8

Systems of Equations and Inequalities

CHAPTER OUTLINE

8.1 Linear Systems in Two Variables with Applications 794

8.2 Linear Systems in Three Variables with Applications 806

8.3 Nonlinear Systems of Equations and Inequalities 819

8.4 Systems of Inequalities and Linear Programming 826

CHAPTER CONNECTIONS

The disposal of hazardous waste is a growing concern for today's communities, and with many budgets stretched to the breaking point, there is a cost/benefit analysis involved. One major hauler uses trucks with a carrying capacity of 800 ft^3, and can transport at most 10 tons. A full container of liquid waste weighs 800 lb and has a volume of 20 ft^3, while a full container of solid waste weighs 600 lb and has a volume of 30 ft^3. If the hauler makes $300 for disposing of liquid waste and $400 for disposing of solid waste, what is the maximum revenue that can be generated per truck? Chapter 8 outlines a systematic process for answering this question. This application appears as Exercise 58 in Section 8.4.

Check out these other real-world connections:

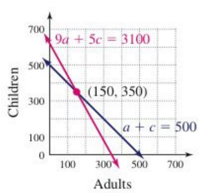
- Appropriate Measurements in Dietetics (Section 8.1, Exercise 64)
- Allocating Winnings to Different Investments (Section 8.2, Exercise 54)
- Minimizing Shipping Costs (Section 8.4, Exercise 61)
- Market Pricing for Organic Produce (Section 8.3, Exercise 56)

8.1 Linear Systems in Two Variables with Applications

Learning Objectives

In Section 8.1 you will learn how to:

- ☐ **A.** Verify ordered pair solutions
- ☐ **B.** Solve linear systems by graphing
- ☐ **C.** Solve linear systems by substitution
- ☐ **D.** Solve linear systems by elimination
- ☐ **E.** Recognize inconsistent systems and dependent systems
- ☐ **F.** Use a system of equations to model and solve applications



In earlier chapters, we used linear equations in two variables to model a number of real-world situations. Graphing these equations gave us a visual image of how the variables were related, and helped us better understand this relationship. In many applications, two different measures of the independent variable must be considered simultaneously, leading to a **system of two linear equations in two unknowns.** Here, a graphical presentation once again supports a better understanding, as we explore systems and their many applications.

A. Solutions to a System of Equations

A **system of equations** is a set of two or more equations for which a common solution is sought. Systems are widely used to model and solve applications when the information given enables the relationship between variables to be stated in different ways. For example, consider an amusement park that brought in \$3100 in revenue by charging \$9.00 for adults and \$5.00 for children, while selling 500 tickets. Using a for adult and c for children, we could write one equation modeling the number of tickets sold: $a + c = 500$, and a second modeling the amount of revenue brought in: $9a + 5c = 3100$. To show that we're considering both equations simultaneously, a large "left brace" is used and the result is called a **system of two equations in two variables:**

$$\begin{cases} a + c = 500 & \text{number of tickets} \\ 9a + 5c = 3100 & \text{amount of revenue} \end{cases}$$

We note that both equations are linear and will have different slope values, so their graphs must intersect at some point. Since every point on a line satisfies the equation of that line, this point of intersection must satisfy *both* equations simultaneously and is the solution to the system. The figure that accompanies Example 1 shows the point of intersecion for this system is (150, 350).

EXAMPLE 1 ► **Verifying Solutions to a System**

Verify that (150, 350) is a solution to $\begin{cases} a + c = 500 \\ 9a + 5c = 3100 \end{cases}$.

Solution ► Substitute the **150** for ***a*** and **350** for ***c*** in each equation.

$$\begin{aligned} a + c &= 500 && \text{first equation} \\ (150) + (350) &= 500 \\ 500 &= 500 \checkmark \end{aligned} \qquad \begin{aligned} 9a + 5c &= 3100 && \text{second equation} \\ 9(150) + 5(350) &= 3100 \\ 3100 &= 3100 \checkmark \end{aligned}$$

Since (150, 350) satisfies both equations, it is the solution to the system and we find the park sold 150 adult tickets and 350 tickets for children.

☑ **A. You've just learned how to verify ordered pair solutions**

Now try Exercises 7 through 18 ►

B. Solving Systems Graphically

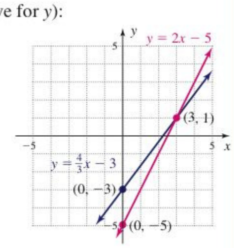
To **solve a system of equations** means we apply various methods in an attempt to find ordered pair solutions. As Example 1 suggests, one method for finding solutions is to graph the system. Any method for graphing the lines can be employed, but to keep important concepts fresh, the slope-intercept method is used here.

EXAMPLE 2 ▶ Solving a System Graphically

Solve the system by graphing: $\begin{cases} 4x - 3y = 9 \\ -2x + y = -5 \end{cases}$.

Solution ▶ First write each equation in slope-intercept form (solve for y):

$$\begin{cases} 4x - 3y = 9 \\ -2x + y = -5 \end{cases} \rightarrow \begin{cases} y = \dfrac{4}{3}x - 3 \\ y = 2x - 5 \end{cases}$$



For the first line, $\frac{\Delta y}{\Delta x} = \frac{4}{3}$ with y-intercept $(0, -3)$. The second equation yields $\frac{\Delta y}{\Delta x} = \frac{2}{1}$ with $(0, -5)$ as the y-intercept. Both are then graphed on the grid as shown. The point of intersection appears to be $(3, 1)$, and checking this point in both equations gives

$$\begin{aligned} 4x - 3y &= 9 \\ 4(3) - 3(1) &= 9 \\ 9 &= 9 \checkmark \end{aligned} \qquad \text{substitute 3 for } x \text{ and 1 for } y \qquad \begin{aligned} -2x + y &= -5 \\ -2(3) + (1) &= -5 \\ -5 &= -5 \checkmark \end{aligned}$$

This verifies that $(3, 1)$ is the solution to the system.

B. You've just learned how to solve linear systems by graphing

Now try Exercises 19 through 22 ▶

C. Solving Systems by Substitution

While a graphical approach best illustrates *why* the solution must be an ordered pair, it does have one obvious drawback—noninteger solutions are difficult to spot. The ordered pair $(\frac{2}{5}, \frac{12}{5})$ is the solution to $\begin{cases} 4x + y = 4 \\ y = x + 2 \end{cases}$, but this would be difficult to "pinpoint" as a precise location on a hand-drawn graph. To overcome this limitation, we next consider a method known as **substitution.** The method involves converting a system of two equations in two variables into a single equation in one variable by using an appropriate substitution. For $\begin{cases} 4x + y = 4 \\ y = x + 2 \end{cases}$, the second equation says "y is two more than x." We reason that *all* points on this line are related this way, *including the point where this line intersects the other.* For this reason, we can substitute $x + 2$ for y *in the first equation,* obtaining a single equation in x.

EXAMPLE 3 ▶ Solving a System Using Substitution

Solve using substitution: $\begin{cases} 4x + y = 4 \\ y = x + 2 \end{cases}$.

Solution ▶ Since $y = x + 2$, we can replace y with $x + 2$ in the first equation.

$$\begin{aligned} 4x + y &= 4 && \text{first equation} \\ 4x + (x + 2) &= 4 && \text{substitute } x + 2 \text{ for } y \\ 5x + 2 &= 4 && \text{simplify} \\ x &= \frac{2}{5} && \text{result} \end{aligned}$$

The x-coordinate is $\frac{2}{5}$. To find the y-coordinate, substitute $\frac{2}{5}$ for x into either of the original equations. Substituting in the second equation gives

$$\begin{aligned} y &= x + 2 && \text{second equation} \\ &= \frac{2}{5} + 2 && \text{substitute } \tfrac{2}{5} \text{ for } x \\ &= \frac{12}{5} && \tfrac{2}{1} = \tfrac{10}{5}, \tfrac{10}{5} + \tfrac{2}{5} = \tfrac{12}{5} \end{aligned}$$

The solution to the system is $(\frac{2}{5}, \frac{12}{5})$. Verify by substituting $\frac{2}{5}$ for x and $\frac{12}{5}$ for y into both equations.

Now try Exercises 23 through 32 ▶

If neither equation allows an immediate substitution, we first solve for one of the variables, either x or y, and *then* substitute. The method is summarized here, and can actually be used with either like variables or like variable expressions. **See Exercises 57 to 60.**

Solving Systems Using Substitution

1. Solve one of the equations for x in terms of y or y in terms of x.
2. Substitute for the appropriate variable in the *other* equation and solve for the variable that remains.
3. Substitute the value from step 2 into either of the original equations and solve for the other unknown.
4. Write the answer as an ordered pair and check the solution in both original equations.

C. You've just learned how to solve linear systems by substitution

D. Solving Systems Using Elimination

Now consider the system $\begin{cases} -2x + 5y = 13 \\ 2x - 3y = -7 \end{cases}$, where solving for any one of the variables will result in fractional values. The substitution method can still be used, but often the **elimination method** is more efficient. The method takes its name from what happens when you add certain equations in a system (by adding the like terms from each). If the coefficients of either x or y are additive inverses—they sum to zero and are *eliminated*. For the system shown, adding the equations produces $2y = 6$, giving $y = 3$, then $x = 1$ using back-substitution (verify).

When neither variable term meets this condition, we can multiply one or both equations by a nonzero constant to "match up" the coefficients, so an elimination will take place. In doing so, we create an **equivalent system of equations,** meaning one that has the same solution as the original system. For $\begin{cases} 7x - 4y = 16 \\ -3x + 2y = -6 \end{cases}$, multiplying the second equation by 2 produces $\begin{cases} 7x - 4y = 16 \\ -6x + 4y = -12 \end{cases}$, giving $x = 4$ after "adding the equations." Note the three systems produced are equivalent, and have the solution (4, 3) ($y = 3$ was found using back-substitution).

