as one that required more than a single component to describe its attributes. The complex number $z = a + bi$ certainly fits this description, since both a real number "component" and an imaginary "component" are needed to define it. In many respects, we can treat complex numbers in the same way we treated vectors and in fact, there is much we can learn from this connection.

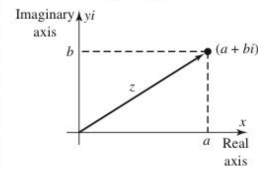
Since both axes in the xy-plane have real number values, it's not possible to graph a complex number in $\mathbb{R}$ (the real plane). However, in the same way we used

WORTHY OF NOTE

Surprisingly, the study of complex numbers matured much earlier than the study of vectors, and representing complex numbers as directed line segments actually preceded their application to a vector quantity.

the x-axis for the horizontal component of a vector and the y-axis for the vertical, we can let the x-axis represent the real valued part of a complex number and the y-axis the imaginary part. The result is called the **complex plane** $\mathbb{C}$. Every point (a, b) in $\mathbb{C}$ can be associated with a complex number $a + bi$, and any complex number $a + bi$ can be associated with a point (a, b) in $\mathbb{C}$ (Figure 7.69). The point (a, b) can also be regarded as the terminal point of a position vector representing the complex number, generally named using the letter z.

Figure 7.69



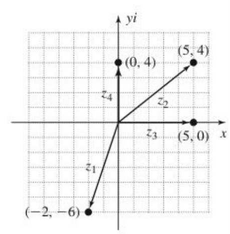
EXAMPLE 1 ► Graphing Complex Numbers

Graph the complex numbers below on the same complex plane.

a. $z_1 = -2 - 6i$ **b.** $z_2 = 5 + 4i$

c. $z_3 = 5$ **d.** $z_4 = 4i$

Solution ► The graph of each complex number is shown in the figure.

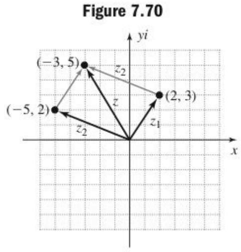


Now try Exercises 7 through 10 ►

In Example 1, you likely noticed that from a vector perspective, z_2 is the "resultant vector" for the sum $z_3 + z_4$. To investigate further, consider $z_1 = (2 + 3i)$, $z_2 = (-5 + 2i)$, and the sum $z_1 + z_2 = z$ shown in Figure 7.70. The figure helps to confirm that the sum of complex numbers can be illustrated geometrically using the parallelogram (tail-to-tip) method employed for vectors in Section 7.4.

A. You've just learned how to graph a complex number

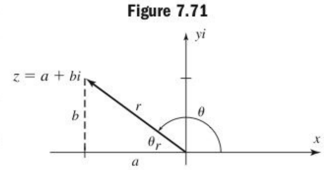
Figure 7.70



B. Complex Numbers in Trigonometric Form

The complex number $z = a + bi$ is said to be in **rectangular form** since it can be graphed using the rectangular coordinates of the complex plane. Complex numbers can also be written in **trigonometric form.** Similar to how $|x|$ represents the distance between the real number x and zero, $|z|$ represents the distance between (a, b) and the origin in the complex plane, and is computed as $|z| = \sqrt{a^2 + b^2}$. With any nonzero z, we can also associate an angle θ, which is the angle in standard position whose terminal side coincides with the graph of z. If we let r represent $|z|$, Figure 7.71 shows $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$, yielding $r\cos\theta = a$ and $r\sin\theta = b$. The appropriate substitutions into $a + bi$ give the trigonometric form:

Figure 7.71



$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + r\sin\theta \cdot i \end{aligned}$$

Factoring out r and writing the imaginary unit as the lead factor of $\sin\theta$ gives the relationship in its more common form, $z = r(\cos\theta + i\sin\theta)$, where $\tan\theta = \frac{b}{a}$.

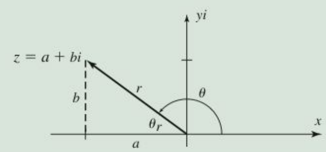
WORTHY OF NOTE

While it is true the trigonometric form can more generally be written as $z = r[\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k)]$ for $k \in \mathbb{Z}$, the result is identical for any integer k and we will select θ so that $0 \le \theta < 2\pi$ or $0° \le \theta < 360°$, depending on whether we are working in radians or degrees.

The Trigonometric Form of a Complex Number

For the complex number $z = a + bi$ and angle θ shown, $z = r(\cos\theta + i\sin\theta)$ is the trigonometric form of z, where $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$; $a \neq 0$.

- $r = |z|$ represents the magnitude of z (also called the **modulus**).
- θ is often referred to as the **argument** of z.



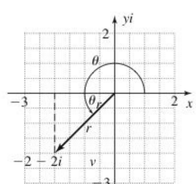
Be sure to note that for $\tan\theta = \frac{b}{a}$, $\tan^{-1}\left(\frac{b}{a}\right)$ is equal to θ_r (the reference angle for θ) and the value of θ will ultimately *depend on the quadrant of z.*

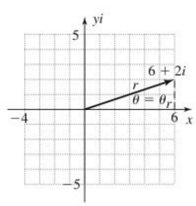
EXAMPLE 2 ► **Converting a Complex Number from Rectangular to Trigonometric Form**

State the quadrant of the complex number, then write each in trigonometric form.

a. $z_1 = -2 - 2i$ **b.** $z_2 = 6 + 2i$

Solution ► Knowing that modulus r and angle θ are needed for the trigonometric form, we first determine these values. Once again, to find the correct value of θ, *it's important to note the quadrant of the complex number.*





a. $z_1 = -2 - 2i$; QIII

$r = \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{8} = 2\sqrt{2}$

$\theta_r = \tan^{-1}\left(\frac{-2}{-2}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

with z_1 in QIII, $\theta = \frac{5\pi}{4}$.

$z_1 = 2\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$

See the figure.

b. $z = 6 + 2i$; QI

$r = \sqrt{(6)^2 + (2)^2}$

$= \sqrt{40} = 2\sqrt{10}$

$\theta_r = \tan^{-1}\left(\frac{2}{6}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right)$

z is in QI, so $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

$z = 2\sqrt{10}\left(\cos\left[\tan^{-1}\left(\frac{1}{3}\right)\right] + i\sin\left[\tan^{-1}\left(\frac{1}{3}\right)\right]\right)$

Now try Exercises 11 through 26 ►

B. You've just learned how to write a complex number in trigonometric form

WORTHY OF NOTE

Using the triangle diagrams from Section 6.5, $\cos\left[\tan^{-1}\left(\frac{1}{3}\right)\right]$ and $\sin\left[\tan^{-1}\left(\frac{1}{3}\right)\right]$ can easily be evaluated and used to verify $2\sqrt{10}\,\text{cis}\left[\tan^{-1}\left(\frac{1}{3}\right)\right] = 6 + 2i$.

Since the angle θ is repeated for both cosine and sine, we often use an abbreviated notation for the trigonometric form, called "cis" (sis) notation: $z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta$. The results of Example 2(a) and 2(b) would then be written $z = 2\sqrt{2}\,\text{cis}\left(\frac{5\pi}{4}\right)$ and $z = 2\sqrt{10}\,\text{cis}\left[\tan^{-1}\left(\frac{1}{3}\right)\right]$, respectively.

As in Example 2b, when $\theta_r = \tan^{-1}\left(\frac{b}{a}\right)$ is not a standard angle we either answer in exact form as shown, or use a four-decimal-place approximation: $2\sqrt{10}\,\text{cis}(0.3218)$.

C. Converting from Trigonometric Form to Rectangular Form

Converting from trigonometric form back to rectangular form is simply a matter of evaluating $r\,\text{cis}\,\theta$. This can be done regardless of whether θ is a standard angle or in the form $\tan^{-1}\left(\frac{b}{a}\right)$, since in the latter case we can construct a right triangle with side b opposite θ and side a adjacent θ, and find the needed values as in Section 6.5.

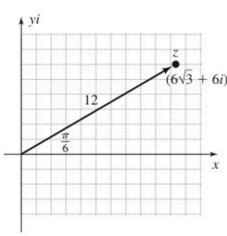
EXAMPLE 3 ► Converting a Complex Number from Trigonometric to Rectangular Form

Graph the following complex numbers, then write them in rectangular form.

a. $z = 12\,\text{cis}\left(\frac{\pi}{6}\right)$ **b.** $z = 13\,\text{cis}\left[\tan^{-1}\left(\frac{5}{12}\right)\right]$

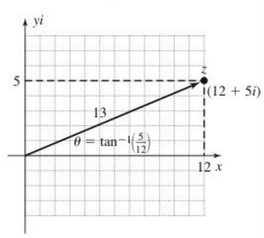
Solution ► **a.** We have $r = 12$ and $\theta = \frac{\pi}{6}$, which yields the graph in Figure 7.72. In the nonabbreviated form we have $z = 12\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$. Evaluating within the brackets gives $z = 12\left[\frac{\sqrt{3}}{2} + \frac{1}{2}i\right] = 6\sqrt{3} + 6i$.

Figure 7.72



b. For $r = 13$ and $\theta = \tan^{-1}\left(\frac{5}{12}\right)$, we have the graph shown in Figure 7.73. Here we obtain the rectangular form directly from the diagram with $z = 12 + 5i$. Verify by noting that for $\theta = \tan^{-1}\left(\frac{5}{12}\right)$, $\cos\theta = \frac{12}{13}$, and $\sin\theta = \frac{5}{13}$, meaning $z = 13(\cos\theta + i\sin\theta) = 13\left[\frac{12}{13} + \frac{5}{13}i\right] = 12 + 5i$.

Figure 7.73



C. You've just learned how to convert from trigonometric form to rectangular form

Now try Exercises 27 through 34 ►

D. Interpreting Products and Quotients Geometrically

The multiplication and division of complex numbers has some geometric connections that can help us understand their computation in trigonometric form. Note the relationship between the modulus and argument of the following product, with the moduli (plural of modulus) and arguments from each factor.

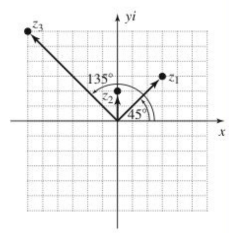
EXAMPLE 4 ► **Noting Graphical Connections for the Product of Two Complex Numbers**

For $z_1 = 3 + 3i$ and $z_2 = 0 + 2i$,

a. Graph the complex numbers and compute their moduli and arguments.

b. Compute and graph the product z_1z_2 and find its modulus and argument. Discuss any connections you see between the factors and the resulting product.

Solution ► **a.** The graphs of z_1 and z_2 are shown in the figure. For the modulus and argument we have:



$z_1 = 3 + 3i$; QI	$z_2 = 0 + 2i$;
$r = \sqrt{(3)^2 + (3)^2}$	(quadrantal)
$= \sqrt{18} = 3\sqrt{2}$	$r = 2$ directly
$\theta = \tan^{-1}1$	$\theta = 90°$ directly
$\Rightarrow \theta = 45°$	

b. The product z_1z_2 is $(3 + 3i)(2i) = -6 + 6i$, which is in QII. The modulus is $\sqrt{(-6)^2 + (6)^2} = \sqrt{72} = 6\sqrt{2}$, with an argument of $\theta_r = \tan^{-1}(-1)$ or 135° (QII). Note the product of the two moduli is *equal to the modulus of the final product:* $2 \cdot 3\sqrt{2} = 6\sqrt{2}$. Also note that the sum of the arguments for z_1 and z_2 *is equal to the argument of the product:* $45° + 90° = 135°$!

Now try Exercises 35 and 36 ►

☑ **D. You've just learned how to interpret products and quotients geometrically**

A similar geometric connection exists for the division of complex numbers. This connection is explored in **Exercises 37 and 38** of the exercise set.

E. Products and Quotients in Trigonometric Form

The connections in Example 4 are not a coincidence, and can be proven to hold for all complex numbers. Consider any two nonzero complex numbers $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2(\cos\beta + i\sin\beta)$. For the product z_1z_2 we have

$$\begin{aligned}
z_1z_2 &= r_1(\cos\alpha + i\sin\alpha)\,r_2(\cos\beta + i\sin\beta) && \text{product in trig form}\\
&= r_1r_2[(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)] && \text{rearrange factors}\\
&= r_1r_2[\cos\alpha\cos\beta + i\sin\beta\cos\alpha + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta] && \text{F-O-I-L}\\
&= r_1r_2[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\beta\cos\alpha + \sin\alpha\cos\beta)] && \text{commute terms; } i^2 = -1\\
&= r_1r_2[\cos(\alpha + \beta) + i\sin(\alpha + \beta)] && \text{use sum/difference identities for sine/cosine}
\end{aligned}$$

In words, the proof says that to *multiply* complex numbers in trigonometric form, we *multiply* the moduli and *add* the arguments. For *division*, we *divide* the moduli and *subtract* the arguments. The proof for division resembles that for multiplication and is asked for in **Exercise 71.**

Products and Quotients of Complex Numbers in Trigonometric Form

For the complex numbers $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and
$z_2 = r_2(\cos\beta + i\sin\beta)$,

$$z_1z_2 = r_1r_2[\cos(\alpha + \beta) + i\sin(\alpha + \beta)]$$

and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)],\ z_2 \neq 0.$$

EXAMPLE 5 ► **Multiplying Complex Numbers in Trigonometric Form**

For $z_1 = -3 + \sqrt{3}i$ and $z_2 = \sqrt{3} + 1i$,

a. Write z_1 and z_2 in trigonometric form and compute z_1z_2.

b. Compute the quotient $\frac{z_1}{z_2}$ in trigonometric form.

c. Verify the product using the rectangular form.

Solution ► **a.** For z_1 in QII we find $r = 2\sqrt{3}$ and $\theta = 150°$, for z_2 in QI, $r = 2$ and $\theta = 30°$. In trigonometric form,

$$\begin{aligned} z_1 &= 2\sqrt{3}(\cos 150° + i\sin 150°) \text{ and} \\ z_2 &= 2(\cos 30° + i\sin 30°): \\ z_1z_2 &= 2\sqrt{3}(\cos 150° + i\sin 150°)\cdot 2(\cos 30° + i\sin 30°) \\ &= 2\sqrt{3}\cdot 2[\cos(150° + 30°) + i\sin(150° + 30°)] && \text{multiply moduli, add arguments} \\ &= 4\sqrt{3}(\cos 180° + i\sin 180°) \\ &= 4\sqrt{3}(-1 + 0i) \\ &= -4\sqrt{3} \end{aligned}$$

b.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\sqrt{3}(\cos 150° + i\sin 150°)}{2(\cos 30° + i\sin 30°)} \\ &= \sqrt{3}[\cos(150° - 30°) + i\sin(150° - 30°)] && \text{divide moduli, subtract arguments} \\ &= \sqrt{3}(\cos 120° + i\sin 120°) \\ &= \sqrt{3}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\frac{\sqrt{3}}{2} + \frac{3}{2}i \end{aligned}$$

c.

$$\begin{aligned} z_1z_2 &= (-3 + \sqrt{3}i)(\sqrt{3} + 1i) \\ &= -3\sqrt{3} - 3i + 3i + \sqrt{3}i^2 \\ &= -4\sqrt{3} \end{aligned}$$

Now try Exercises 39 through 46 ►

Converting to trigonometric form for multiplication and division seems too clumsy for practical use, as we can often compute these results more efficiently in rectangular form. However, this approach leads to powers and roots of complex numbers, *an indispensable part of advanced equation solving*, and these are not easily found in rectangular form. In any case, note that the power and simplicity of computing products/quotients in trigonometric form is highly magnified when the complex numbers are *given in trig form*:

$$(12 \text{ cis } 50°)(3 \text{ cis } 20°) = 36 \text{ cis } 70° \qquad \frac{12 \text{ cis } 50°}{3 \text{ cis } 20°} = 4 \text{ cis } 30°.$$

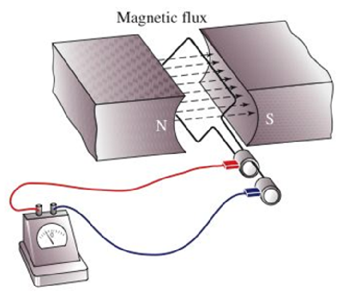
E. You've just learned how to compute products and quotients in trigonometric form

See Exercises 47 through 50.

F. (Optional) Applications of Complex Numbers

Somewhat surprisingly, complex numbers have several applications in the real world. Many of these involve a study of electricity, and in particular **AC (alternating current) circuits.**

Figure 7.74



In simplistic terms, when an armature (molded wire) is rotated in a uniform magnetic field, a voltage V is generated that depends on the strength of the field. As the armature is rotated, the voltage varies between a maximum and a minimum value, with the amount of voltage modeled by $V(\theta) = V_{max}\sin(B\theta)$, with θ in degrees. Here, V_{max} represents the maximum voltage attained, and the input variable θ represents the angle the armature makes with the **magnetic flux,** indicated in Figure 7.74 by the dashed arrows between the magnets.

When the armature is perpendicular to the flux, we say $\theta = 0°$. At $\theta = 0°$ and $\theta = 180°$, no voltage is produced, while at $\theta = 90°$ and $\theta = 270°$, the voltage reaches its maximum and minimum values respectively (hence the name *alternating current*). Many electric dryers and other large appliances are labeled as 220 volt (V) appliances, but use an alternating current that varies from 311 V to −311 V (see *Worthy of Note*). This means when $\theta = 52°$, $V(52°) = 311\sin(52°) = 245$ V is being generated. In practical applications, we use time t as the independent variable, rather the angle of the armature. These large appliances usually operate with a frequency of 60 cycles per second, or 1 cycle every $\frac{1}{60}$ of a second $\left(P = \frac{1}{60}\right)$. Using $B = \frac{2\pi}{P}$, we obtain $B = 120\pi$ and our equation model becomes $V(t) = 311\sin(120\pi t)$ with t in radians. This variation in voltage is an excellent example of a simple harmonic model.

WORTHY OF NOTE

You may have wondered why we're using an amplitude of 311 for a 220-V appliance. Due to the nature of the sine wave, the average value of an alternating current is always zero and gives no useful information about the voltage generated. Instead, the root-mean-square (rms) of the voltage is given on most appliances. While the maximum voltage is 311 V, the rms voltage is $\frac{311}{\sqrt{2}} \approx 220$ V. **See Exercise 72.**

EXAMPLE 6 ▶ Analyzing Alternating Current Using Trigonometry

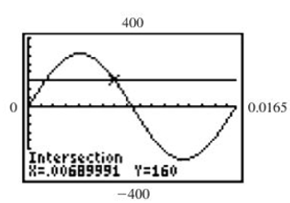
Use the equation $V(t) = 311\sin(120\pi t)$ to:

a. Create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle $\left(t = \frac{1}{120} \approx 0.008\right)$.

b. Use a graphing calculator to find the times t in this half-cycle when 160 V is being produced.

Solution ▶ **a.** Starting at $t = 0$ and using increments of 0.001 sec produces the table shown.

Time t	Voltage
0	0
0.001	114.5
0.002	212.9
0.003	281.4
0.004	310.4
0.005	295.8
0.006	239.6
0.007	149.8
0.008	39.9



b. From the table we note $V(t) = 160$ when $t \in (0.001, 0.002)$ and $t \in (0.006, 0.007)$. Using the intersection of graphs method places these values at $t \approx 0.0014$ and $t \approx 0.0069$ (see graph).

Now try Exercises 53 and 54 ►

The chief components of AC circuits are **voltage** (V) and **current** (I). Due to the nature of how the current is generated, V and I can be modeled by sine functions. Other characteristics of electricity include pure **resistance** (R), **inductive reactance** (X_L), and **capacitive reactance** (X_C) (see Figure 7.75). Each of these is measured in a unit called ohms (Ω), while current I is measured in amperes (A), and voltages are measured in volts (V). These components of electricity *are related by fixed and inherent traits*, which include the following: (1) voltage across a resistor is always *in phase* with the current, meaning the phase shift or **phase angle** between them is 0° (Figure 7.76);(2) voltage across an inductor *leads the current* by 90° (Figure 7.77); (3) voltage across a capacitor *lags the current* by 90° (Figure 7.78); and (4) voltage is equal to the product of the current times the resistance or reactance: $V = IR$, $V = IX_L$, and $V = IX_C$.

Figure 7.75

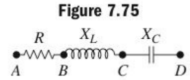


Figure 7.76

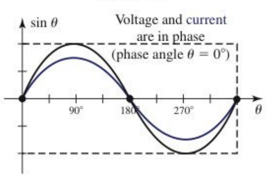


Figure 7.77

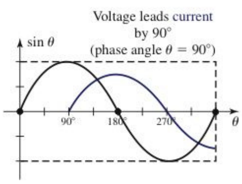
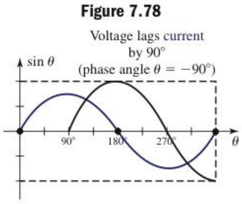
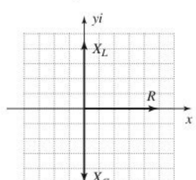


Figure 7.78



Different combinations of R, X_L, and X_C in a combined (series) circuit alter the phase angle and the resulting voltage. Since voltage across a resistance is always in phase with the current (trait 1), we can model the resistance as a vector along the positive real axis (since the phase angle is 0°). For traits (2) and (3), X_L is modeled on the positive imaginary axis since voltage leads current by 90°, and X_C on the negative imaginary axis since voltage lags current by 90° (see Figure 7.79). These natural characteristics make the complex plane *a perfect fit for describing the characteristics of the circuit.*

Figure 7.79



Consider a series circuit (Figure 7.75), where $R = 12\,\Omega$, $X_L = 9\,\Omega$, and $X_C = 4\,\Omega$. For a current of $I = 2$ amps through this circuit, the voltage across each individual element would be $V_R = (2)(12) = 24$ V (A to B), $V_L = (2)(9) = 18$ V (B to C), and $V_C = (2)(4) = 8$ V (C to D). However, the resulting voltage across this circuit *cannot be an arithmetic sum*, since R is real while X_L and X_C are represented by imaginary numbers. The joint effect of resistance (R) and reactance (X_L, X_C) in a circuit is called the **impedance,** denoted by the letter Z, and is a measure of the total resistance to the flow of electrons. It is computed $Z = R + X_L j - X_C j$ (see *Worthy of Note*), due to the phase angle relationship of the voltage in each element (X_L and X_C point in opposite directions, hence the subtraction). The expression for Z is more commonly written $R + (X_L - X_C)j$, where we more clearly note *Z is a complex number* whose magnitude and angle with the x-axis can be found as before: $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$ and $\theta_r = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$. The angle θ represents the phase angle between the voltage and current brought about by this combination of elements. The resulting voltage of the circuit is then calculated as the product of the current with the magnitude of the impedance, or $V_{RLC} = I|Z|$ (Z is also measured in ohms, Ω).

WORTHY OF NOTE

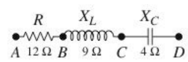
While mathematicians generally use the symbol i to represent $\sqrt{-1}$, the "i" is used in other fields to represent an electric current so the symbol $j = \sqrt{-1}$ is used instead. In conformance with this convention, we will temporarily use j for $\sqrt{-1}$ as well.

EXAMPLE 7 ► Finding the Impedence and Phase Angle of the Current in a Circuit

For the circuit diagrammed in the figure, (a) find the magnitude of Z, the phase angle between current and voltage, and write the result in trigonometric form; and (b) find the total voltage across this circuit.

Solution ▶ **a.** Using the values given, we find $Z = R + (X_L - X_C)j = 12 + (9 - 4)j = 12 + 5j$ (QI). This gives a magnitude of $|Z| = \sqrt{(12)^2 + (5)^2} = \sqrt{169} = 13\ \Omega$, with a phase angle of $\theta = \tan^{-1}\left(\frac{5}{12}\right) \approx 22.6°$ (voltage leads the current by about 22.6°). In trigonometric form $Z \approx 13 \text{ cis } 22.6°$.

b. With $I = 2$ amps, the total voltage across this circuit is $V_{RLC} = I|Z| = 2(13) = 26$ V.



☑ **F.** You have just learned how to solve applications involving complex numbers

Now try Exercises 55 through 68 ▶

7.5 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For a complex number written in the form $z = r(\cos\theta + i\sin\theta)$, r is called the ________ and θ is called the ________.

2. The complex number $z = 2\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ can be written as the abbreviated "cis" notation as ________.

3. To multiply complex numbers in trigonometric form, we ________ the moduli and ________ the arguments.

4. To divide complex numbers in trigonometric form, we ________ the moduli and ________ the arguments.

5. Write $z = -1 - \sqrt{3}i$ in trigonometric form and explain why the argument is $\theta = 240°$ instead of 60° as indicated by your calculator.

6. Discuss the similarities between finding the components of a vector and writing a complex number in trigonometric form.

▶ DEVELOPING YOUR SKILLS

Graph the complex numbers z_1, z_2, and z_3 given, then express one as the sum of the other two.