1. $\begin{cases} 7x - 4y = 16 \\ -3x + 2y = -6 \end{cases}$ **2.** $\begin{cases} 7x - 4y = 16 \\ -6x + 4y = -12 \end{cases}$ **3.** $\begin{cases} 7x - 4y = 16 \\ x = 4 \end{cases}$

In summary,

Operations that Produce an Equivalent System

1. Changing the order of the equations.
2. Replacing an equation by a nonzero constant multiple of that equation.
3. Replacing an equation with the sum of two equations from the system.

Before beginning a solution using elimination, check to make sure the equations are written in the **standard form** $Ax + By = C$, so that like terms will appear above/below each other. Throughout this chapter, we will use R1 to represent the equation in *row 1* of the system, R2 to represent the equation in *row 2*, and so on. These designations are used to help describe and document the steps being used to solve a system, as in Example 4 where 2R1 + R2 indicates the first equation has been multiplied by two, with the result added to the second equation.

EXAMPLE 4 ► **Solving a System by Elimination**

Solve using elimination: $\begin{cases} 2x - 3y = 7 \\ 6y + 5x = 4 \end{cases}$

Solution ► The second equation is not in standard form, so we re-write the system as $\begin{cases} 2x - 3y = 7 \\ 5x + 6y = 4 \end{cases}$. If we "add the equations" now, we would get $7x + 3y = 11$, with neither variable eliminated. However, if we multiply *both sides* of the first equation by 2, the y-coefficients will be additive inverses. The sum then results in an equation with x as the only unknown.

$$\begin{array}{rl} 2\text{R1} & \begin{cases} 4x - 6y = 14 \\ 5x + 6y = 4 \end{cases} \\ + \\ \text{R2} & \\ \text{sum} & 9x + 0y = 18 \end{array} \quad \text{add}$$

$$9x = 18$$

$$x = 2 \quad \text{solve for } x$$

Substituting 2 for x back into either of the original equations yields $y = -1$. The ordered pair solution is $(2, -1)$. Verify using the original equations.

Now try Exercises 33 through 38 ►

WORTHY OF NOTE

As the elimination method involves adding two equations, it is sometimes referred to as the *addition method* for solving systems.

The elimination method is summarized here. If either equation has fraction or decimal coefficients, we can "clear" them using an appropriate constant multiplier.

Solving Systems Using Elimination

1. Write each equation in standard form: $Ax + By = C$.
2. Multiply one or both equations by a constant that will create coefficients of x (or y) that are additive inverses.
3. Combine the two equations using vertical addition and solve for the variable that remains.
4. Substitute the value from step 3 into either of the original equations and solve for the other unknown.
5. Write the answer as an ordered pair and check the solution in both original equations.

EXAMPLE 5 ► **Solving a System Using Elimination**

Solve using elimination: $\begin{cases} \frac{5}{8}x - \frac{3}{4}y = \frac{1}{4} \\ \frac{1}{2}x - \frac{2}{3}y = 1 \end{cases}$.

Solution ► Multiplying the first equation by 8(8R1) and the second equation by 6(6R2) will clear the fractions from each.

$$\begin{array}{l} 8\text{R1} \\ 6\text{R2} \end{array} \begin{cases} \frac{8}{1}\left(\frac{5}{8}\right)x - \frac{8}{1}\left(\frac{3}{4}\right)y = \frac{8}{1}\left(\frac{1}{4}\right) \\ \frac{6}{1}\left(\frac{1}{2}\right)x - \frac{6}{1}\left(\frac{2}{3}\right)y = 6(1) \end{cases} \rightarrow \begin{cases} 5x - 6y = 2 \\ 3x - 4y = 6 \end{cases}$$

The x-terms can now be eliminated if we use $3\text{R1} + (-5\text{R2})$.

$$\begin{array}{rl} 3\text{R1} & \begin{cases} 15x - 18y = 6 \\ -15x + 20y = -30 \end{cases} \\ + \\ -5\text{R2} & \\ \text{sum} & 0x + 2y = -24 \end{array} \quad \text{add}$$

$$y = -12 \quad \text{solve for } y$$

Substituting $y = -12$ in either of the original equations yields $x = -14$, and the solution is $(-14, -12)$. Verify by substituting in both equations.

☑ **D.** You've just learned how to solve linear systems by elimination

Now try Exercises 39 through 44 ►

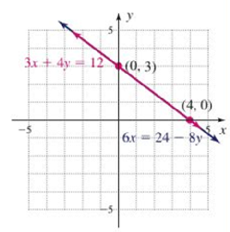
CAUTION ► Be sure to multiply *all* terms (on both sides) of the equation when using a constant multiplier. Also, note that for Example 5, we could have eliminated the y-terms using 2R1 with −3R2.

E. Inconsistent and Dependent Systems

A system having *at least one* solution is called a **consistent system.** As seen in Example 2, if the lines have different slopes, they intersect at a single point and the system has exactly one solution. Here, the lines are *independent* of each other and the system is called an **independent system.** If the lines have equal slopes *and* the same y-intercept, they are identical or **coincident lines.** Since one is right atop the other, they *intersect at all points,* and the system has an infinite number of solutions. Here, one line *depends* on the other and the system is called a **dependent system.** Using substitution or elimination on a dependent system results in the elimination of all variable terms and leaves a statement that is *always true,* such as $0 = 0$ or some other simple identity.

EXAMPLE 6 ► Solving a Dependent System

Solve using elimination: $\begin{cases} 3x + 4y = 12 \\ 6x = 24 - 8y \end{cases}$.



Solution ► Writing the system in standard form gives $\begin{cases} 3x + 4y = 12 \\ 6x + 8y = 24 \end{cases}$. By applying −2R1, we can eliminate the variable x:

$$\begin{array}{r} -2\text{R1} \\ + \\ \text{R2} \\ \text{sum} \end{array} \begin{cases} -6x - 8y = -24 \\ 6x + 8y = 24 \end{cases} \quad \text{add}$$
$$0x + 0y = 0 \quad \text{variables are eliminated}$$
$$0 = 0 \quad \text{true statement}$$

Although we didn't expect it, both variables were eliminated and the final statement is true $(0 = 0)$. This indicates the system is dependent, which the graph verifies (the lines are coincident). Writing both equations in slope-intercept form shows they represent the same line.

$$\begin{cases} 3x + 4y = 12 \\ 6x + 8y = 24 \end{cases} \longrightarrow \begin{cases} 4y = -3x + 12 \\ 8y = -6x + 24 \end{cases} \longrightarrow \begin{cases} y = -\dfrac{3}{4}x + 3 \\ y = -\dfrac{3}{4}x + 3 \end{cases}$$

The solutions of a dependent system are often written in set notation as the set of ordered pairs (x, y), where y is a specified function of x. For Example 6 the solution would be $\{(x, y) \mid y = -\frac{3}{4}x + 3\}$. Using an ordered pair with an arbitrary variable, called a **parameter,** is also common: $\left(p, \dfrac{-3p}{4} + 3\right)$.

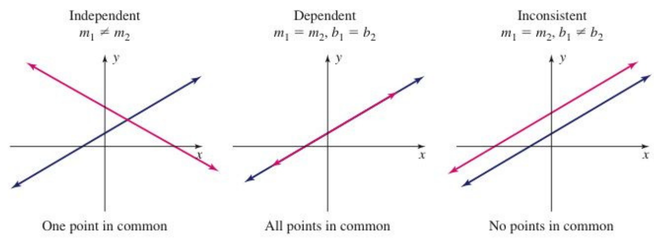
WORTHY OF NOTE

When writing the solution to a dependent system using a parameter, the solution can be written in many different ways. For instance, if we let $p = 4b$ for the first coordinate of the solution to Example 6, we have $\dfrac{-3(4b)}{4} + 3 = -3b + 3$ as the second coordinate, and the solution becomes $(4b, -3b + 3)$ for any constant b.

Now try Exercises 45 through 56 ►

Finally, if the lines have equal slopes and *different y-intercepts,* they are parallel and the system will have no solution. A system with no solutions is called an **inconsistent system.** An "inconsistent system" produces an "inconsistent answer," such as $12 = 0$ or some other false statement when substitution or elimination is applied. In other words, all variable terms are once again eliminated, but the remaining statement is *false.* A summary of the three possibilities is shown here for arbitrary slope m and y-intercept $(0, b)$.

☑ **E.** You've just learned how to recognize inconsistent and dependent systems



F. Systems and Modeling

In previous chapters, we solved numerous real-world applications by writing all given relationships in terms of a single variable. Many situations are easier to model using a system of equations with each relationship modeled independently using *two* variables. We begin here with a **mixture** application. Although they appear in many different forms (coin problems, metal alloys, investments, merchandising, and so on), mixture problems all have a similar theme. Generally one equation is related to *quantity* (how much of each item is being combined) and one equation is related to *value* (what is the value of each item being combined).

EXAMPLE 7 ▶ Solving a Mixture Application

A jeweler is commissioned to create a piece of artwork that will weigh 14 oz and consist of 75% gold. She has on hand two alloys that are 60% and 80% gold, respectively. How much of each should she use?

Solution ▶ Let x represent ounces of the 60% alloy and y represent ounces of the 80% alloy. The first equation must be $x + y = 14$, since the piece of art must weigh exactly 14 oz (this is the *quantity* equation). The x ounces are 60% gold, the y ounces are 80% gold, and the 14 oz will be 75% gold. This gives the *value* equation: $0.6x + 0.8y = 0.75(14)$. The system is $\begin{cases} x + y = 14 \\ 6x + 8y = 105 \end{cases}$ (after clearing decimals). Solving for y in the first equation gives $y = 14 - x$. Substituting $14 - x$ for y in the second equation gives

$$\begin{aligned} 6x + 8y &= 105 && \text{second equation} \\ 6x + 8(14 - x) &= 105 && \text{substitute } 14 - x \text{ for } y \\ -2x + 112 &= 105 && \text{simplify} \\ x &= \frac{7}{2} && \text{solve for } x \end{aligned}$$

Substituting $\frac{7}{2}$ for x in the first equation gives $y = \frac{21}{2}$. She should use 3.5 oz of the 60% alloy and 10.5 oz of the 80% alloy.