7. $z_1 = 7 + 2i$, $z_2 = 8 + 6i$, $z_3 = 1 + 4i$

8. $z_1 = 2 + 7i$, $z_2 = 3 + 4i$, $z_3 = -1 + 3i$

9. $z_1 = -2 - 5i$, $z_2 = 1 - 7i$, $z_3 = 3 - 2i$

10. $z_1 = -2 + 6i$, $z_2 = 7 - 2i$, $z_3 = 5 + 4i$

State the quadrant of each complex number, then write it in trigonometric form. For Exercises 11 through 14, answer in degrees. For 15 through 18, answer in radians.

11. $-2 - 2i$

12. $7 - 7i$

13. $-5\sqrt{3} - 5i$

14. $2 - 2\sqrt{3}i$

15. $-3\sqrt{2} + 3\sqrt{2}i$

16. $5\sqrt{7} - 5\sqrt{7}i$

17. $4\sqrt{3} - 4i$

18. $-6 + 6\sqrt{3}i$

Write each complex number in trigonometric form. For Exercises 19 through 22, answer in degrees using both an exact form and an approximate form, rounding to tenths. For 23 through 26, answer in radians using both an exact form and an approximate form, rounding to four decimal places.

19. $8 + 6i$

20. $-9 + 12i$

21. $-5 - 12i$

22. $-8 + 15i$

23. $6 + 17.5i$

24. $30 - 5.5i$

25. $-6 + 10i$

26. $12 - 4i$

Graph each complex number using its trigonometric form, then convert each to rectangular form.

27. $2 \operatorname{cis}\left(\frac{\pi}{4}\right)$

28. $12 \operatorname{cis}\left(\frac{\pi}{6}\right)$

29. $4\sqrt{3} \operatorname{cis}\left(\frac{\pi}{3}\right)$

30. $5\sqrt{3} \operatorname{cis}\left(\frac{7\pi}{6}\right)$

31. $17 \operatorname{cis}\left[\tan^{-1}\left(\frac{15}{8}\right)\right]$

32. $10 \operatorname{cis}\left[\tan^{-1}\left(\frac{3}{4}\right)\right]$

33. $6 \operatorname{cis}\left[\pi - \tan^{-1}\left(\frac{5}{\sqrt{11}}\right)\right]$

34. $4 \operatorname{cis}\left[\pi + \tan^{-1}\left(\frac{\sqrt{7}}{3}\right)\right]$

For the complex numbers z_1 and z_2 given, find their moduli r_1 and r_2 and arguments θ_1 and θ_2. Then compute their *product* in rectangular form. For modulus r and argument θ of the product, verify that $r_1r_2 = r$ and $\theta_1 + \theta_2 = \theta$.

35. $z_1 = -2 + 2i$; $z_2 = 3 + 3i$

36. $z_1 = 1 + \sqrt{3}i$; $z_2 = 3 + \sqrt{3}i$

For the complex numbers z_1 and z_2 given, find their moduli r_1 and r_2 and arguments θ_1 and θ_2. Then compute their *quotient* in rectangular form. For modulus r and argument θ of the quotient, verify that $\frac{r_1}{r_2} = r$ and $\theta_1 - \theta_2 = \theta$.

37. $z_1 = \sqrt{3} + i$; $z_2 = 1 + \sqrt{3}i$

38. $z_1 = -\sqrt{3} + i$; $z_2 = 3 + 0i$

Compute the product z_1z_2 and quotient $\frac{z_1}{z_2}$ using the trigonometric form. Answer in exact rectangular form where possible, otherwise round all values to two decimal places.

39. $z_1 = -4\sqrt{3} + 4i$
$z_2 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

40. $z_1 = \frac{5\sqrt{3}}{2} + \frac{5}{2}i$
$z_2 = 0 + 6i$

41. $z_1 = -2\sqrt{3} + 0i$
$z_2 = -\frac{21}{2} + \frac{7i\sqrt{3}}{2}$

42. $z_1 = 0 - 6i\sqrt{2}$
$z_2 = \frac{3\sqrt{2}}{2} + \frac{3i\sqrt{6}}{2}$

43. $z_1 = 9\left[\cos\left(\frac{\pi}{15}\right) + i\sin\left(\frac{\pi}{15}\right)\right]$
$z_2 = 1.8\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$

44. $z_1 = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right]$
$z_2 = 8.4\left[\cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)\right]$

45. $z_1 = 10(\cos 60° + i\sin 60°)$
$z_2 = 4(\cos 30° + i\sin 30°)$

46. $z_1 = 7(\cos 120° + i\sin 120°)$
$z_2 = 2(\cos 300° + i\sin 300°)$

47. $z_1 = 5\sqrt{2} \operatorname{cis} 210°$
$z_2 = 2\sqrt{2} \operatorname{cis} 30°$

48. $z_1 = 5\sqrt{3} \operatorname{cis} 240°$
$z_2 = \sqrt{3} \operatorname{cis} 90°$

49. $z_1 = 6 \operatorname{cis} 82°$
$z_2 = 1.5 \operatorname{cis} 27°$

50. $z_1 = 1.6 \operatorname{cis} 59°$
$z_2 = 8 \operatorname{cis} 275°$

▶ WORKING WITH FORMULAS

51. Equilateral triangles in the complex plane:
$\mathbf{u^2 + v^2 + w^2 = uv + uw + vw}$

If the line segments connecting the complex numbers u, v, and w form the vertices of an equilateral triangle, the formula shown above holds true. Verify that $u = 2 + \sqrt{3}i$, $v = 10 + \sqrt{3}i$, and $w = 6 + 5\sqrt{3}i$ form the vertices of an equilateral triangle using the distance formula, then verify the formula given.

52. The cube of a complex number:
$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

The cube of any binomial can be found using the formula here, where A and B are the terms of the binomial. Use the formula to compute the cube of $1 - 2i$ (note $A = 1$ and $B = -2i$).

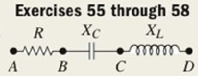
▶ APPLICATIONS

53. Electric current: In the United States, electric power is supplied to homes and offices via a "120 V circuit," using an alternating current that varies from 170 V to −170 V, at a frequency of 60 cycles/sec. (a) Write the voltage equation for U.S. households, (b) create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle, and (c) find the first time t in this half-cycle when exactly 140 V is being produced.

54. Electric current: While the electricity supplied in Europe is still not quite uniform, most countries employ 230-V circuits, using an alternating current that varies from 325 V to −325 V. However, the frequency is only *50 cycles per second.* (a) Write the voltage equation for these European countries, (b) create a table of values illustrating the voltage produced every thousandth of a second for the first half-cycle, and (c) find the first time t in this half-cycle when exactly 215 V is being produced.

AC circuits: For the circuits indicated in Exercises 55 through 60, (a) find the magnitude of Z, the phase angle between current and voltage, and write the result in trigonometric form; and (b) find the total voltage across this circuit. Recall $Z = R + (X_L - X_C)j$ and $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$.

Exercises 55 through 58



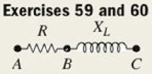
55. $R = 15\ \Omega$, $X_L = 12\ \Omega$, and $X_C = 4\ \Omega$, with $I = 3$ A.

56. $R = 24\ \Omega$, $X_L = 12\ \Omega$, and $X_C = 5\ \Omega$, with $I = 2.5$ A.

57. $R = 7\ \Omega$, $X_L = 6\ \Omega$, and $X_C = 11\ \Omega$, with $I = 1.8$ A.

58. $R = 9.2\ \Omega$, $X_L = 5.6\ \Omega$, and $X_C = 8.3\ \Omega$, with $I = 2.0$ A.

Exercises 59 and 60



59. $R = 12\ \Omega$ and $X_L = 5\ \Omega$, with $I = 1.7$ A.

60. $R = 35\ \Omega$ and $X_L = 12\ \Omega$, with $I = 4$ A.

AC circuits—voltage: The current I and the impedance Z for certain AC circuits are given. Write I and Z in trigonometric form and find the voltage in each circuit. Recall $V = IZ$.

61. $I = \sqrt{3} + 1j$ A and $Z = 5 + 5j\ \Omega$

62. $I = \sqrt{3} - 1j$ A and $Z = 2 + 2j\ \Omega$

63. $I = 3 - 2j$ A and $Z = 2 + 3.75j\ \Omega$

64. $I = 4 + 3j$ A and $Z = 2 - 4j\ \Omega$

AC circuits—current: If the voltage and impedance are known, the current I in the circuit is calculated as the quotient $I = \frac{V}{Z}$. Write V and Z in trigonometric form to find the current in each circuit.

65. $V = 2 + 2\sqrt{3}j$ and $Z = 4 - 4j\ \Omega$

66. $V = 4\sqrt{3} - 4j$ and $Z = 1 - 1j\ \Omega$

67. $V = 3 - 4j$ and $Z = 4 + 7.5j\ \Omega$

68. $V = 2.8 + 9.6j$ and $Z = 1.4 - 4.8j\ \Omega$

Parallel circuits: For AC circuits *wired in parallel*, the total impedance is given by $Z = \frac{Z_1 Z_2}{Z_1 + Z_2}$, where Z_1 and Z_2 represent the impedance in each branch. Find the total impedance for the values given. Compute the product in the numerator using trigonometric form, and the sum in the denominator in rectangular form.

69. $Z_1 = 1 + 2j$ and $Z_2 = 3 - 2j$

70. $Z_1 = 3 - j$ and $Z_2 = 2 + j$

► EXTENDING THE CONCEPT

71. Verify/prove that for the complex numbers $z_1 = r_1(\cos\alpha + i\sin\alpha)$ and $z_2 = r_2(\cos\beta + i\sin\beta)$, $\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]$.

72. Using the Internet, a trade manual, or some other resource, find the voltage and frequency at which electricity is supplied to most of Japan (oddly enough—two different frequencies are in common use). As in Example 6, the voltage given will likely be the root-mean-square (rms) voltage. Use the information to find the true voltage and the equation model for voltage in most of Japan.

73. Recall that two lines are perpendicular if their slopes have a product of −1. For the directed line segment representing the complex number $z_1 = 7 + 24i$, find complex numbers z_2 and z_3 whose directed line segments are perpendicular to z_1 and have a magnitude one-fifth as large.

74. The magnitude of the impedance is $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$. If R, X_L, and X_C are all nonzero, what conditions would make the magnitude of Z as small as possible?

▶ MAINTAINING YOUR SKILLS

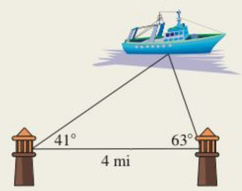
75. **(6.7)** Solve for $x \in [0, 2\pi)$:
$$350 = 750\sin\left(2x - \frac{\pi}{4}\right) - 25$$

76. **(3.6)** Name all asymptotes of the function $h(x) = \frac{1 + x^3}{x^2}$

77. **(2.7)** Graph the piecewise-defined function given:
$$f(x) = \begin{cases} 2 & x < -2 \\ x^2 & -2 \le x < 1 \\ x & x \ge 1 \end{cases}$$

78. **(7.2)** A ship is spotted by two observation posts that are 4 mi apart. Using the line between them for reference, the first post reports the ship is at an angle of 41°, while the second reports an angle of 63°, as shown. How far is the ship from the closest post?

Exercise 78



7.6 De Moivre's Theorem and the Theorem on *n*th Roots

Learning Objectives

In Section 7.6 you will learn how to:

- **A.** Use De Moivre's theorem to raise complex numbers to any power
- **B.** Use De Moivre's theorem to check solutions to polynomial equations
- **C.** Use the *n*th roots theorem to find the *n*th roots of a complex number

The material in this section represents some of the most significant developments in the history of mathematics. After hundreds of years of struggle, mathematical scientists had not only come to recognize the existence of complex numbers, but were able to make operations on them commonplace and routine. This allowed for the unification of many ideas related to the study of polynomial equations, and answered questions that had puzzled scientists from many different fields for centuries. In this section, we will look at two fairly simple theorems that actually represent over 1000 years in the evolution of mathematical thought.