WORTHY OF NOTE

As an estimation tool, note that if equal amounts of the 60% and 80% alloys were used (7 oz each), the result would be a 70% alloy (halfway in between). Since a 75% alloy is needed, more of the 80% gold will be used.

Now try Exercises 63 through 70 ▶

Systems of equations also play a significant role in *cost-based pricing* in the business world. The costs involved in running a business can broadly be understood as either a **fixed cost k** or a **variable cost v.** Fixed costs might include the monthly rent paid for facilities, which remains the same regardless of how many items are produced and sold. Variable costs would include the cost of materials needed to produce the item, which depends on the number of items made. The total cost can then be modeled by

$C(x) = vx + k$ for x number of items. Once a **selling price p** has been determined, the revenue equation is simply $R(x) = px$ (price times number of items sold). We can now set up and solve a system of equations that will determine how many items must be sold to break even, performing what is called a **break-even analysis.**

EXAMPLE 8 ► Solving an Application of Systems: Break-Even Analysis

In home businesses that produce items to sell on Ebay®, fixed costs are easily determined by rent and utilities, and variable costs by the price of materials needed to produce the item. Karen's home business makes large, decorative candles for all occasions. The cost of materials is \$3.50 per candle, and her rent and utilities average \$900 per month. If her candles sell for \$9.50, how many candles must be sold each month to break even?

Solution ► Let x represent the number of candles sold. Her total cost is $C(x) = 3.5x + 900$ (variable cost plus fixed cost), and projected revenue is $R(x) = 9.5x$. This gives the system $\begin{cases} C(x) = 3.5x + 900 \\ R(x) = 9.5x \end{cases}$. To break even, Cost = Revenue which gives

$$9.5x = 3.5x + 900$$
$$6x = 900$$
$$x = 150$$

The analysis shows that Karen must sell 150 candles each month to break even.

Now try Exercises 71 through 74 ►

WORTHY OF NOTE

This break-even concept can also be applied in studies of supply and demand, as well as in the decision to buy a new car or appliance that will enable you to break even over time due to energy and efficiency savings.

Our final example involves an application of uniform motion (distance = rate · time), and explores concepts of great importance to the navigation of ships and airplanes. As a simple illustration, if you've ever walked at your normal rate r on the "moving walkways" at an airport, you likely noticed an increase in your total speed. This is because the resulting speed combines your walking rate r with the speed w of the walkway: *total speed* $= r + w$. If you walk in the *opposite direction* of the walkway, your total speed is much slower, as now *total speed* $= r - w$.

This same phenomenon is observed when an airplane is flying with or against the wind, or a ship is sailing with or against the current.

EXAMPLE 9 ► Solving an Application of Systems — Uniform Motion

An airplane flying due south from St. Louis, Missouri, to Baton Rouge, Louisiana, uses a strong, steady tailwind to complete the trip in only 2.5 hr. On the return trip, the same wind slows the flight and it takes 3 hr to get back. If the flight distance between these cities is 912 km, what is the cruising speed of the airplane (speed with no wind)? How fast is the wind blowing?

Solution ► Let r represent the rate of the plane and w the rate of the wind. Since $D = RT$, the flight to Baton Rouge can be modeled by $912 = (r + w)(2.5)$, and the return flight by $912 = (r - w)(3)$. This produces the system $\begin{cases} 912 = 2.5r + 2.5w \\ 912 = 3r - 3w \end{cases}$. Using $\frac{R1}{2.5}$ and $\frac{R2}{3}$ gives the equivalent system $\begin{cases} 364.8 = r + w \\ 304 = r - w \end{cases}$, which is easily solved using elimination with R1 + R2.

$$668.8 = 2r \quad \text{R1 + R2}$$
$$334.4 = r \quad \text{divide by 2}$$

The cruising speed of the plane (with no wind) is 334.4 kph. Using $r - w = 304$ shows the wind is blowing at 30.4 kph.

Now try Exercises 75 through 78 ►

F. You've just learned how to use a system of equations to model and solve applications

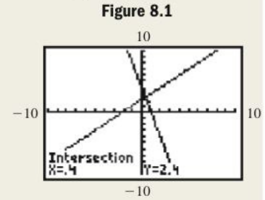
TECHNOLOGY HIGHLIGHT

Solving Systems Graphically

When used with care, graphing calculators offer an accurate way to solve linear systems and to check solution(s) obtained by hand. We'll illustrate using the system from Example 3: $\begin{cases} 4x + y = 4 \\ y = x + 2 \end{cases}$, where we found the solution was $\left(\frac{2}{5}, \frac{12}{5}\right)$.

1. Solve for *y* in both equations: $\begin{cases} y = -4x + 4 \\ y = \quad x + 2 \end{cases}$
2. Enter the equations as $Y_1 = -4x + 4$, $Y_2 = x + 2$
3. Graph using [ZOOM] 6: $Y_1 = -4x + 4$, $Y_2 = x + 2$
4. Press [2nd] [TRACE] **(CALC) 5** [ENTER] [ENTER] [ENTER] to have the calculator compute the point of intersection.

Figure 8.1



The coordinates of the intersection appear as decimal fractions at the bottom of the screen (Figure 8.1). In step 4, The first [ENTER] selects Y_1, the second [ENTER] selects Y_2 and the third [ENTER] bypasses the GUESS option (this option is most often used if the graphs intersect at more than one point). The calculator automatically registers the *x*-coordinate as its most recent entry, and from the home screen, converting it to a standard fraction (using [MATH] **1:** [▶] **Frac** [ENTER]) shows $x = \frac{2}{5}$. You can also get an *approximate solution* by tracing along either line towards the point of intersection using the [TRACE] key and the left or right arrows.

Solve each system graphically, using a graphing calculator.

Exercise 1: $\begin{cases} 3x - y = -7 \\ y + 5x = -1 \end{cases}$ **Exercise 2:** $\begin{cases} 2x - 3y = 3 \\ 6 = 8x - 3y \end{cases}$

8.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Systems that have no solution are called ______ systems.
2. Systems having at least one solution are called ______ systems.
3. If the lines in a system intersect at a single point, the system is said to be ______ and ______.
4. If the lines in a system are coincident, the system is referred to as ______ and ______.
5. The given systems are equivalent. How do we obtain the second system from the first?
$$\begin{cases} \frac{2}{3}x + \frac{1}{2}y = \frac{5}{3} \\ 0.2x + 0.4y = 1 \end{cases} \quad \begin{cases} 4x + 3y = 10 \\ 2x + 4y = 10 \end{cases}$$
6. For $\begin{cases} 2x + 5y = 8 \\ 3x + 4y = 5 \end{cases}$, which solution method would be more efficient, substitution or elimination? Discuss/Explain why.

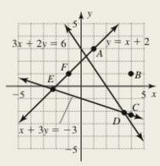
▶ DEVELOPING YOUR SKILLS

Show the lines in each system would intersect in a single point by writing the equations in slope-intercept form.

7. $\begin{cases} 7x - 4y = 24 \\ 4x + 3y = 15 \end{cases}$

8. $\begin{cases} 0.3x - 0.4y = 2 \\ 0.5x + 0.2y = -4 \end{cases}$

An ordered pair is a solution to an equation if it makes the equation true. Given the graph shown here, determine which equation(s) have the indicated point as a solution. If the point satisfies more than one equation, write the system for which it is a solution.



9. *A* **10.** *B*

11. *C* **12.** *D*

13. *E* **14.** *F*

Substitute the *x*- and *y*-values indicated by the ordered pair to determine if it solves the system.

15. $\begin{cases} 3x + y = 11 \\ -5x + y = -13; \end{cases} (3, 2)$

16. $\begin{cases} 3x + 7y = -4 \\ 7x + 8y = -21; \end{cases} (-6, 2)$

17. $\begin{cases} 8x - 24y = -17 \\ 12x + 30y = 2; \end{cases} \left(-\frac{7}{8}, \frac{5}{12}\right)$

18. $\begin{cases} 4x + 15y = 7 \\ 8x + 21y = 11; \end{cases} \left(\frac{1}{2}, \frac{1}{3}\right)$

Solve each system by *graphing*. If the coordinates do not appear to be integers, estimate the solution to the nearest tenth (indicate that your solution is an estimate).

19. $\begin{cases} 3x + 2y = 12 \\ x - y = 9 \end{cases}$ **20.** $\begin{cases} 5x + 2y = -2 \\ -3x + y = 10 \end{cases}$

21. $\begin{cases} 5x - 2y = 4 \\ x + 3y = -15 \end{cases}$ **22.** $\begin{cases} 3x + y = 2 \\ 5x + 3y = 12 \end{cases}$

Solve each system using *substitution*. Write solutions as an ordered pair.

23. $\begin{cases} x = 5y - 9 \\ x - 2y = -6 \end{cases}$ **24.** $\begin{cases} 4x - 5y = 7 \\ 2x - 5 = y \end{cases}$

25. $\begin{cases} y = \frac{2}{3}x - 7 \\ 3x - 2y = 19 \end{cases}$ **26.** $\begin{cases} 2x - y = 6 \\ y = \frac{3}{4}x - 1 \end{cases}$

Identify the equation and variable that makes the substitution method easiest to use. Then solve the system.

27. $\begin{cases} 3x - 4y = 24 \\ 5x + y = 17 \end{cases}$ **28.** $\begin{cases} 3x + 2y = 19 \\ x - 4y = -3 \end{cases}$

29. $\begin{cases} 0.7x + 2y = 5 \\ x - 1.4y = 11.4 \end{cases}$ **30.** $\begin{cases} 0.8x + y = 7.4 \\ 0.6x + 1.5y = 9.3 \end{cases}$

31. $\begin{cases} 5x - 6y = 2 \\ x + 2y = 6 \end{cases}$ **32.** $\begin{cases} 2x + 5y = 5 \\ 8x - y = 6 \end{cases}$

Solve using *elimination*. In some cases, the system must first be written in standard form.

33. $\begin{cases} 2x - 4y = 10 \\ 3x + 4y = 5 \end{cases}$ **34.** $\begin{cases} -x + 5y = 8 \\ x + 2y = 6 \end{cases}$

35. $\begin{cases} 4x - 3y = 1 \\ 3y = -5x - 19 \end{cases}$ **36.** $\begin{cases} 5y - 3x = -5 \\ 3x + 2y = 19 \end{cases}$

37. $\begin{cases} 2x = -3y + 17 \\ 4x - 5y = 12 \end{cases}$ **38.** $\begin{cases} 2y = 5x + 2 \\ -4x = 17 - 6y \end{cases}$

39. $\begin{cases} 0.5x + 0.4y = 0.2 \\ 0.3y = 1.3 + 0.2x \end{cases}$ **40.** $\begin{cases} 0.2x + 0.3y = 0.8 \\ 0.3x + 0.4y = 1.3 \end{cases}$

41. $\begin{cases} 0.32m - 0.12n = -1.44 \\ -0.24m + 0.08n = 1.04 \end{cases}$

42. $\begin{cases} 0.06g - 0.35h = -0.67 \\ -0.12g + 0.25h = 0.44 \end{cases}$

43. $\begin{cases} -\frac{1}{6}u + \frac{1}{4}v = 4 \\ \frac{1}{2}u - \frac{2}{3}v = -11 \end{cases}$ **44.** $\begin{cases} \frac{3}{4}x + \frac{1}{3}y = -2 \\ \frac{3}{2}x + \frac{1}{5}y = 3 \end{cases}$

Solve using any method and identify the system as consistent, inconsistent, or dependent.

45. $\begin{cases} 4x + \frac{3}{4}y = 14 \\ -9x + \frac{5}{8}y = -13 \end{cases}$ **46.** $\begin{cases} \frac{2}{3}x + y = 2 \\ 2y = \frac{5}{6}x - 9 \end{cases}$

47. $\begin{cases} 0.2y = 0.3x + 4 \\ 0.6x - 0.4y = -1 \end{cases}$ **48.** $\begin{cases} 1.2x + 0.4y = 5 \\ 0.5y = -1.5x + 2 \end{cases}$

49. $\begin{cases} 6x - 22 = -y \\ 3x + \frac{1}{2}y = 11 \end{cases}$ **50.** $\begin{cases} 15 - 5y = -9x \\ -3x + \frac{5}{3}y = 5 \end{cases}$

51. $\begin{cases} -10x + 35y = -5 \\ y = 0.25x \end{cases}$ **52.** $\begin{cases} 2x + 3y = 4 \\ x = -2.5y \end{cases}$

53. $\begin{cases} 7a + b = -25 \\ 2a - 5b = 14 \end{cases}$ **54.** $\begin{cases} -2m + 3n = -1 \\ 5m - 6n = 4 \end{cases}$

55. $\begin{cases} 4a = 2 - 3b \\ 6b + 2a = 7 \end{cases}$

56. $\begin{cases} 3p - 2q = 4 \\ 9p + 4q = -3 \end{cases}$

57. $\begin{cases} 2x + 4y = 6 \\ x + 12 = 4y \end{cases}$

58. $\begin{cases} 8x = 3y + 24 \\ 8x - 5y = 36 \end{cases}$

The substitution method can be used for like variables *or for like expressions.* Solve the following systems, *using the expression* common to both equations (do not solve for *x* or *y* alone).

59. $\begin{cases} 5x - 11y = 21 \\ 11y = 5 - 8x \end{cases}$

60. $\begin{cases} -6x = 5y - 16 \\ 5y - 6x = 4 \end{cases}$

▶ WORKING WITH FORMULAS

61. Uniform motion with current: $\begin{cases} (R + C)T_1 = D_1 \\ (R - C)T_2 = D_2 \end{cases}$

The formula shown can be used to solve uniform motion problems involving a *current,* where *D* represents distance traveled, *R* is the rate of the object with no current, *C* is the speed of the current, and *T* is the time. Chan-Li rows 9 mi up river (against the current) in 3 hr. It only took him 1 hr to row 5 mi downstream (with the current). How fast was the current? How fast can he row in still water?

62. Fahrenheit and Celsius temperatures:
$\begin{cases} y = \frac{9}{5}x + 32 & \text{°F} \\ y = \frac{5}{9}(x - 32) & \text{°C} \end{cases}$

Many people are familiar with temperature measurement in degrees Celsius and degrees Fahrenheit, but few realize that the equations are linear and there is one temperature at which the two scales agree. Solve the system using the method of your choice and find this temperature.

▶ APPLICATIONS

Solve each application by modeling the situation with a linear system. Be sure to clearly indicate what each variable represents.

Mixture

63. Theater productions: At a recent production of *A Comedy of Errors,* the Community Theater brought in a total of \$30,495 in revenue. If adult tickets were \$9 and children's tickets were \$6.50, how many tickets of each type were sold if 3800 tickets in all were sold?

64. Milk-fat requirements: A dietician needs to mix 10 gal of milk that is $2\frac{1}{2}\%$ milk fat for the day's rounds. He has some milk that is 4% milk fat and some that is $1\frac{1}{2}\%$ milk fat. How much of each should be used?

65. Filling the family cars: Cherokee just filled both of the family vehicles at a service station. The total cost for 20 gal of regular unleaded and 17 gal of premium unleaded was \$144.89. The premium gas was \$0.10 more per gallon than the regular gas. Find the price per gallon for each type of gasoline.

66. Household cleaners: As a cleaning agent, a solution that is 24% vinegar is often used. How much pure (100%) vinegar and 5% vinegar must be mixed to obtain 50 oz of a 24% solution?

67. Alumni contributions: A wealthy alumnus donated \$10,000 to his alma mater. The college used the funds to make a loan to a science major at 7% interest and a loan to a nursing student at 6% interest. That year the college earned \$635 in interest. How much was loaned to each student?

68. Investing in bonds: A total of \$12,000 is invested in two municipal bonds, one paying 10.5% and the other 12% simple interest. Last year the annual interest earned on the two investments was \$1335. How much was invested at each rate?

69. Saving money: Bryan has been doing odd jobs around the house, trying to earn enough money to buy a new Dirt-Surfer©. He saves all quarters and dimes in his piggy bank, while he places all nickels and pennies in a drawer to spend. So far, he has 225 coins in the piggy bank, worth a total of \$45.00. How many of the coins are quarters? How many are dimes?

70. Coin investments: In 1990, Molly attended a coin auction and purchased some rare "Flowing Hair" fifty-cent pieces, and a number of very rare two-cent pieces from the Civil War Era. If she bought 47 coins with a face value of \$10.06, how many of each denomination did she buy?

71. Lawn service: Dave and his sons run a lawn service, which includes mowing, edging, trimming, and aerating a lawn. His fixed cost includes insurance, his salary, and monthly payments on equipment, and amounts to \$4000/mo. The variable costs include gas, oil, hourly wages for his employees, and miscellaneous expenses, which run about \$75 per lawn. The average charge for full-service lawn care is \$115 per visit. Do a break-even analysis to (a) determine how many lawns Dave must service each month to break even and (b) the revenue required to break even.

72. Production of mini-microwave ovens: Due to high market demand, a manufacturer decides to introduce a new line of mini-microwave ovens for personal and office use. By using existing factory space and retraining some employees, fixed costs are estimated at \$8400/mo. The components to assemble and test each microwave are expected to run \$45 per unit. If market research shows consumers are willing to pay at least \$69 for this product, find (a) how many units must be made and sold each month to break even and (b) the revenue required to break even.

In a market economy, the availability of goods is closely related to the market price. Suppliers are willing to produce more of the item at a higher price (the supply), with consumers willing to buy more of the item at a lower price (the demand). This is called the law of supply and demand. When supply and demand are equal, both the buyer and seller are satisfied with the current price and we have *market equilibrium*.

73. Farm commodities: One area where the law of supply and demand is clearly at work is farm commodities. Both growers and consumers watch this relationship closely, and use data collected by government agencies to track the relationship and make adjustments, as when a farmer decides to convert a large portion of her farmland from corn to soybeans to improve profits. Suppose that for x billion bushels of soybeans, supply is modeled by $y = 1.5x + 3$, where y is the current market price (in dollars per bushel). The related demand equation might be $y = -2.20x + 12$. (a) How many billion bushels will be supplied at a market price of \$5.40? What will the demand be at this price? Is supply less than demand? (b) How many billion bushels will be supplied at a market price of \$7.05? What will the demand be at this price? Is demand less than supply? (c) To the nearest cent, at what price does the market reach equilibrium? How many bushels are being supplied/demanded?

74. Digital music: Market research has indicated that by 2010, sales of MP3 portables will mushroom into a \$70 billion dollar market. With a market this large, competition is often fierce—with suppliers fighting to earn and hold market shares. For x million MP3 players sold, supply is modeled by $y = 10.5x + 25$, where y is the current market price (in dollars). The related demand equation might be $y = -5.20x + 140$. (a) How many million MP3 players will be supplied at a market price of \$88? What will the demand be at this price? Is supply less than demand? (b) How many million MP3 players will be supplied at a market price of \$114? What will the demand be at this price? Is demand less than supply? (c) To the nearest cent, at what price does the market reach equilibrium? How many units are being supplied/demanded?