A. De Moivre's Theorem

Having found acceptable means for applying the four basic operations to complex numbers, our attention naturally shifts to the computation of powers and roots. Without them, we'd remain wholly unable to offer complete solutions to polynomial equations and find solutions for many applications. The computation of powers, squares, and cubes offer little challenge, as they can be computed easily using the formula for binomial squares $[(A + B)^2 = A^2 + 2AB + B^2]$ or by applying the **binomial theorem.** For larger powers, the binomial theorem becomes too time consuming and a more efficient method is desired. The key here is to use the trigonometric form of the complex number. In Section 7.5, we noted the product of two complex numbers involved multiplying their moduli and adding their arguments:

For $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$z_1z_2 = r_1r\,[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

For the square of a complex number, $r_1 = r_2$ and $\theta_1 = \theta_2$. Using θ itself yields

$$\begin{aligned} z^2 &= r^2[\cos(\theta + \theta) + i\sin(\theta + \theta)] \\ &= r^2[\cos(2\theta) + i\sin(2\theta)] \end{aligned}$$

Multiplying this result by $z = r(\cos\theta + i\sin\theta)$ to compute z^3 gives

$$r^2[\cos(2\theta) + i\sin(2\theta)]\, r(\cos\theta + i\sin\theta) = r^3[\cos(2\theta + \theta) + i\sin(2\theta + \theta)]$$
$$= r^3[\cos(3\theta) + i\sin(3\theta)].$$

The result can be extended further and generalized into **De Moivre's theorem.**

WORTHY OF NOTE

Sometimes the argument of cosine and sine becomes very large after applying De Moivre's theorem. In these cases, we use the fact that $\theta = \theta \pm 360°k$ and $\theta = \theta \pm 2\pi k$ represent coterminal angles for integers k, and use the coterminal angle θ where $0 \le \theta < 360°$ or $0 \le \theta < 2\pi$.

De Moivre's Theorem

For any positive integer n, and $z = r(\cos\theta + i\sin\theta)$,

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$

For a proof of the theorem where n is an integer and $n \ge 1$, see Appendix V.

EXAMPLE 1 ► Using De Moivre's Theorem to Compute the Power of a Complex Number

Use De Moivre's theorem to compute z^9, given $z = -\frac{1}{2} - \frac{1}{2}i$.

Solution ► Here we have $r = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$. With z in QIII, $\tan\theta = 1$ yields $\theta = \frac{5\pi}{4}$. The trigonometric form is $z = \frac{\sqrt{2}}{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$ and applying the theorem with $n = 9$ gives

$$z^9 = \left(\frac{\sqrt{2}}{2}\right)^9\left[\cos\left(9 \cdot \frac{5\pi}{4}\right) + i\sin\left(9 \cdot \frac{5\pi}{4}\right)\right] \quad \text{De Moivre's theorem}$$
$$= \frac{\sqrt{2}}{32}\left[\cos\left(\frac{45\pi}{4}\right) + i\sin\left(\frac{45\pi}{4}\right)\right] \quad \text{simplify}$$
$$= \frac{\sqrt{2}}{32}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \quad \text{coterminal angles}$$
$$= \frac{\sqrt{2}}{32}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \quad \text{evaluate functions}$$
$$= -\frac{1}{32} - \frac{1}{32}i \quad \text{result}$$

A. You've just learned how to use De Moivre's theorem to raise complex numbers to any power

Now try Exercises 7 through 14 ►

As with products and quotients, if the complex number is *given* in trigonometric form, computing any power of the number is both elegant and efficient. For instance, if $z = 2 \text{ cis } 40°$, then $z^4 = 16 \text{ cis } 160°$. **See Exercises 15 through 18.**

For cases where θ is not a standard angle, De Moivre's theorem requires an intriguing application of the skills developed in Chapter 6, including the use of multiple angle identities and working from a right triangle drawn relative to $\theta_r = \tan^{-1}\left(\frac{b}{a}\right)$. **See Exercises 57 and 58.**

B. Checking Solutions to Polynomial Equations

One application of De Moivre's theorem is checking the complex roots of a polynomial, as in Example 2.

EXAMPLE 2 ► **Using De Moivre's Theorem to Check Solutions to a Polynomial Equation**

Use De Moivre's theorem to show that $z = -2 - 2i$ is a solution to $z^4 - 3z^3 - 38z^2 - 128z - 144 = 0$.

Solution ► We will apply the theorem to the third and fourth degree terms, and compute the square directly. Since z is in QIII, the trigonometric form is $z = 2\sqrt{2}\text{ cis } 225°$. In the following illustration, note that 900° and 180° are coterminal, as are 675° and 315°.

$$\begin{aligned}&(-2-2i)^4\\&=(2\sqrt{2})^4\text{cis}(4\cdot 225°)\\&=(2\sqrt{2})^4\text{cis }900°\\&=64\text{ cis }180°\\&=64(-1+0i)\\&=-64\end{aligned}\qquad\begin{aligned}&(-2-2i)^3\\&=(2\sqrt{2})^3\text{cis}(3\cdot 225°)\\&=(2\sqrt{2})^3\text{cis }675°\\&=(2\sqrt{2})^3\text{cis }315°\\&=16\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)\\&=16-16i\end{aligned}\qquad\begin{aligned}&(-2-2i)^2\\&=4+8i+(2i)^2\\&=4+8i+4i^2\\&=4+8i-4\\&=0+8i\\&=8i\end{aligned}$$

Substituting back into the original equation gives

$$\begin{aligned}1z^4 - 3z^3 - 38z^2 - 128z - 144 &= 0\\1(-64) - 3(16-16i) - 38(8i) - 128(-2-2i) - 144 &= 0\\-64 - 48 + 48i - 304i + 256 + 256i - 144 &= 0\\(-64 - 48 + 256 - 144) + (48 - 304 + 256)i &= 0\\0 &= 0✓\end{aligned}$$

B. You've just learned to use De Moivre's theorem to check solutions to polynomial equations

Now try Exercises 19 through 26 ►

Regarding Example 2, we know from a study of algebra that complex roots must occur in conjugate pairs, meaning $-2 + 2i$ is also a root. This equation actually has two real and two complex roots, with $z = 9$ and $z = -2$ being the two real roots.

C. The *n*th Roots Theorem

Having looked at De Moivre's theorem, which raises a complex number to any power, we now consider the ***n*th roots theorem,** which will compute the nth roots of a complex number. If we allow that De Moivre's theorem also holds for rational values $\frac{1}{n}$, instead of only the integers n illustrated previously, the formula for computing an nth root would be a direct result:

$$\begin{aligned}z^{\frac{1}{n}} &= r^{\frac{1}{n}}\left[\cos\left(\frac{1}{n}\theta\right) + i\sin\left(\frac{1}{n}\theta\right)\right] && \text{De Moivre's theorem}\\&= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{n}\right)\right] && \text{simplify}\end{aligned}$$

However, this formula would *find only the principal nth root*! In other words, periodic solutions would be ignored. As in Section 7.5, it's worth noting the most general form of a complex number is $z = r[\cos(\theta + 360°k) + i\sin(\theta + 360°k)]$, for $k \in \mathbb{Z}$. When De Moivre's theorem is applied to this form for *integers n*, we obtain $z^n = r^n[\cos(n\theta + 360°kn) + i\sin(n\theta + 360°kn)]$, which returns a result identical to $r^n[\cos(n\theta) + i\sin(n\theta)]$. However, for the rational exponent $\frac{1}{n}$, the general form takes additional solutions into account and will return all n, nth roots.

$$\begin{aligned}z^{\frac{1}{n}} &= r^{\frac{1}{n}}\left\{\cos\left[\frac{1}{n}(\theta + 360°k)\right] + i\sin\left[\frac{1}{n}(\theta + 360°k)\right]\right\} && \text{De Moivre's theorem for rational exponents}\\&= \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{360°k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360°k}{n}\right)\right] && \text{simplify}\end{aligned}$$

The *n*th Roots Theorem

For $z = r(\cos\theta + i\sin\theta)$, a positive integer n, and $r \in \mathbb{R}$, z has exactly n distinct nth roots determined by

$$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{360°k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360°k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$.

For ease of computation, it helps to note that once the argument for the principal root is found using $k = 0$, $\frac{\theta}{n} + \frac{360°k}{n}$ simply adds $\frac{360}{n}\left(\text{or } \frac{2\pi}{n}\right)$ to the previous argument for $k = 1, 2, 3, \ldots, n - 1$.

In Example 3 you're asked to find the three cube roots of 1, also called the **cube roots of unity,** and graph the results. The nth roots of unity play a significant role in the solution of many polynomial equations. For an in-depth study of this connection, visit www.mhhe.com/coburn and go to **Section 7.8: Trigonometry, Complex Numbers and Cubic Equations.**

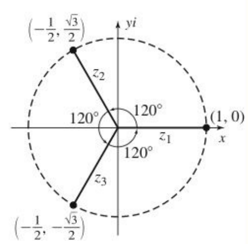
EXAMPLE 3 ► **Finding *n*th Roots**

Use the nth roots theorem to solve the equation $x^3 - 1 = 0$. Write the results in rectangular form and graph.

Solution ► From $x^3 - 1 = 0$, we have $x^3 = 1$ and must find the three cube roots of unity. As before, we begin in trigonometric form: $1 + 0i = 1(\cos 0° + i\sin 0°)$. With $n = 3$, $r = 1$, and $\theta = 0°$, we have $\sqrt[3]{r} = \sqrt[3]{1} = 1$, and $\frac{0°}{3} + \frac{360°k}{3} = 0° + 120°k$. The principal root $(k = 0)$ is $z_0 = 1(\cos 0° + i\sin 0°) = 1$. Adding 120° to each previous argument, we find the other roots are

$$z_1 = 1(\cos 120° + i\sin 120°)$$
$$z_2 = 1(\cos 240° + i\sin 240°).$$

In rectangular form these are $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, as shown in the figure.



Now try Exercises 27 through 40 ►

EXAMPLE 4 ► **Finding *n*th Roots**

Use the nth roots theorem to find the five fifth roots of $z = 16\sqrt{3} + 16i$.

Solution ► In trigonometric form, $16\sqrt{3} + 16i = 32(\cos 30° + i\sin 30°)$. With $n = 5$, $r = 32$, and $\theta = 30°$, we have $\sqrt[5]{r} = \sqrt[5]{32} = 2$, and $\frac{30°}{5} + \frac{360°k}{5} = 6° + 72°k$. The principal root is $z_0 = 2(\cos 6° + i\sin 6°)$. Adding 72° to each previous argument, we find the other four roots are

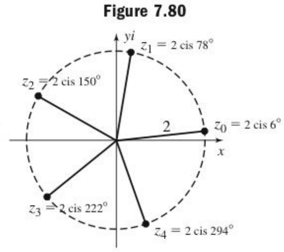
$$z_1 = 2(\cos 78° + i\sin 78°) \qquad z_2 = 2(\cos 150° + i\sin 150°)$$
$$z_3 = 2(\cos 222° + i\sin 222°) \qquad z_4 = 2(\cos 294° + i\sin 294°)$$

Now try Exercises 41 through 44 ►

Of the five roots in Example 4, only $z_2 = 2(\cos 150° + i \sin 150°)$ uses a standard angle. Applying De Moivre's theorem with $n = 5$ gives $(2 \text{ cis } 150°)^5 = 32 \text{ cis } 750° = 32 \text{ cis } 30°$ or $16\sqrt{3} + 16i$.✓ **See Exercise 54.**

As a consequence of the arguments in a solution being uniformly separated by $\frac{360°}{n}$, the graphs of complex roots are equally spaced about a circle of radius r. The five fifth roots from Example 3 are shown in Figure 7.80 (note each argument differs by 72°).

Figure 7.80



For additional insight into roots of complex numbers, we reason that the nth roots of a complex number must also be complex. To find the four fourth roots of $z = 8 + 8\sqrt{3}i = 16(\cos 60° + i \sin 60°)$, we seek a number of the form $r(\cos \alpha + i \sin \alpha)$ such that $[r(\cos \alpha + i \sin \alpha)]^4 = 16(\cos 60° + i \sin 60°)$. Applying De Moivre's theorem to the left-hand side and equating equivalent parts we obtain

$$r^4[\cos(4\alpha) + i \sin(4\alpha)] = 16(\cos 60° + i \sin 60°), \text{ which leads to}$$
$$r^4 = 16 \text{ and}$$
$$4\alpha = 60°$$

From this it is obvious that $r = 2$, but as with similar equations solved in Chapter 6, the equation $4\alpha = 60°$ has multiple solutions. To find them, we first add $360°k$ to 60°, *then* solve for α.

$$4\alpha = 60° + 360°k \quad \text{add } 360°k$$
$$\alpha = \frac{60° + 360°k}{4} \quad \text{divide by 4}$$
$$= 15° + 90°k \quad \text{result}$$

For convenience, we start with $k = 0, 1, 2$, and so on, which leads to

For $k = 0$: $\alpha = 15° + 90°(0) = 15°$ For $k = 1$: $\alpha = 15° + 90°(1) = 105°$

For $k = 2$: $\alpha = 15° + 90°(2) = 195°$ For $k = 3$: $\alpha = 15° + 90°(3) = 285°$

At this point it should strike us that we have four roots—exactly the number required. Indeed, using $k = 4$ gives $\alpha = 15° + 90°(4) = 375°$, which is coterminal with the 15° obtained when $k = 0$. Hence, the four fourth roots are

$$z_0 = 2(\cos 15° + i \sin 15°) \qquad z_1 = 2(\cos 105° + i \sin 105°)$$
$$z_2 = 2(\cos 195° + i \sin 195°) \qquad z_3 = 2(\cos 285° + i \sin 285°).$$

The check for these solutions is asked for in **Exercise 53.**

C. You've just learned how to use the *n*th roots theorem to find the *n*th roots of a complex number

As a final note, it must have struck the mathematicians who pioneered these discoveries with some amazement that complex numbers and the trigonometric functions should be so closely related. The amazement must have been all the more profound upon discovering an additional connection between complex numbers and *exponential functions*. For more on these connections, visit www.mhhe.com/coburn and review **Section 7.7: Complex Numbers in Exponential Form.**

7.6 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section if needed.