Uniform Motion

75. Canoeing on a stream: On a recent camping trip, it took Molly and Sharon 2 hr to row 4 mi upstream from the drop in point to the campsite. After a leisurely weekend of camping, fishing, and relaxation, they rowed back downstream to the drop in point in just 30 min. Use this information to find (a) the speed of the current and (b) the speed Sharon and Molly would be rowing in still water.

76. Taking a luxury cruise: A luxury ship is taking a Caribbean cruise from Caracas, Venezuela, to just off the coast of Belize City on the Yucatan Peninsula, a distance of 1435 mi. En route they encounter the Caribbean Current, which flows to the northwest, parallel to the coastline. From Caracas to the Belize coast, the trip took 70 hr. After a few days of fun in the sun, the ship leaves for Caracas, with the return trip taking 82 hr. Use this information to find (a) the speed of the Caribbean Current and (b) the cruising speed of the ship.

77. Airport walkways: As part of an algebra field trip, Jason takes his class to the airport to use their moving walkways for a demonstration. The class measures the longest walkway, which turns out to be 256 ft long. Using a stop watch, Jason shows it takes him just 32 sec to complete the walk going in the same direction as the walkway. Walking in a direction opposite the walkway, it takes him 320 sec—10 times as long! The next day in class, Jason hands out a two-question quiz: (1) What was the speed of the walkway in feet per second? (2) What is my (Jason's) normal walking speed? Create the answer key for this quiz.

78. Racing pigeons: The American Racing Pigeon Union often sponsors opportunities for owners to fly their birds in friendly competitions. During a recent competition, Steve's birds were liberated in Topeka, Kansas, and headed almost due north to their loft in Sioux Falls, South Dakota, a distance of 308 mi. During the flight, they encountered a steady wind from the north and the trip took 4.4 hr. The next month, Steve took his birds to a competition in Grand Forks, North Dakota, with the birds heading almost due south to home, also a distance of 308 mi. This time the birds were aided by the same wind from the north, and the trip took only 3.5 hr. Use this information to (a) find the racing speed of Steve's birds and (b) find the speed of the wind.

Descriptive Translation

79. Important dates in U.S. history: If you sum the year that the Declaration of Independence was signed and the year that the Civil War ended, you get 3641. There are 89 yr that separate the two events. What year was the Declaration signed? What year did the Civil War end?

80. Architectual wonders: When it was first constructed in 1889, the Eiffel Tower in Paris, France, was the tallest structure in the world. In 1975, the CN Tower in Toronto, Canada, became the world's tallest structure. The CN Tower is 153 ft less than twice the height of the Eiffel Tower, and the sum of their heights is 2799 ft. How tall is each tower?

81. Pacific islands land area: In the South Pacific, the island nations of Tahiti and Tonga have a combined land area of 692 mi^2. Tahiti's land area is 112 mi^2 more than Tonga's. What is the land area of each island group?

82. Card games: On a cold winter night, in the lobby of a beautiful hotel in Sante Fe, New Mexico, Marc and Klay just barely beat John and Steve in a close game of Trumps. If the sum of the team scores was 990 points, and there was a 12-point margin of victory, what was the final score?

▶ EXTENDING THE CONCEPT

83. Answer using observations only—no calculations. Is the given system consistent/independent, consistent/dependent, or inconsistent? Explain/Discuss your answer. $\begin{cases} y = 5x + 2 \\ y = 5.01x + 1.9 \end{cases}$

84. Federal income tax reform has been a hot political topic for many years. Suppose tax plan A calls for a flat tax of 20% tax on all income (no deductions or loopholes). Tax plan B requires taxpayers to pay $5000 plus 10% of all income. For what income level do both plans require the same tax?

85. Suppose a certain amount of money was invested at 6% per year, and another amount at 8.5% per year, with a total return of $1250. If the amounts invested at each rate were switched, the yearly income would have been $1375. To the nearest whole dollar, how much was invested at each rate?

▶ MAINTAINING YOUR SKILLS

86. (2.6) Given the parent function $f(x) = |x|$, sketch the graph of $F(x) = -|x + 3| - 2$.

87. (5.1) Find two positive and two negative angles that are coterminal with $\theta = 112°$.

88. (4.4) Solve for x (rounded to the nearest thousandth): $33 = 77.5e^{-0.0052x} - 8.37$

89. (6.2) Verify that $\dfrac{\sin x - \csc x}{\csc x} = -\cos^2 x$ is an identity.

8.2 Linear Systems in Three Variables with Applications

Learning Objectives

In Section 8.2 you will learn how to:

- ☐ **A.** Visualize a solution in three dimensions
- ☐ **B.** Check ordered triple solutions
- ☐ **C.** Solve linear systems in three variables
- ☐ **D.** Recognize inconsistent and dependent systems
- ☐ **E.** Use a system of three equations in three variables to solve applications

The transition to systems of three equations in three variables requires a fair amount of "visual gymnastics" along with good organizational skills. Although the techniques used are identical and similar results are obtained, the third equation and variable give us more to track, and we must work more carefully toward the solution.

A. Visualizing Solutions in Three Dimensions

The solution to an equation in one variable is the single number that satisfies the equation. For $x + 1 = 3$, the solution is $x = 2$ and its graph is a single *point* on the number line, a **one-dimensional graph.** The solution to an equation in two variables, such as $x + y = 3$, is an ordered pair (x, y) that satisfies the equation. When we graph this solution set, the result is a *line* on the *xy*-coordinate grid, a **two-dimensional graph.** The solutions to an equation in three variables, such as $x + y + z = 6$, are the **ordered triples** (x, y, z) that satisfy the equation. When we graph this solution set, the result is a **plane** in **space,** a *graph in three dimensions.* Recall a plane is a flat surface having infinite length and width, but no depth. We can graph this plane using the intercept method and the result is shown in Figure 8.2. For graphs in three dimensions, the *xy*-plane is parallel to the ground (the *y*-axis points to the right) and *z* is the **vertical axis.** To find an additional point on this plane, we use any three numbers whose sum is 6, such as (2, 3, 1). Move 2 units along the *x*-axis, 3 units parallel to the *y*-axis, and 1 unit parallel to the *z*-axis, as shown in Figure 8.3.

WORTHY OF NOTE

We can visualize the location of a point in space by considering a large rectangular box 2 ft long × 3 ft wide × 1 ft tall, placed snugly in the corner of a room. The floor is the *xy*-plane, one wall is the *xz*-plane, and the other wall is the *yz*-plane. The *z*-axis is formed where the two walls meet and the corner of the room is the origin (0, 0, 0). To find the corner of the box located at (2, 3, 1), first locate the point (2, 3) in the *xy*-plane (the floor), then move up 1 ft.

Figure 8.2

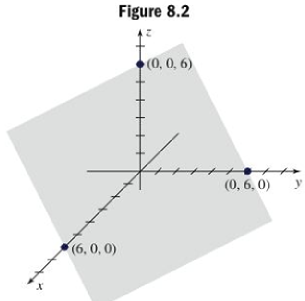
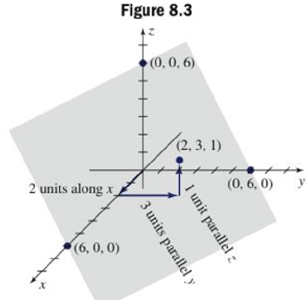


Figure 8.3



EXAMPLE 1 ► **Finding Solutions to an Equation in Three Variables**

Use a guess-and-check method to find four additional points on the plane determined by $x + y + z = 6$.

Solution ► We can begin by letting $x = 0$, then use any combination of y and z that sum to 6. Two examples are (0, 2, 4) and (0, 5, 1). We could also select any two values for x and y, then determine a value for z that results in a sum of 6. Two examples are $(-2, 9, -1)$ and $(8, -3, 1)$.

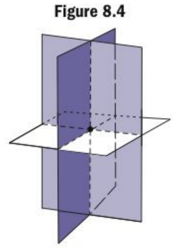
☑ **A.** You've just learned how to visualize a solution in three dimensions

Now try Exercises 7 through 10 ►

B. Solutions to a System of Three Equations in Three Variables

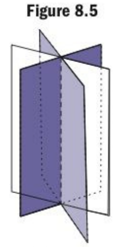
When solving a system of three equations in three variables, remember each equation represents a plane in space. These planes can intersect in various ways, creating different possibilities for a solution set (see Figures 8.4 to 8.7). The system could have a **unique solution** (a, b, c), if the planes intersect at a single point (Figure 8.4) (the point satisfies all three equations simultaneously). If the planes intersect in a line (Figure 8.5), the system is **linearly dependent** and there are an infinite number of solutions. Unlike the two-dimensional case, the equation of a line in three dimensions is somewhat complex, and the coordinates of all points on this line are usually *represented* by a specialized ordered triple, which we use to state the solution set. If the planes intersect at all points, the system has **coincident dependence** (see Figure 8.6). This indicates the equations of the system differ by only a constant multiple—they are all "disguised forms" of the *same equation*. The solution set is any ordered triple (a, b, c) satisfying this equation. Finally, the system may have no solutions. This can happen a number of different ways, most notably if the planes intersect as shown in Figure 8.7 (other possibilities are discussed in the exercises). In the case of "no solutions," an ordered triple may satisfy none of the equations, only one of the equations, only two of the equations, but not all three equations.

Figure 8.4



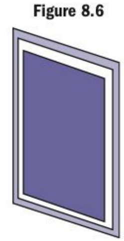
Unique solution

Figure 8.5



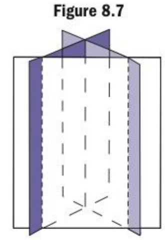
Linear dependence

Figure 8.6



Coincident dependence

Figure 8.7



No solutions

EXAMPLE 2 ▶ Determining If an Ordered Triple Is a Solution

Determine if the ordered triple $(1, -2, 3)$ is a solution to the systems shown.

a. $\begin{cases} x + 4y - z = -10 \\ 2x + 5y + 8z = 4 \\ x - 2y - 3z = -4 \end{cases}$ **b.** $\begin{cases} 3x + 2y - z = -4 \\ 2x - 3y - 2z = 2 \\ x - y + 2z = 9 \end{cases}$

Solution ▶ Substitute 1 for x, -2 for y, and 3 for z in the first system.

a. $\begin{cases} x + 4y - z = -10 \\ 2x + 5y + 8z = 4 \\ x - 2y - 3z = -4 \end{cases} \rightarrow \begin{cases} (1) + 4(-2) - (3) = -10 \\ 2(1) + 5(-2) + 8(3) = 4 \\ (1) - 2(-2) - 3(3) = -4 \end{cases} \rightarrow \begin{cases} -10 = -10 \text{ true} \\ 16 = 4 \text{ false} \\ -4 = -4 \text{ true} \end{cases}$

No, the ordered triple $(1, -2, 3)$ is not a solution to the first system. Now use the same substitutions in the second system.

b. $\begin{cases} 3x + 2y - z = -4 \\ 2x - 3y - 2z = 2 \\ x - y + 2z = 9 \end{cases} \rightarrow \begin{cases} 3(1) + 2(-2) - (3) = -4 \\ 2(1) - 3(-2) - 2(3) = 2 \\ (1) - (-2) + 2(3) = 9 \end{cases} \rightarrow \begin{cases} -4 = -4 \text{ true} \\ 2 = 2 \text{ true} \\ 9 = 9 \text{ true} \end{cases}$

The ordered triple $(1, -2, 3)$ is a solution to the second system only.

B. You've just learned how to check ordered triple solutions

Now try Exercises 11 and 12 ▶

C. Solving Systems of Three Equations in Three Variables Using Elimination

From Section 8.1, we know that two systems of equations are **equivalent** if they have the same solution set. The systems

$$\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ -2y - z = -3 \end{cases} \text{ and } \begin{cases} 2x + y - 2z = -7 \\ y + 4z = 5 \\ z = 1 \end{cases}$$

are equivalent, as both have the unique solution $(-3, 1, 1)$. In addition, it is evident that the second system can be solved more easily, since R2 and R3 have fewer variables than the first system. In the simpler system, mentally substituting 1 for z into R2 immediately gives $y = 1$, and these values can be back-substituted into the first equation to find that $x = -3$. This observation guides us to a general approach for solving larger systems—we would like to *eliminate variables in the second and third equations, until we obtain an equivalent system that can easily be solved by back-substitution.* To begin, let's review the three operations that "transform" a given system, and produce an equivalent system.

Operations That Produce an Equivalent System

1. Changing the order of the equations.
2. Replacing an equation by a nonzero constant multiple of that equation.
3. Replacing an equation with the sum of two equations from the system.

Building on the ideas from Section 8.1, we develop the following approach for solving a system of three equations in three variables.

Solving a System of Three Equations in Three Variables

1. Write each equation in standard form: $Ax + By + Cz = D$.
2. If the "x" term in any equation has a coefficient of 1, interchange equations (if necessary) so this equation becomes R1.
3. Use the x-term in R1 to eliminate the x-terms from R2 and R3. The original R1, with the new R2 and R3, form an equivalent system that contains a smaller "subsystem" of two equations in two variables.
4. Solve the subsystem and keep the result as the new R3. The result is an equivalent system that can be solved using back-substitution.

We'll begin by solving the system $\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ -2y - z = -3 \end{cases}$ using the elimination method and the procedure outlined. In Example 3, the notation $-2\text{R1} + \text{R2} \rightarrow \text{R2}$ indicates the equation in row 1 has been multiplied by -2 and added to the equation in row 2, with the result placed in the system as the new row 2.

EXAMPLE 3 ► Solving a System of Three Equations in Three Variables

Solve using elimination: $\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1. \\ -2y - z = -3 \end{cases}$

Solution ►

1. The system is in standard form.
2. If the x-term in any equation has a coefficient of 1, interchange equations so this equation becomes R1.

$$\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ -2y - z = -3 \end{cases} \xrightarrow{R2 \leftrightarrow R1} \begin{cases} x + y + z = -1 \\ 2x + y - 2z = -7 \\ -2y - z = -3 \end{cases}$$

3. Use R1 to eliminate the x-term in R2 and R3. Since R3 has no x-term, the only elimination needed is the x-term from R2. Using $-2R1 + R2$ will eliminate this term:

$$\begin{array}{lrl} -2R1 & -2x - 2y - 2z = & 2 \\ + & & \\ \underline{R2} & \underline{2x + y - 2z} = & \underline{-7} \\ & 0x - 1y - 4z = & -5 \quad \text{sum} \\ & y + 4z = & 5 \quad \text{simplify} \end{array}$$

The new R2 is $y + 4z = 5$. The original R1 and R3, along with the new R2 form an equivalent system that contains a smaller **subsystem**

$$\begin{cases} x + y + z = -1 \\ 2x + y - 2z = -7 \\ -2y - z = -3 \end{cases} \xrightarrow[R3 \rightarrow R3]{-2R1 + R2 \rightarrow R2} \begin{cases} x + y + z = -1 \\ y + 4z = 5 \\ -2y - z = -3 \end{cases} \begin{array}{l} \text{new} \\ \text{equivalent} \\ \text{system} \end{array}$$

4. Solve the subsystem for either y or z, and keep the result as a *new* R3. We choose to eliminate y using $2R2 + R3$:

$$\begin{array}{lrl} 2R2 & 2y + 8z = & 10 \\ + & & \\ \underline{R3} & \underline{-2y - z} = & \underline{-3} \\ & 0y + 7z = & 7 \quad \text{sum} \\ & z = & 1 \quad \text{simplify} \end{array}$$

The new R3 is $z = 1$.

$$\begin{cases} x + y + z = -1 \\ y + 4z = 5 \\ -2y - z = -3 \end{cases} \xrightarrow{2R2 + R3 \rightarrow R3} \begin{cases} x + y + z = -1 \\ y + 4z = 5 \\ z = 1 \end{cases} \begin{array}{l} \text{new} \\ \text{equivalent} \\ \text{system} \end{array}$$

The new R3, along with the original R1 and R2 from step 3, form an equivalent system that can be solved using back-substitution. Substituting 1 for z in R2 yields $y = 1$. Substituting 1 for z and 1 for y in R1 yields $x = -3$. The solution is $(-3, 1, 1)$.

Now try Exercises 13 through 18 ►

While not absolutely needed for the elimination process, there are two reasons for wanting the coefficient of x to be "1" in R1. First, it makes the elimination method more efficient since we can more easily see what to use as a multiplier. Second, it lays the foundation for developing other methods of solving larger systems. If no equation has an x-coefficient of 1, we simply use the y- or z-variable instead (see Example 7). Since solutions to larger systems generally are worked out in stages, we will sometimes track the transformations used by writing them *between* the original system and the equivalent system, rather than to the left as we did in Section 8.1.

Here is an additional example illustrating the elimination process, but in *abbreviated form*. Verify the calculations indicated using a separate sheet.

EXAMPLE 4 ► **Solving a System of Three Equations in Three Variables**

Solve using elimination: $\begin{cases} -5y + 2x - z = -8 \\ -x + 3z + 2y = 13 \\ -z + 3y + x = 5 \end{cases}$.

Solution ►

1. Write the equations in standard form: $\begin{cases} 2x - 5y - z = -8 \\ -x + 2y + 3z = 13 \\ x + 3y - z = 5 \end{cases}$

2. $\begin{cases} 2x - 5y - z = -8 \\ -x + 2y + 3z = 13 \\ x + 3y - z = 5 \end{cases} \xrightarrow{R3 \leftrightarrow R1} \begin{cases} x + 3y - z = 5 \\ -x + 2y + 3z = 13 \\ 2x - 5y - z = -8 \end{cases}$ equivalent system

3. Using R1 + R2 will eliminate the x-term from R2, yielding $5y + 2z = 18$. Using -2R1 + R3 eliminates the x-term from R3, yielding $-11y + z = -18$.

$\begin{cases} x + 3y - z = 5 \\ -x + 2y + 3z = 13 \\ 2x - 5y - z = -8 \end{cases} \xrightarrow[-2R1 + R3 \to R3]{R1 + R2 \to R2} \begin{cases} x + 3y - z = 5 \\ 5y + 2z = 18 \\ -11y + z = -18 \end{cases}$ equivalent system

4. Using -2R3 + R2 will eliminate z from the subsystem, leaving $27y = 54$.

$\begin{cases} x + 3y - z = 5 \\ 5y + 2z = 18 \\ -11y + z = -18 \end{cases} \xrightarrow{-2R3 + R2 \to R3} \begin{cases} x + 3y - z = 5 \\ 5y + 2z = 18 \\ 27y = 54 \end{cases}$ equivalent system

Solving for y in R3 shows $y = 2$. Substituting 2 for y in R2 yields $z = 4$, and substituting 2 for y and 4 for z in R1 shows $x = 3$. The solution is (3, 2, 4).