1. For $z = r(\cos\theta + i\sin\theta)$, z^5 is computed as ________ according to ________ theorem.

2. If $z = 6i$, then z raised to an ______ power will be real and z raised to an ______ power will be ______ since $\theta =$ ______.

3. One application of De Moivre's theorem is to check ________ solutions to a polynomial equation.

4. The nth roots of a complex number are equally spaced on a circle of radius r, since their arguments all differ by ______ degrees or ______ radians.

5. From Example 4, go ahead and compute the value of z_5, z_6, and z_7. What do you notice? Discuss how this reaffirms that there are exactly n, nth roots.

6. Use a calculator to find $(1 - 3i)^4$. Then use it again to find the fourth root of the result. What do you notice? Explain the discrepancy and then resolve it using the nth roots theorem to find all four roots.

▶ DEVELOPING YOUR SKILLS

Use De Moivre's theorem to compute the following. Clearly state the value of r, n, and θ before you begin.

7. $(3 + 3i)^4$

8. $(-2 + 2i)^6$

9. $(-1 + \sqrt{3}i)^3$

10. $(\sqrt{3} - i)^3$

11. $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^5$

12. $\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^6$

13. $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)^6$

14. $\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^5$

15. $(4\ \text{cis}\ 330°)^3$

16. $(4\ \text{cis}\ 300°)^3$

17. $\left(\frac{\sqrt{2}}{2}\ \text{cis}\ 135°\right)^5$

18. $\left(\frac{\sqrt{2}}{2}\ \text{cis}\ 135°\right)^8$

Use De Moivre's theorem to verify the solution given for each polynomial equation.

19. $z^4 + 3z^3 - 6z^2 + 12z - 40 = 0;\ z = 2i$

20. $z^4 - z^3 + 7z^2 - 9z - 18 = 0;\ z = -3i$

21. $z^4 + 6z^3 + 19z^2 + 6z + 18 = 0;\ z = -3 - 3i$

22. $2z^4 + 3z^3 - 4z^2 + 2z + 12 = 0;\ z = 1 - i$

23. $z^5 + z^4 - 4z^3 - 4z^2 + 16z + 16 = 0;\ z = \sqrt{3} - i$

24. $z^5 + z^4 - 16z^3 - 16z^2 + 256z + 256 = 0;$ $z = 2\sqrt{3} + 2i$

25. $z^4 - 4z^3 + 7z^2 - 6z - 10 = 0;\ z = 1 + 2i$

26. $z^4 - 2z^3 - 7z^2 + 28z + 52 = 0;\ z = 3 - 2i$

Find the nth roots indicated by writing and solving the related equation.

27. five fifth roots of unity

28. six sixth roots of unity

29. five fifth roots of 243

30. three cube roots of 8

31. three cube roots of $-27i$

32. five fifth roots of $32i$

Solve each equation using the nth roots theorem.

33. $x^5 - 32 = 0$

34. $x^5 - 243 = 0$

35. $x^3 - 27i = 0$

36. $x^3 + 64i = 0$

37. $x^5 - \sqrt{2} - \sqrt{2}i = 0$

38. $x^5 - 1 + \sqrt{3}i = 0$

39. Solve the equation $x^3 - 1 = 0$ by factoring it as the difference of cubes and applying the quadratic formula. Compare results to those obtained in Example 3.

40. Use the nth roots theorem to find the four fourth roots of unity, then find all solutions to $x^4 - 1 = 0$ by factoring it as a difference of squares. What do you notice?

Use the nth roots theorem to find the *n*th roots. Clearly state *r*, *n*, and θ (from the trigonometric form of *z*) as you begin. Answer in exact form when possible, otherwise use a four decimal place approximation.

41. four fourth roots of $-8 + 8\sqrt{3}i$

42. five fifth roots of $16 - 16\sqrt{3}i$

43. four fourth roots of $-7 - 7i$

44. three cube roots of $9 + 9i$

▶ WORKING WITH FORMULAS

The discriminant of a cubic equation: $D = \dfrac{4p^3 + 27q^2}{108}$

For cubic equations of the form $z^3 + pz + q = 0$, where p and q are real numbers, one solution has the form $z = \sqrt[3]{-\frac{q}{2} + \sqrt{D}} + \sqrt[3]{-\frac{q}{2} - \sqrt{D}}$, where D is called the discriminant. Compute the value of D for the cubic equations given, then use the *n*th roots theorem to find the three cube roots of $-\frac{q}{2} + \sqrt{D}$ and $-\frac{q}{2} - \sqrt{D}$ in trigonometric form (also see Exercises 61 and 62).

45. $z^3 - 6z + 4 = 0$

46. $z^3 - 12z - 8 = 0$

▶ APPLICATIONS

47. Powers and roots: Just after Example 4, the four fourth roots of $z = 8 + 8\sqrt{3}i$ were given as

$z_0 = 2(\cos 15° + i \sin 15°)$

$z_1 = 2(\cos 105° + i \sin 105°)$

$z_2 = 2(\cos 195° + i \sin 195°)$

$z_3 = 2(\cos 285° + i \sin 285°)$.

Verify these are the four fourth roots of $z = 8 + 8\sqrt{3}i$ using a calculator and De Moivre's theorem.

48. Powers and roots: In Example 4 we found the five fifth roots of $z = 16\sqrt{3} + 16i$ were

$z_0 = 2(\cos 6° + i \sin 6°)$

$z_1 = 2(\cos 78° + i \sin 78°)$

$z_2 = 2(\cos 150° + i \sin 150°)$

$z_3 = 2(\cos 222° + i \sin 222°)$

$z_4 = 2(\cos 294° + i \sin 294°)$

Verify these are the five fifth roots of $16\sqrt{3} + 16i$ using a calculator and De Moivre's theorem.

Electrical circuits: For an AC circuit with three branches wired in parallel, the total impedance is given by $Z_T = \dfrac{Z_1Z_2Z_3}{Z_1Z_2 + Z_1Z_3 + Z_2Z_3}$, where Z_1, Z_2, and Z_3 represent the impedance in each branch of the circuit. If the impedance in each branch is identical, $Z_1 = Z_2 = Z_3 = Z$, and the numerator becomes Z^3 and the denominator becomes $3Z^2$, (a) use De Moivre's theorem to calculate the numerator and denominator for each value of Z given, (b) find the total impedance by computing the quotient $\dfrac{Z^3}{3Z^2}$, and (c) verify your result is identical to $\dfrac{Z}{3}$.

49. $Z = 3 + 4j$ in all three branches

50. $Z = 5\sqrt{3} + 5j$ in all three branches

▶ EXTENDING THE CONCEPT

In Chapter 6, you were asked to verify that $\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$ and $\cos(4\theta) = 8\cos^2\theta - 8\cos^2\theta + 1$ were identities (Section 6.4, Exercises 21 and 22). For $z = 3 + \sqrt{7}i$, verify $|z| = 4$ and $\theta = \tan^{-1}\left(\frac{\sqrt{7}}{3}\right)$, then draw a right triangle with $\sqrt{7}$ opposite θ and 3 adjacent to θ. Discuss how this right triangle and the identities given can be used in conjunction with De Moivre's theorem to find the exact value of the powers given (also see Exercises 53 and 54).

51. $(3 + \sqrt{7}i)^3$

52. $(3 + \sqrt{7}i)^4$

For cases where θ is not a standard angle, working toward an exact answer using De Moivre's theorem requires the use of multiple angle identities and drawing the right triangle related to $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. For Exercises 53 and 54, use De Moivre's theorem to compute the complex powers by (a) constructing the related right triangle for θ, (b) evaluating $\sin(4\theta)$ using two applications of double-angle identities, and (c) evaluating $\cos(4\theta)$ using a Pythagorean identity and the computed value of $\sin(4\theta)$.

53. $z = (1 + 2i)^4$

54. $(2 + \sqrt{5}i)^4$

The solutions to the cubic equations in Exercises 45 and 46 (repeated in Exercises 55 and 56) can be found by adding the cube roots of $-\frac{q}{2} + \sqrt{D}$ and $-\frac{q}{2} - \sqrt{D}$ that have arguments summing to 360°.

55. Find the roots of $z^3 - 6z + 4 = 0$

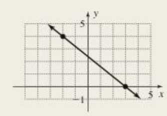
56. Find the roots of $z^3 - 12z - 8 = 0$

▶ MAINTAINING YOUR SKILLS

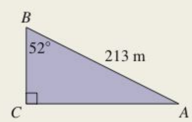
57. (6.2) Prove the following is a identity:
$$\frac{\tan^2 x}{\sec x + 1} = \frac{1 - \cos x}{\cos x}$$

58. (3.3) Given $f(x) = 2x^2 - 3x$, determine: $f(-1), f\left(\frac{1}{3}\right), f(a)$ and $f(a + h)$.

59. (2.3) Find the equation of the line whose graph is given.



60. (5.2) Solve the triangle given. Round lengths to hundredths of a meter.



SUMMARY AND CONCEPT REVIEW

SECTION 7.1 Oblique Triangles and the Law of Sines

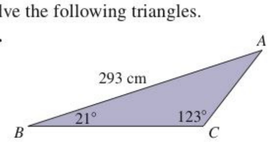
KEY CONCEPTS

- In any triangle, the ratio of the sine of an angle to its opposite side is constant: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.
- The law of sines requires a known angle, a side opposite this angle and an additional side or angle, hence cannot be applied for SSS and SAS triangles.
- For AAS and ASA triangles, the law of sines yields a unique solution.
- When given two sides of a triangle and an angle opposite one of these sides (SSA), the number of solutions is *in doubt,* giving rise to the designation, "the ambiguous case."
- SSA triangles may have no solution, one solution, or two solutions, depending on the length of the side opposite the given angle.
- When solving triangles, always remember:
  - The sum of all angles must be 180°: $\angle A + \angle B + \angle C = 180°$.
  - The sum of any two sides must exceed the length of the remaining side.
  - Longer sides are opposite larger angles.
  - $k = \sin^{-1}\theta$ has no solution for $k > 1$.
  - $k = \sin^{-1}\theta$ has two solutions in $[0, 360°)$ for $0 < |k| < 1$.

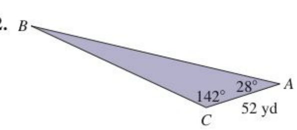
EXERCISES

Solve the following triangles.

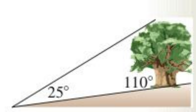
1.



2.

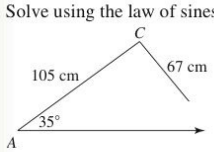


3. A tree is growing vertically on a hillside. Find the height of the tree if it makes an angle of 110° with the hillside and the angle of elevation from the base of the hill to the top of the tree is 25° at a distance of 70 ft.

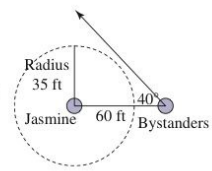


4. Find two values of θ that will make the equation true: $\frac{\sin\theta}{14} = \frac{\sin 50}{31}$.

5. Solve using the law of sines. If two solutions exist, find both (figure not drawn to scale).



6. Jasmine is flying her tethered, gas-powered airplane at a local park, where a group of bystanders is watching from a distance of 60 ft, as shown. If the tether has a radius of 35 ft and one of the bystanders walks away at an angle of 40°, will he get hit by the plane? What is the smallest angle of exit he could take (to the nearest whole) without being struck by Jasmine's plane?



SECTION 7.2 The Law of Cosines; the Area of a Triangle

KEY CONCEPTS

- The law of cosines is used to solve SSS and SAS triangles.
- The law of cosines states that in any triangle, the square of any side is equal to the sums of the squares of the other two sides, minus twice their product times the cosine of the included angle:

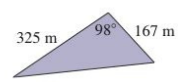
$$a^2 = b^2 + c^2 - 2bc\cos A$$

- When using the law of cosines to solve a SSS triangle, always begin with the largest angle or the angle opposite the largest side.
- The area of a nonright triangle can be found using the following formulas. The choice of formula depends on the information given.
  - two sides a and b with included angle c $\mathcal{A} = \frac{1}{2}ab\sin C$
  - two angles A and B with included side c $\mathcal{A} = \frac{c^2 \sin A \sin B}{2\sin C}$
  - three sides a, b, and c with $S = \frac{a+b+c}{2}$ $\mathcal{A} = \sqrt{s(s-a)(s-b)(s-c)}$

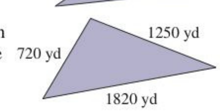
EXERCISES

7. Solve for B: $9^2 = 12^2 + 15^2 - 2(12)(15)\cos B$

8. Use the law of cosines to find the missing side.



9. While preparing for the day's orienteering meet, Rick finds that the distances between the first three markers he wants to pick up are 1250 yd, 1820 yd, and 720 yd. Find the measure of each angle in the triangle formed so that Rick is sure to find all three markers.



10. The Great Pyramid of Giza, also known Khufu's pyramid, is the sole remaining member of the Seven Wonders of the Ancient World. It was built as a tomb for the Egyptian pharaoh Khufu from the fourth dynasty. This square pyramid is made up of four isosceles triangles, each with a base of 230.0 m and a slant height of about 218.7 m. Approximate the total surface area of Khufu's pyramid (excluding the base).

SECTION 7.3 Vectors and Vector Diagrams

KEY CONCEPTS

- Quantities/concepts that can be described using a single number are called scalar quantities. Examples are time, perimeter, area, volume, energy, temperature, weight, and so on.
- Quantities/concepts that require more than a single number to describe their attributes are called vector quantities. Examples are force, velocity, displacement, pressure, and so on.

- Vectors can be represented using directed line segments to indicate magnitude and direction. The origin of the segment is called the initial point, with the arrowhead pointing to the terminal point. When used solely for comparative analysis, they are called geometric vectors.
- Two vectors are equal if they have the same magnitude and direction.
- Vectors can be represented graphically in the *xy*-plane by naming the initial and terminal points of the vector or by giving the magnitude and angle of the related position vector [initial point at (0, 0)].
- For a vector with initial point (x_1, y_1) and terminal point (x_2, y_2), the related position vector can be written in the component form $\langle a, b\rangle$, where $a = x_2 - x_1$ and $b = y_2 - y_1$.
- For a vector written in the component form $\langle a, b\rangle$, a is called the horizontal component and b is called the vertical component of the vector.
- For vector $\mathbf{v} = \langle a, b\rangle$, the magnitude of $\mathbf{v}$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$.
- Vector components can also be written in trigonometric form. See page 739.
- For $\mathbf{u} = \langle a, b\rangle$, $\mathbf{v} = \langle c, d\rangle$, and any scalar k, we have the following operations defined:

  $$\mathbf{u} + \mathbf{v} = \langle a + c, b + d\rangle \qquad \mathbf{u} - \mathbf{v} = \langle a - c, b - d\rangle \qquad k\mathbf{u} = \langle ka, kb\rangle \text{ for } k \in \mathbb{R}$$

  If $k > 0$, the new vector has the same direction as $\mathbf{u}$; $k < 0$, the opposite direction.
- Vectors can be written in algebraic form using **i**, **j** notation, where **i** is an *x*-axis unit vector and **j** is a *y*-axis unit vector. The vector $\langle a, b\rangle$ is written as a linear combination of **i** and **j**: $\langle a, b\rangle = a\mathbf{i} + b\mathbf{j}$.
- For any nonzero vector **v**, vector $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the same direction as **v**.
- In aviation and shipping, the heading of a ship or plane is understood to be the amount of rotation from due north in the clockwise direction.

EXERCISES

11. Graph the vector $\mathbf{v} = \langle 9, 5\rangle$, then compute its magnitude and direction angle.

12. Write the vector $\mathbf{u} = \langle -8, 3\rangle$ in **i**, **j** form and compute its magnitude and direction angle.

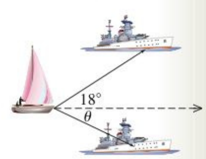
13. Approximate the horizontal and vertical components of the vector **u**, where $|\mathbf{u}| = 18$ and $\theta = 52°$.

14. Compute $2\mathbf{u} + \mathbf{v}$, then find the magnitude and direction of the resultant: $\mathbf{u} = \langle -3, -5\rangle$ and $\mathbf{v} = \langle 2, 8\rangle$.

15. Find a unit vector that points in the same direction as $\mathbf{u} = 7\mathbf{i} + 12\mathbf{j}$.

16. Without computing, if $\mathbf{u} = \langle -9, 2\rangle$ and $\mathbf{v} = \langle 2, 8\rangle$, will the resultant sum lie in Quadrant I or II? Why?

17. It's once again time for the Great River Race, a $\frac{1}{2}$-mi swim across the Panache River. If Karl fails to take the river's 1-mph current into account and he swims the race at 3 mph, how far from the finish marker does he end up when he makes it to the other side?



18. Two Coast Guard vessels are towing a large yacht into port. The first is pulling with a force of 928 N and the second with a force of 850 N. Determine the angle θ for the second Coast Guard vessel that will keep the ship moving safely in a straight line.

SECTION 7.4 Vector Applications and the Dot Product

KEY CONCEPTS

- Vector forces are in equilibrium when the sum of their components is the zero vector.
- When the components of vector **u** are nonquadrantal, with one of its components lying along vector **v**, we call this component the "component of **u** along **v**" or $\text{comp}_{\mathbf{v}}\mathbf{u}$.
- For vectors **u** and **v**, $\text{comp}_{\mathbf{v}}\mathbf{u} = |\mathbf{u}|\cos\theta$, where θ is the angle between **u** and **v**.
- Work done is computed as the product of the constant force **F** applied, times the distance D the force is applied: $W = \mathbf{F} \cdot D$.
- If force is not applied parallel to the direction of movement, only the component of the force in the direction of movement is used in the computation of work. If **u** is a force vector not parallel to the direction of vector **v**, the equation becomes $W = \text{comp}_{\mathbf{v}}\mathbf{u} \cdot |\mathbf{v}|$.

- For vectors $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$, the dot product $\mathbf{u} \cdot \mathbf{v}$ is defined as the scalar $ac + bd$.
- The dot product $\mathbf{u} \cdot \mathbf{v}$ is equivalent to $\text{comp}_{\mathbf{u}}\mathbf{v} \cdot |\mathbf{v}|$ and to $|\mathbf{u}||\mathbf{v}|\cos\theta$.
- The angle between two vectors can be computed using $\cos\theta = \frac{\mathbf{u}}{|\mathbf{u}|} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.
- Given vectors **u** and **v**, the projection of **u** along **v** is the *vector* $\mathbf{proj_v u}$ defined by $\mathbf{proj_v u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2}\right)\mathbf{v}$.
- Given vectors **u** and **v** and $\mathbf{proj_v u}$, **u** can be resolved into the orthogonal components $\mathbf{u}_1$ and $\mathbf{u}_2$ where $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2$, $\mathbf{u}_1 = \mathbf{proj_v u}$, and $\mathbf{u}_2 = \mathbf{u} - \mathbf{u}_1$.
- The horizontal distance x a projectile travels in t seconds is $x = (|\mathbf{v}|\cos\theta)t$.
- The vertical height y of a projectile after t seconds is $y = (|\mathbf{v}|\sin\theta)t - 16t^2$, where $|\mathbf{v}|$ is the magnitude of the initial velocity, and θ is the angle of projection.

EXERCISES

19. For the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ given, find the resultant and an additional force vector so that equilibrium takes place: $\mathbf{F}_1 = \langle -20, 70\rangle$; $\mathbf{F}_2 = \langle 45, 53\rangle$.

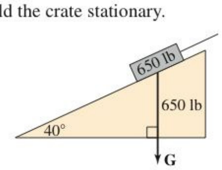
20. Find $\text{comp}_{\mathbf{v}}\mathbf{u}$ for $\mathbf{u} = -12\mathbf{i} - 16\mathbf{j}$ and $\mathbf{v} = 19\mathbf{i} - 13\mathbf{j}$.

21. Find the component d that ensures vectors **u** and **v** are orthogonal: $\mathbf{u} = \langle 2, 9\rangle$ and $\mathbf{v} = \langle -18, d\rangle$.

22. Compute $\mathbf{p} \cdot \mathbf{q}$ and find the angle between them: $\mathbf{p} = \langle -5, -2\rangle$; $\mathbf{q} = \langle 4, -7\rangle$.

23. Given force vector $\mathbf{F} = \langle 50, 15\rangle$ and $\mathbf{v} = \langle 85, 6\rangle$, find the work required to move an object along the entire length of **v**. Assume force is in pounds and distance in feet.

24. A 650-lb crate is sitting on a ramp that is inclined at 40°. Find the force needed to hold the crate stationary.

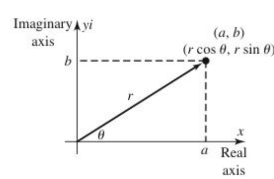


25. An arctic explorer is hauling supplies from the supply hut to her tent, a distance of 120 ft, in a sled she is dragging behind her. If the straps used make an angle of 25° with the snow-covered ground and she pulls with a constant force of 75 lb, find the amount of work done.

26. A projectile is launched from a sling-shot with an initial velocity of $v_0 = 280$ ft/sec at an angle of $\theta = 50°$. Find (a) the position of the object after 1.5 sec and (b) the time required to reach a height of 150 ft.

SECTION 7.5 Complex Numbers in Trigonometric Form

KEY CONCEPTS



- A complex number $a + bi = (a, b)$ can be written in trigonometric form by noting (from its graph) that $a = r\cos\theta$ and $b = r\sin\theta$: $a + bi = r(\cos\theta + i\sin\theta)$.
- The angle θ is called the argument of z and r is called the modulus of z.
- The argument of a complex number z is not unique, since any rotation of $\theta + 2\pi k$ (k an integer) will yield a coterminal angle.
- To convert from trigonometric to rectangular form, evaluate $\cos\theta$ and $\sin\theta$ and multiply by the modulus.
- To multiply complex numbers in trig form, multiply the moduli and add the arguments. To divide complex numbers in trig form, divide the moduli and subtract the arguments.
- Complex numbers have numerous real-world applications, particularly in a study of AC electrical circuits.
- The impedance of an AC circuit is given as $Z = R + j(X_L - X_C)$, where R is a pure resistance, X_C is the capacitive reactance, X_L is the inductive reactance, and $j = \sqrt{-1}$.
- Z is a complex number with magnitude $|Z| = \sqrt{R^2 + (X_L - X_C)^2}$ and phase angle $\theta = \tan^{-1}\left(\frac{X_L - X_C}{R}\right)$ (θ represents the angle between the voltage and current).
- In an AC circuit, voltage $V = IZ$; current $I = \frac{V}{Z}$.

EXERCISES

27. Write in trigonometric form: $z = -1 - \sqrt{3}i$

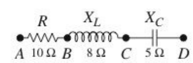
28. Write in rectangular form: $z = 3\sqrt{2}\left[\operatorname{cis}\left(\frac{\pi}{4}\right)\right]$

29. Graph in the complex plane: $z = 5(\cos 30° + i \sin 30°)$

30. For $z_1 = 8\operatorname{cis}\left(\frac{\pi}{4}\right)$ and $z_2 = 2\operatorname{cis}\left(\frac{\pi}{6}\right)$, compute $z_1 z_2$ and $\frac{z_1}{z_2}$.

31. Find the current I in a circuit where $V = 4\sqrt{3} - 4j$ and $Z = 1 - \sqrt{3}j\ \Omega$.

32. In the V_{RLC} series circuit shown, $R = 10\ \Omega$, $X_L = 8\ \Omega$, and $X_C = 5\ \Omega$. Find the magnitude of Z and the phase angle between current and voltage. Express the result in trigonometric form.



SECTION 7.6 De Moivre's Theorem and the Theorem on *n*th Roots

KEY CONCEPTS

- For complex number $z = r(\cos\theta + i\sin\theta)$, $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$ (De Moivre's theorem).
- De Moivre's theorem can be used to check complex solutions of polynomial equations.
- For complex number $z = r(\cos\theta + i\sin\theta)$, $\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$, for $k = 1, 2, 3, \ldots, n - 1$ (*n*th roots theorem).
- The *n*th roots of a complex number are equally spaced around a circle of radius r in the complex plane.