C. You've learned just how to solve linear systems in three variables

Now try Exercises 19 through 24 ►

D. Inconsistent and Dependent Systems

As mentioned, it is possible for larger systems to have no solutions or an infinite number of solutions. As with our work in Section 8.1, an inconsistent system (no solutions) will produce inconsistent results, ending with a statement such as $0 = -3$ or some other **contradiction.**

EXAMPLE 5 ► **Attempting to Solve an Inconsistent System**

Solve using elimination: $\begin{cases} 2x + y - 3z = -3 \\ 3x - 2y + 4z = 2 \\ 4x + 2y - 6z = -7 \end{cases}$.

Solution ►

1. This system has no equation where the coefficient of x is 1.

2. We can still use R1 to begin the solution process, but this time we'll use the variable y since it *does* have coefficient 1.

Using 2R1 + R2 eliminates the y-term from R2, leaving $7x - 2z = -4$. But using -2R1 + R3 to eliminate the y-term from R3 results in a contradiction:

$$\begin{array}{rl} 2\text{R1} & 4x + 2y - 6z = -6 \\ + & \\ \underline{\text{R2}} & \underline{3x - 2y + 4z = 2} \\ & 7x \qquad\quad - 2z = -4 \end{array} \qquad \begin{array}{rl} -2\text{R1} & -4x - 2y + 6z = 6 \\ + & \\ \underline{\text{R3}} & \underline{4x + 2y - 6z = -7} \\ & 0x + 0y + 0z = -1 \\ & \qquad\qquad\; 0 = -1 \end{array}$$

contradiction

We conclude the system is inconsistent. The answer is the empty set $\varnothing$, and we need work no further.

Now try Exercises 25 and 26 ►

Unlike our work with systems having only two variables, systems in three variables can have two forms of dependence—*linear dependence* or *coincident dependence*. To help understand linear dependence, consider a system of two equations in three variables: $\begin{cases} -2x + 3y - z = 5 \\ x - 3y + 2z = -1 \end{cases}$. Each of these equations represents a plane, and unless the planes are parallel, their intersection will be a line (see Figure 8.5). As in Section 8.1, we can state solutions to a dependent system using set notation with two of the variables written in terms of the third, or as an ordered triple using a parameter. The relationships named can then be used to generate specific solutions to the system.

Systems with two equations and two variables or three equations and three variables are called **square systems,** meaning there are exactly as many equations as there are variables. A system of linear equations cannot have a unique solution unless there are at least as many equations as there are variables in the system.

EXAMPLE 6 ► **Solving a Dependent System**

Solve using elimination: $\begin{cases} -2x + 3y - z = 5 \\ x - 3y + 2z = -1 \end{cases}$.

Solution ► Using R1 + R2 eliminates the y-term from R2, yielding $-x + z = 4$. This means (x, y, z) will satisfy both equations only when $x = z - 4$ (the x-coordinate must be 4 less than the z-coordinate). Since x is written in terms of z, we substitute $z - 4$ for x *in either equation* to find how y is related to z. Using R2 we have: $(z - 4) - 3y + 2z = -1$, which yields $y = z - 1$ (verify). This means the y-coordinate of the solution must be 1 less than z. In set notation the solution is $\{(x, y, z,) \mid x = z - 4, y = z - 1, z \in \mathbb{R}\}$. For $z = -2$, 0, and 3, the solutions would be $(-6, -3, -2)$, $(-4, -1, 0)$, and $(-1, 2, 3)$, respectively. Verify that these satisfy both equations. Using p as our parameter, the solution could be written $(p - 4, p - 1, p)$ in parameterized form.

Now try Exercises 27 through 30 ►

The system in Example 6 was nonsquare, and we knew ahead of time the system would be dependent. The system in Example 7 *is* square, but only by applying the elimination process can we determine the nature of its solution(s).

EXAMPLE 7 ► **Solving a Dependent System**

Solve using elimination: $\begin{cases} 3x - 2y + z = -1 \\ 2x + y - z = 5 \\ 10x - 2y = 8 \end{cases}$.

Solution ► This system has no equation where the coefficient of x is 1. We will still use R1, but we'll try to eliminate z in R2 (there is no z-term in R3).

Using R1 + R2 eliminates the z-term from R2, yielding $5x - y = 4$.

$$\begin{cases} 3x - 2y + z = -1 \\ 2x + y - z = 5 \\ 10x - 2y = 8 \end{cases} \xrightarrow[\text{R3} \to \text{R3}]{\text{R1} + \text{R2} \to \text{R2}} \begin{cases} 3x - 2y + z = -1 \\ 5x - y = 4 \\ 10x - 2y = 8 \end{cases}$$

We next solve the subsystem. Using $-2\text{R2} + \text{R3}$ eliminates the y-term in R3, but also all other terms:

$$\begin{array}{rrl} -2\text{R2} & -10x + 2y = -8 & \\ + & & \\ \underline{\text{R3}} & \underline{10x - 2y = 8} & \\ & 0x + 0y = 0 & \text{sum} \\ & 0 = 0 & \text{result} \end{array}$$

Since R3 is the same as 2R2, the system is linearly dependent and equivalent to $\begin{cases} 3x - 2y + z = -1 \\ 5x - y \quad = 4 \end{cases}$. We can solve for y in R2 to write y in terms of x: $y = 5x - 4$. Substituting $5x - 4$ for y in R1 enables us to also write z in terms of x:

$$\begin{aligned} 3x - 2y \quad + z &= -1 && \text{R1} \\ 3x - 2(5x - 4) + z &= -1 && \text{substitute } 5x - 4 \text{ for } y \\ 3x - 10x + 8 + z &= -1 && \text{distribute} \\ -7x + z &= -9 && \text{simplify} \\ z &= 7x - 9 && \text{solve for } z \end{aligned}$$

The solution set is $\{(x, y, z) | x \in \mathbb{R}, y = 5x - 4, z = 7x - 9)\}$. Three of the infinite number of solutions are $(0, -4, -9)$ for $x = 0$, $(2, 6, 5)$ for $x = 2$, and $(-1, -9, -16)$ for $x = -1$. Verify these triples satisfy all three equations. Again using the parameter p, the solution could be written as $(p, 5p - 4, 7p - 9)$ in parameterized form.

D. You've just learned how to recognize inconsistent and dependent systems

Now try Exercises 31 through 34 ►

Solutions to linearly dependent systems can actually be written in terms of either x, y, or z, depending on which variable is eliminated in the first step and the variable we elect to solve for afterward.

For **coincident dependence** the equations in a system differ by only a constant multiple. After applying the elimination process—all variables are eliminated from the other equations, leaving statements that are always true (such as $2 = 2$ or some other). See **Exercises 35 and 36.** For additional practice solving various kinds of systems, see **Exercises 37 to 51.**

E. Applications

Applications of larger systems are simply an extension of our work with systems of two equations in two variables. Once again, the applications come in a variety of forms and from many fields. In the world of business and finance, systems can be used to diversify investments or spread out liabilities, a financial strategy hinted at in Example 8.

EXAMPLE 8 ► **Modeling the Finances of a Business**

A small business borrowed \$225,000 from three different lenders to expand their product line. The interest rates were 5%, 6%, and 7%. Find how much was borrowed at each rate if the annual interest came to \$13,000 and twice as much was borrowed at the 5% rate than was borrowed at the 7% rate.

Solution ► Let x, y, and z represent the amount borrowed at 5%, 6%, and 7%, respectively. This means our first equation is $x + y + z = 225$ (in thousands). The second equation is determined by the total interest paid, which was \$13,000: $0.05x + 0.06y + 0.07z = 13$. The third is found by carefully reading the problem.

"*twice as much was borrowed at the 5% rate than was borrowed at the 7% rate*", or $x = 2z$.

These equations form the system: $\begin{cases} x + y + z = 225 \\ 0.05x + 0.06y + 0.07z = 13 \\ x = 2z \end{cases}$. The x-term of the first equation has a coefficient of 1. Written in standard form we have:

$$\begin{cases} x + y + z = 225 & \text{R1} \\ 5x + 6y + 7z = 1300 & \text{R2} \\ x \qquad - 2z = 0 & \text{R3} \end{cases}$$ (multiplied by 100)

Using $-5\text{R1} + \text{R2}$ will eliminate the x term in R2, while $-\text{R1} + \text{R3}$ will eliminate the x-term in R3.

$$\begin{array}{rr} -5\text{R1} & -5x - 5y - 5z = -1125 \\ + & \\ \text{R2} & 5x + 6y + 7z = 1300 \\ \hline & y + 2z = 175 \end{array} \qquad \begin{array}{rr} -\text{R1} & -x - y - z = -225 \\ + & \\ \text{R3} & x \qquad - 2z = 0 \\ \hline & -y - 3z = -225 \end{array}$$

The new R2 is $y + 2z = 175$, and the new R3 (after multiplying by -1) is $y + 3z = 225$, yielding the equivalent system $\begin{cases} x + y + z = 225 \\ y + 2z = 175. \\ y + 3z = 225 \end{cases}$

E. You've just learned how to use a system of three equations in three variables to solve applications

Solving the 2×2 subsystem using $-\text{R2} + \text{R3}$ yields $z = 50$. Back-substitution shows $y = 75$ and $x = 100$, yielding the solution (100, 75, 50). This means \$50,000 was borrowed at the 7% rate, \$75,000 was borrowed at 6%, and \$100,000 at 5%.