EXERCISES

33. Use De Moivre's theorem to compute the value of $(-1 + i\sqrt{3})^5$.

34. Use De Moivre's theorem to verify that $z = 1 - i$ is a solution of $z^4 + z^3 - 2z^2 + 2z + 4 = 0$.

35. Use the *n*th roots theorem to find the three cube roots of $125i$.

36. Solve the equation using the *n*th roots theorem: $x^3 - 216 = 0$.

37. Given that $z = 2 + 2i$ is a fourth root of -64, state the other three roots.

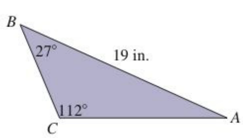
38. Solve using the quadratic formula and the *n*th roots theorem: $z^4 + 6z^2 + 25 = 0$.

39. Use De Moivre's theorem to verify the three roots of $125i$ found in Exercise 35.

MIXED REVIEW

Solve each triangle using either the law of sines or law of cosines (whichever is appropriate) then find the area of each.

1.



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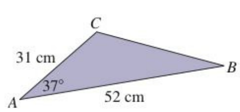
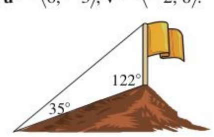


3. Find the horizontal and vertical components of the vector **u**, where $|\mathbf{u}| = 21$ and $\theta = 40°$.

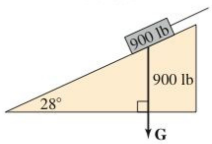
4. Compute $2\mathbf{u} + \mathbf{v}$, then find the magnitude and direction of the resultant: $\mathbf{u} = \langle 6, -3 \rangle$, $\mathbf{v} = \langle -2, 8 \rangle$.

5. Find the height of a flagpole that sits atop a hill, if it makes an angle of 122° with the hillside, and the angle of elevation between the side of the hill to the top of the flagpole is 35° at a distance of 120 ft.



6. A 900-lb crate is sitting on a ramp that is inclined at 28°. Find the force needed to hold the object stationary.

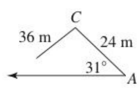
Exercise 6



7. A jet plane is flying at 750 mph on a heading of 30°. There is a strong, 50-mph wind blowing from due south (heading 0°). What is the true course and speed of the plane (relative to the ground)?

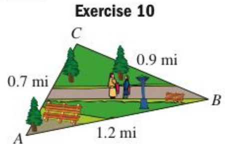
8. Graph the vector $\mathbf{v} = \langle -8, 5 \rangle$, then compute its magnitude and direction.

9. Solve using the law of sines. If two solutions exist, find both.

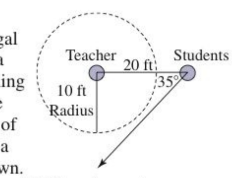


10. A local Outdoors Club sponsors a treasure hunt activity for its members, and has placed surprise packages at the corners of the triangular park shown. Find the measure of each angle to help club members find their way to the treasure.

Exercise 10



11. As part of a lab demonstrating centrifugal and centripetal forces, a physics teacher is whirling a tethered weight above her head while a group of students looks on from a distance of 20 ft as shown. If the tether has a radius of 10 ft and a student departs at the 35° angle shown, will the student be struck by the weight? What is the smallest angle of exit the student could take (to the nearest whole) without being struck by the whirling weight?



12. Given the vectors $\mathbf{p} = \langle -5, 2 \rangle$ and $\mathbf{q} = \langle 4, 7 \rangle$, use the dot product $\mathbf{p} \cdot \mathbf{q}$ to find the angle between them.

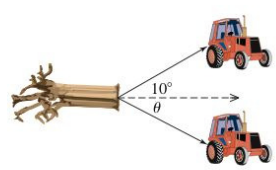
13. a. Graph the complex number using the rectangular form, then convert to trigonometric form: $z = 4 - 4i$.

b. Graph the complex number using the trigonometric form, then convert to rectangular form: $z = 6(\cos 120° + i \sin 120°)$.

14 a. Verify that $z = 4 - 5i$ *and* its conjugate are solutions to $z^2 - 8z + 41 = 0$.

b. Solve using the quadratic formula: $z^2 - 6iz + 7 = 0$

15. Two tractors are dragging a large, fallen tree into the brush pile that's being prepared for a large Fourth of July bonfire. The first is pulling with a force of 418 N and the second with a force of 320 N. Determine the angle θ for the second tractor that will keep the tree headed straight for the brush pile.



16. Given $z_1 = 8(\cos 45° + i \sin 45°)$ and $z_2 = 4(\cos 15° + i \sin 15°)$ compute:

a. the product $z_1 z_2$ **b.** the quotient $\frac{z_1}{z_2}$

17. Given the vectors $\mathbf{u} = -12\mathbf{i} - 16\mathbf{j}$ and $\mathbf{v} = 19\mathbf{i} - 13\mathbf{j}$, find $\text{comp}_{\mathbf{v}}\mathbf{u}$ and $\mathbf{proj}_{\mathbf{v}}\mathbf{u}$.

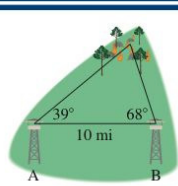
18. Find the result using De Moivre's theorem: $(2\sqrt{3} - 2i)^6$.

19. Use the *n*th roots theorem to find the four fourth roots of $-2 + 2i\sqrt{3}$.

20. The impedance of an AC circuit is $Z = R + j(X_L - X_C)$. The voltage across the circuit is $V_{RLC} = I|Z|$. Given $R = 12\ \Omega$, $X_L = 15.2\ \Omega$, and $X_C = 9.4\ \Omega$, write Z in trigonometric form and find the voltage in the circuit if the current is $I = 6.5$ A.

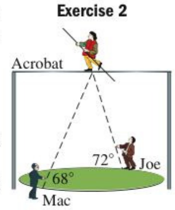
PRACTICE TEST

1. Within the Kilimanjaro Game Reserve, a fire is spotted by park rangers stationed in two towers known to be 10 mi apart. Using the line between them as a baseline, tower A reports the fire is at an angle of 39°, while tower B reports an angle of 68°. How far is the fire from the closer tower?

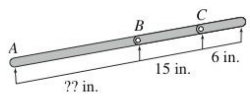


2. At the circus, Mac and Joe are watching a high-wire act from first-row seats on opposite sides of the center ring. Find the height of the performing acrobat at the instant Mac measures an angle of elevation of 68° while Joe measures an angle of 72°. Assume Mac and Joe are sitting 100 ft apart.

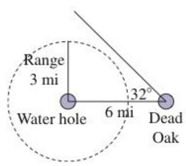
Exercise 2



3. Three rods are attached via two joints and shaped into a triangle. How many triangles can be formed if the angle at the joint B must measure 20°? If two triangles can be formed, solve both.

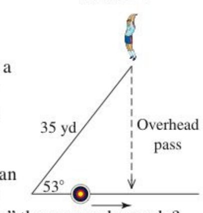


4. Jackie and Sam are rounding up cattle in the brush country, and are communicating via walkie-talkie. Jackie is at the water hole and Sam is at Dead Oak, which are 6 mi apart. Sam finds some strays and heads them home at the 32° indicated. (a) If the maximum range of Jackie's unit is 3 mi, will she be able to communicate with Sam as he heads home? (b) If the maximum range were 4 mi, how far from Dead Oak is Sam when he is first contacted by Jackie?

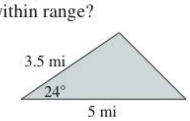


5. As part of an All-Star competition, a group of soccer players (forwards) stand where shown in the diagram and attempt to hit a moving target with a two-handed overhead pass. If a player has a maximum effective range of approximately (a) 25 yd, can the target be hit? (b) about 28 yd, how many "effective" throws can be made? (c) 35 yd and the target is moving at 5 yd/sec, how many seconds is the target within range?

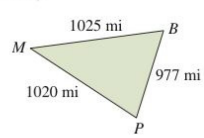
Exercise 5



6. The summit of Triangle Peak can only be reached from one side, using a trail straight up the side that is approximately 3.5 mi long. If the mountain is 5 mi wide at its base and the trail makes a 24° angle with the horizontal, (a) what is the approximate length of the opposing side? (b) How tall is the peak (in feet)?

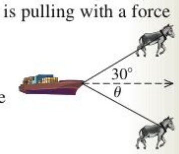


7. The Bermuda Triangle is generally thought to be the triangle formed by Miami, Florida, San Juan, Puerto Rico, and Bermuda itself. If the distances between these locations are the 1025 mi, 1020 mi, and 977 mi indicated, find the measure of each angle and the area of the Bermuda Triangle.

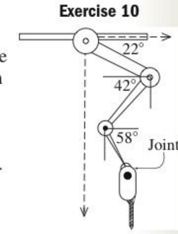


8. A helicopter is flying at 90 mph on a heading of 40°. A 20-mph wind is blowing from the NE on a heading of 190°. What is the true course and speed of the helicopter relative to the ground? Draw a diagram as part of your solution.

9. Two mules walking along a river bank are pulling a heavy barge up river. The first is pulling with a force of 250 N and the second with a force of 210 N. Determine the angle θ for the second mule that will ensure the barge stays midriver and does not collide with the shore.

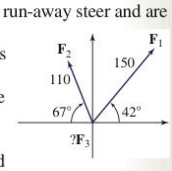


Exercise 10



10. Along a production line, various tools are attached to the ceiling with a multijointed arm so that workers can draw one down, position it for use, then move it up out of the way for the next tool (see the diagram). If the first segment is 100 cm, the second is 75 cm, and the third is 50 cm, determine the approximate coordinates of the last joint.

11. Three ranch hands have roped a run-away steer and are attempting to hold him steady. The first and second ranch hands are pulling with the magnitude and at the angles indicated in the diagram. If the steer is held fast by the efforts of all three, find the magnitude of the tension and angle of the rope from the third cowhand.



12. For $\mathbf{u} = \langle -9, 5\rangle$ and $\mathbf{v} = \langle -2, 6\rangle$, (a) compute the angle between $\mathbf{u}$ and $\mathbf{v}$; (b) find the projection of $\mathbf{u}$ along $\mathbf{v}$ (find $\mathbf{proj_v u}$; and (c) resolve $\mathbf{u}$ into vectors $\mathbf{u}_1$ and $\mathbf{u}_2$, where $\mathbf{u}_1 \| \mathbf{v}$ and $\mathbf{u}_2 \perp \mathbf{v}$.

13. A lacrosse player flips a long pass to a teammate way down field who is near the opponent's goal. If the initial velocity of the pass is 110 ft/sec and the ball is released at an angle of 50° with level ground, how high is the ball after 2 sec? How long until the ball again reaches this same height?

14. Compute the quotient $\frac{z_1}{z_2}$, given

$$z_1 = 6\sqrt{5}\,\text{cis}\left(\frac{\pi}{8}\right) \text{ and } z_2 = 3\sqrt{5}\,\text{cis}\left(\frac{\pi}{12}\right).$$

15. Compute the product $z = z_1 z_2$ in trigonometric form, then verify $|z_1||z_2| = |z|$ and $\theta_1 + \theta_2 = \theta$: $z_1 = -6 + 6i$; $z_2 = 4 - 4\sqrt{3}i$

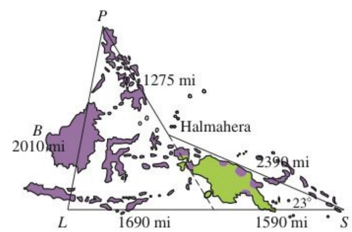
16. Use De Moivre's theorem to compute the value of $(\sqrt{3} - i)^4$.

17. Use De Moivre's theorem to verify $2 + 2\sqrt{3}i$ is a solution to $z^5 + 3z^3 + 64z^2 + 192 = 0$.

18. Use the *n*th roots theorem to solve $x^3 - 125i = 0$.

19. Solve using u-substitution, the quadratic formula, and the nth roots theorem: $z^4 - 6z^2 + 58 = 0$.

20. Due to its huge biodiversity, preserving Southeast Asia's Coral Triangle has become a top priority for conservationists. Stretching from the northern Philippines (P), south along the coast of Borneo (B) to the Lesser Sunda Islands (L), then eastward to the Solomon Islands (S), this area is home to over 75% of all coral species known. Use Heron's formula to help find the total area of this natural wonderland, given the dimensions shown.



CALCULATOR EXPLORATION AND DISCOVERY

Investigating Projectile Motion

There are two important aspects of projectile motion that were not discussed earlier, the **range** of the projectile and the **optimum angle** θ that will maximize this range. Both can be explored using the equations for the horizontal and vertical components of the projectile's position: horizontal $\rightarrow (|\mathbf{v}|\cos\theta)t$ and vertical $\rightarrow (|\mathbf{v}|\sin\theta)t - 16t^2$. In Example 10 of Section 7.4, an arrow was shot from a bow with initial velocity $|\mathbf{v}| = 150$ ft/sec at an angle of $\theta = 50°$. Enter the equations above on the Y= screen as Y_1 and Y_2, using these values (Figure 7.81). Then set up the TABLE using TblStart = 0, ΔTbl = 0.5 and the AUTO mode. The resulting table is shown in Figure 7.82, where Y_1 represents the horizontal distance the arrow has traveled, and Y_2 represents the height of the arrow. To find the *range* of the arrow, scroll downward ▼ until the height (Y_2) shows a value that is less than or equal to zero (the arrow has hit the ground). As Figure 7.83 shows, this happens somewhere between $t = 7$ and $t = 7.5$ sec. We could now change the TBLSET settings to TblStart = 0 and ΔTbl = 0.1 to get a better approximation of the time the arrow is in flight (it's just less than 7.2 sec) and the horizontal range of the arrow (about 692.4 ft), but our main interest is how to *compute these values exactly*. We begin with the equation for the arrow's vertical position $y = (|\mathbf{v}|\sin\theta)t - 16t^2$. Since the object returns to Earth when $y = 0$, we substitute 0 for y and factor out t: $0 = t(|\mathbf{v}|\sin\theta - 16t)$. Solving for t gives $t = 0$ or $t = \frac{|\mathbf{v}|\sin\theta}{16}$. Since the component of velocity in the horizontal direction is $|\mathbf{v}|\cos\theta$, the basic distance relationship $D = r \cdot t$ gives the horizontal range of $R = |\mathbf{v}|\cos\theta \cdot \frac{|\mathbf{v}|\sin\theta}{16}$ or $\frac{|\mathbf{v}|^2\sin\theta\cos\theta}{16}$. Checking the values given for the arrow ($|\mathbf{v}| = 150$ ft/sec and $\theta = 50°$) verifies the range is $R \approx 692.4$. But what about the *maximum possible range* for the arrow? Using $|\mathbf{v}| = 150$ for R results in an equation in theta only: $R(\theta) = \frac{150^2\sin\theta\cos\theta}{16}$, which we can enter as Y_3 and investigate for various θ. After carefully entering $R(\theta)$ as Y_3 and resetting TBLSET to TblStart = 30 and ΔTbl = 5, the TABLE in Figure 7.84 shows a maximum range of about 703 ft at 45°. Resetting TBLSET to TblStart = 40 and ΔTbl = 1 verifies this fact.

Figure 7.81

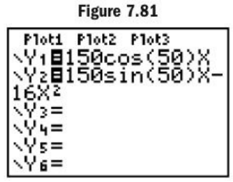


Figure 7.82

X	Y1	Y2
0	0	0
.5	48.209	53.453
1	96.418	98.907
1.5	144.63	136.36
2	192.84	165.81
2.5	241.05	187.27
3	289.25	200.72
X=0		

Figure 7.83

X	Y1	Y2
5	482.09	174.53
5.5	530.3	147.99
6	578.51	113.44
6.5	626.72	70.893
7	674.93	20.347
7.5	723.14	⁻38.2
8	771.35	⁻104.7
X=7.5		

For each of the following exercises, find (a) the height of the projectile after 1.75 sec, (b) the maximum height of the projectile, (c) the range of the projectile, and (d) the number of seconds the projectile is airborne.

Figure 7.84

X	Y3	
30	608.92	
35	660.72	
40	692.44	
45	703.13	
50	692.44	
55	660.72	
60	608.92	
X=30		

Exercise 1: A javelin is thrown with an initial velocity of 85 ft/sec at an angle of 42°.

Exercise 2: A cannon ball is shot with an initial velocity of 1120 ft/sec at an angle of 30°.

Exercise 3: A baseball is hit with an initial velocity of 120 ft/sec at an angle of 50°. Will it clear the center field fence, 10 ft high and 375 ft away?

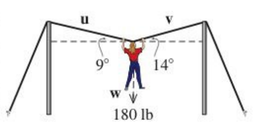
Exercise 4: A field goal (American football) is kicked with an initial velocity of 65 ft/sec at an angle of 35°. Will it clear the crossbar, 10 ft high and 40 yd away?

STRENGTHENING CORE SKILLS

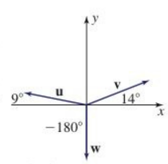
Vectors and Static Equilibrium

In Sections 7.3 and 7.4, the concepts of vector forces, resultant forces, and equilibrium were studied extensively. A nice extension of these concepts involves what is called **static equilibrium.** Assuming that only coplanar forces are acting on an object, the object is said to be in static equilibrium if *the sum of all vector forces acting on it is* ***0***. This implies that the object is stationary, since the forces all counterbalance each other. The methods involved are simple and direct, with a wonderful connection to the systems of equations you've likely seen previously. Consider the following example.

Illustration 1 ▶ As part of their training, prospective FBI agents must move hand-over-hand across a rope strung between two towers. An agent-in-training weighing 180 lb is two-thirds of the way across, causing the rope to deflect from the horizontal at the angles shown. What is the tension in each part of the rope at this point?



Solution ▶ We have three concurrent forces acting on the point where the agent grasps the rope. Begin by drawing a vector diagram and computing the components of each force, using the **i**, **j** notation. Note that $\mathbf{w} = -180\mathbf{j}$.



$$\mathbf{u} = -|\mathbf{u}|\cos(9°)\mathbf{i} + |\mathbf{u}|\sin(9°)\mathbf{j}$$
$$\approx -0.9877|\mathbf{u}|\mathbf{i} + 0.1564|\mathbf{u}|\mathbf{j}$$
$$\mathbf{v} = |\mathbf{v}|\cos(14°)\mathbf{i} + |\mathbf{v}|\sin(14°)\mathbf{j}$$
$$\approx 0.9703|\mathbf{v}|\mathbf{i} + 0.2419|\mathbf{v}|\mathbf{j}$$

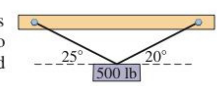
For equilibrium, all vector forces must sum to the zero vector: $\mathbf{u} + \mathbf{v} + \mathbf{w} = 0$, which results in the following equation: $-0.9877|\mathbf{u}|\mathbf{i} + 0.1564|\mathbf{u}|\mathbf{j} + 0.9703|\mathbf{v}|\mathbf{i} + 0.2419|\mathbf{v}|\mathbf{j} - 180\mathbf{j} = 0\mathbf{i} + 0\mathbf{j}$. Factoring out **i** and **j** from the left-hand side yields $(-0.9877|\mathbf{u}| + 0.9703|\mathbf{v}|)\mathbf{i} + (0.1564|\mathbf{u}| + 0.2419|\mathbf{v}| - 180)\mathbf{j} = \ \mathbf{i} + 0\mathbf{j}$. Since any two vectors are equal only when corresponding components are equal, we obtain a system in the two variables

$$|\mathbf{u}| \text{ and } |\mathbf{v}|: \begin{cases} -0.9877|\mathbf{u}| + 0.9703|\mathbf{v}|) = 0 \\ 0.1564|\mathbf{u}| + 0.2419|\mathbf{v}| - 180 = 0 \end{cases}.$$

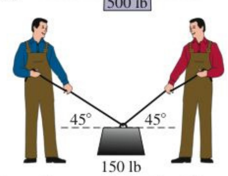
Solving the system using matrix equations and a calculator (or any desired method), gives $|\mathbf{u}| \approx 447$ lb and $|\mathbf{v}| \approx 455$ lb.

At first it may seem surprising that the vector forces (tension) in each part of the rope are so much greater than the 180-lb the agent weighs. But with a 180-lb object hanging from the middle of the rope, the tension required to keep the rope taut (with small angles of deflection) must be very great. This should become more obvious to you after you work Exercise 2.

Exercise 1: A 500-lb crate is suspended by two ropes attached to the ceiling rafters. Find the tension in each rope.



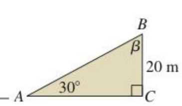
Exercise 2: Two people team up to carry a 150-lb weight by passing a rope through an eyelet in the object. Find the tension in each rope.



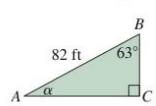
Exercise 3: Referring to Illustration 1, if the rope has a tension limit of 600-lb (before it snaps), can a 200-lb agent make it across?

CUMULATIVE REVIEW CHAPTERS 1–7

1. Solve using a standard triangle.
$a = 20$, $b =$ ____, $c =$ ____
$\alpha = 30°$, $\beta =$ ____, $\gamma =$ ____



2. Solve using trigonometric ratios.
$a \approx$ ____, $b \approx$ ____, $c = 82$
$\alpha =$ ____, $\beta = 63°$, $\gamma =$ ____



3. A torus is a donut-shaped solid figure. Its surface area is given by the formula $A = \pi^2(R^2 - r^2)$, where R is the outer radius of the donut, and r is the inner radius. Solve the formula for R in terms of r and A.

4. For a complex number $a + bi$, (a) verify the sum of a complex number and its conjugate is a real number, and (b) verify the product of a complex number and its conjugate is a real number.

5. State the value of all six trig functions given $\tan \alpha = -\frac{3}{4}$ with $\cos \alpha > 0$.

6. Sketch the graph of $y = 4\cos\left(\frac{\pi}{6}x - \frac{\pi}{3}\right)$ using transformations of $y = \cos x$.

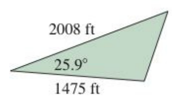
7. Solve using the quadratic formula: $5x^2 + 8x + 2 = 0$.

8. Solve by completing the square: $3x^2 - 72x + 427 = 0$.

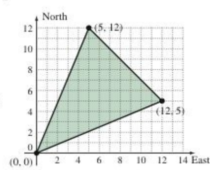
9. Given $\cos 53° \approx 0.6$ and $\cos 72° \approx 0.3$, approximate the value of $\cos 19°$ and $\cos 125°$ without using a calculator.

10. Find all real values of x that satisfy the equation $\sqrt{3} + 2\sin(2x) = 2\sqrt{3}$. State the answer in degrees.

11. **a.** Given that 1 acre = 43,560 ft², find the cost of a lot with the dimensions shown (to the nearest dollar) if land in this area is going for \$4500 per acre.



b. After an accident at sea, a search and rescue team decides to focus their efforts on the area shown due to prevailing winds and currents. Find the distances between each vertex (use Pythagorean triples and a special triangle) and the number of square miles in the search area.



12. State the domain of each function:

a. $f(x) = \sqrt{2x - 3}$

b. $g(x) = \log_b(x + 3)$

c. $h(x) = \frac{x + 3}{x^2 - 5}$

d. $v(x) = \sqrt{x^2 - x - 6}$

13. Write the following formulas from memory:

a. slope formula

b. midpoint formula

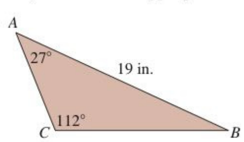
c. quadratic formula

d. distance formula

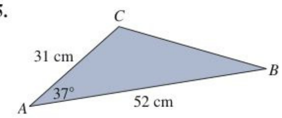
e. interest formula (compounded continuously)

Solve each triangle using the law of sines or the law of cosines, whichever is appropriate.

14.

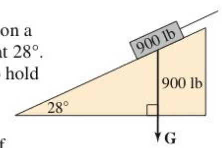


15.



16. A commercial fishery stocks a lake with 250 fish. Based on previous experience, the population of fish is expected to grow according to the model $P(t) = \frac{12{,}000}{1 + 25e^{-0.2t}}$, where t is the time in months. From on this model, (a) how many months are required for the population to grow to 7500 fish? (b) If the fishery expects to harvest three-fourths of the fish population in 2 yr, approximately how many fish will be taken?

17. A 900-lb crate is sitting on a ramp which is inclined at 28°. Find the force needed to hold the object stationary.



18. A jet plane is flying at 750 mph on a heading of 30°. There is a strong, 50-mph wind blowing from due south (heading 0°). What is the true course and speed of the plane (relative to the ground)?

19. Use the *Guidelines for Graphing* to sketch the graph of function f given, then use it to solve $f(x) < 0$.

$$f(x) = x^3 - 4x^2 + x + 6$$

20. Use the *Guidelines for Graphing* to sketch the graph of function g given, then use it to name the intervals where $g(x)\downarrow$ and $g(x)\uparrow$.

$$g(x) = \frac{x^2 - 4}{x^2 - 1}$$

21. Find $(1 - \sqrt{3}i)^8$ using De Moivre's theorem.

22. Solve $\ln(x + 2) + \ln(x - 3) = \ln(4x)$.

23. If I saved \$200 each month in an annuity program that paid 8% annual interest compounded monthly, how long would it take to save \$10,000?

24. Mount Tortolas lies on the Argentine-Chilean border. When viewed from a distance of 5 mi, the angle of elevation to the top of the peak is 38°. How tall is Mount Tortolas? State the answer in feet.

25. The graph given is of the form $y = A\sin(Bx + C)$. Find the values of A, B, and C.