Now try Exercises 54 through 63 ►

TECHNOLOGY HIGHLIGHT

More on Parameterized Solutions

For linearly dependent systems, a graphing calculator can be used to both find and check possible solutions using the parameters Y_1, Y_2, and Y_3. This is done by assigning the chosen parameter to Y_1, then using Y_2 and Y_3 to form the other coordinates of the solution. We can then build the equations in the system using Y_1, Y_2, and Y_3 in place of x, y, and z. The system from Example 7 is $\begin{cases} 3x - 2y + z = -1 \\ 2x + y - z = 5 \\ 10x - 2y = 8 \end{cases}$, which we found had solutions of the form $(x, 5x - 4, 7x - 9)$. We first form the solution using $Y_1 = X$, $Y_2 = 5Y_1 - 4$ (for y), and $Y_3 = 7Y_1 - 9$ (for z). Then we form the equations in the system using $Y_4 = 3Y_1 - 2Y_2 + Y_3$, $Y_5 = 2Y_1 + Y_2 - Y_3$, and $Y_6 = 10Y_1 - 2Y_2$ (see Figure 8.8). After setting up the table (set on **AUTO**), solutions can be found by enabling only Y_1, Y_2, and Y_3, which gives values of x, y, and z, respectively (see Figure 8.9—use the right arrow ► to view Y_3). By enabling Y_4, Y_5, and Y_6 you can verify that for any value of the parameter, the first equation is equal to -1, the second is equal to 5, and the third is equal to 8 (see Figure 8.10—use the right arrow ► to view Y_6).

Figure 8.8

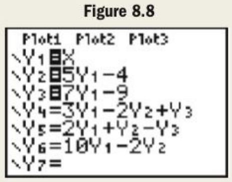


Figure 8.9

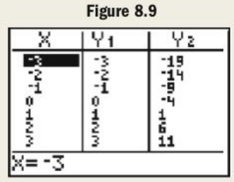
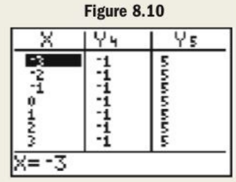


Figure 8.10



Exercise 1: Use the ideas from this Technology Highlight to (a) find four specific solutions to Example 6, (b) check multiple variations of the solution given, and (c) determine if $(-9, -6, -5)$, $(-2, 1, 2)$, and $(6, 2, 4)$ are solutions.

8.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The solution to an equation in three variables is an ordered ______.
2. The graph of the solutions to an equation in three variables is a(n) ______.
3. Systems that have the same solution set are called ______ ______.
4. If a 3×3 system is linearly dependent, the ordered triple solutions can be written in terms of a single variable called a(n) ______.
5. Find a value of z that makes the ordered triple $(2, -5, z)$ a solution to $2x + y + z = 4$. Discuss/Explain how this is accomplished.
6. Explain the difference between linear dependence and coincident dependence, and describe how the equations are related.

▶ DEVELOPING YOUR SKILLS

Find any four ordered triples that satisfy the equation given.

7. $x + 2y + z = 9$
8. $3x + y - z = 8$
9. $-x + y + 2z = -6$
10. $2x - y + 3z = -12$

Determine if the given ordered triples are solutions to the system.

11. $\begin{cases} x + y - 2z = -1 \\ 4x - y + 3z = 3 \\ 3x + 2y - z = 4 \end{cases}$; $(0, 3, 2)$, $(-3, 4, 1)$

12. $\begin{cases} 2x + 3y + z = 9 \\ 5x - 2y - z = -32 \\ x - y - 2z = -13 \end{cases}$; $(-4, 5, 2)$, $(5, -4, 11)$

Solve each system using elimination and back-substitution.

13. $\begin{cases} x - y - 2z = -10 \\ x - z = 1 \\ z = 4 \end{cases}$

14. $\begin{cases} x + y + 2z = -1 \\ 4x - y = 3 \\ 3x = 6 \end{cases}$

15. $\begin{cases} x + 3y + 2z = 16 \\ -2y + 3z = 1 \\ 8y - 13z = -7 \end{cases}$

16. $\begin{cases} -x + y + 5z = 1 \\ 4x + y = 1 \\ -3x - 2y = 8 \end{cases}$

17. $\begin{cases} 2x - y + 4z = -7 \\ x + 2y - 5z = 13 \\ y - 4z = 9 \end{cases}$

18. $\begin{cases} 2x + 3y + 4z = -18 \\ x - 2y + z = 4 \\ 4x + z = -19 \end{cases}$

19. $\begin{cases} -x + y + 2z = -10 \\ x + y - z = 7 \\ 2x + y + z = 5 \end{cases}$

20. $\begin{cases} x + y - 2z = -1 \\ 4x - y + 3z = 3 \\ 3x + 2y - z = 4 \end{cases}$

21. $\begin{cases} 3x + y - 2z = 3 \\ x - 2y + 3z = 10 \\ 4x - 8y + 5z = 5 \end{cases}$

22. $\begin{cases} 2x - 3y + 2z = 0 \\ 3x - 4y + z = -20 \\ x + 2y - z = 16 \end{cases}$

23. $\begin{cases} 3x - y + z = 6 \\ 2x + 2y - z = 5 \\ 2x - y + z = 5 \end{cases}$

24. $\begin{cases} 2x - 3y - 2z = 7 \\ x - y + 2z = -5 \\ 2x - 2y + 3z = -7 \end{cases}$

Solve using the elimination method. If a system is inconsistent or dependent, so state. For systems with linear dependence, write solutions in set notation and as an ordered triple in terms of a parameter.

25. $\begin{cases} 3x + y + 2z = 3 \\ x - 2y + 3z = 1 \\ 4x - 8y + 12z = 7 \end{cases}$

26. $\begin{cases} 2x - y + 3z = 8 \\ 3x - 4y + z = 4 \\ -4x + 2y - 6z = 5 \end{cases}$

27. $\begin{cases} 4x + y + 3z = 8 \\ x - 2y + 3z = 2 \end{cases}$

28. $\begin{cases} 4x - y + 2z = 9 \\ 3x + y + 5z = 5 \end{cases}$

29. $\begin{cases} 6x - 3y + 7z = 2 \\ 3x - 4y + z = 6 \end{cases}$

30. $\begin{cases} 2x - 4y + 5z = -2 \\ 3x - 2y + 3z = 7 \end{cases}$

Solve using elimination. If the system is linearly dependent, state the general solution in terms of a parameter. Different forms of the solution are possible.

31. $\begin{cases} 3x - 4y + 5z = 5 \\ -x + 2y - 3z = -3 \\ 3x - 2y + z = 1 \end{cases}$

32. $\begin{cases} 5x - 3y + 2z = 4 \\ -9x + 5y - 4z = -12 \\ -3x + y - 2z = -12 \end{cases}$

33. $\begin{cases} x + 2y - 3z = 1 \\ 3x + 5y - 8z = 7 \\ x + y - 2z = 5 \end{cases}$

34. $\begin{cases} -2x + 3y - 5z = 3 \\ 5x - 7y + 12z = -8 \\ x - y + 2z = -2 \end{cases}$

Solve using elimination. If the system has coincident dependence, state the solution in set notation.

35. $\begin{cases} -0.2x + 1.2y - 2.4z = -1 \\ 0.5x - 3y + 6z = 2.5 \\ x - 6y + 12z = 5 \end{cases}$

36. $\begin{cases} 6x - 3y + 9z = 21 \\ 4x - 2y + 6z = 14 \\ -2x + y - 3z = -7 \end{cases}$

Solve using the elimination method. If a system is inconsistent or dependent, so state. For systems with linear dependence, write the answer in terms of a parameter. For coincident dependence, state the solution in set notation.

37. $\begin{cases} x + 2y - z = 1 \\ x + z = 3 \\ 2x - y + z = 3 \end{cases}$

38. $\begin{cases} 3x + 5y - z = 11 \\ 2x + y - 3z = 12 \\ y + 2z = -4 \end{cases}$

39. $\begin{cases} 2x - 5y - 4z = 6 \\ x - 2.5y - 2z = 3 \\ -3x + 7.5y + 6z = -9 \end{cases}$

40. $\begin{cases} x - 2y + 2z = 6 \\ 2x - 6y + 3z = 13 \\ 3x + 4y - z = -11 \end{cases}$

41. $\begin{cases} 4x - 5y - 6z = 5 \\ 2x - 3y + 3z = 0 \\ x + 2y - 3z = 5 \end{cases}$

42. $\begin{cases} x - 5y - 4z = 3 \\ 2x - 9y - 7z = 2 \\ 3x - 14y - 11z = 5 \end{cases}$

43. $\begin{cases} 2x + 3y - 5z = 4 \\ x + y - 2z = 3 \\ x + 3y - 4z = -1 \end{cases}$

44. $\begin{cases} \frac{1}{6}x + \frac{1}{3}y - \frac{1}{2}z = 2 \\ \frac{3}{4}x - \frac{1}{3}y + \frac{1}{2}z = 9 \\ \frac{1}{2}x - y + \frac{1}{2}z = 2 \end{cases}$

45. $\begin{cases} \frac{x}{2} + \frac{y}{3} - \frac{z}{2} = 2 \\ \frac{2x}{3} - y - z = 8 \\ \frac{x}{6} + 2y + \frac{3z}{2} = 6 \end{cases}$

Some applications of systems lead to systems similar to those that follow. Solve using elimination.

46. $\begin{cases} -2A - B - 3C = 21 \\ B - C = 1 \\ A + B = -4 \end{cases}$

47. $\begin{cases} -A + 3B + 2C = 11 \\ 2B + C = 9 \\ B + 2C = 8 \end{cases}$

48. $\begin{cases} A + 2C = 7 \\ 2A - 3B = 8 \\ 3A + 6B - 8C = -33 \end{cases}$

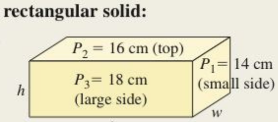
49. $\begin{cases} A - 2B = 5 \\ B + 3C = 7 \\ 2A - B - C = 1 \end{cases}$

50. $\begin{cases} C = -2 \\ 5A - 2C = 5 \\ -4B - 9C = 16 \end{cases}$

51. $\begin{cases} C = 3 \\ 2A + 3C = 10 \\ 3B - 4C = -11 \end{cases}$

▶ WORKING WITH FORMULAS

52. Dimensions of a rectangular solid: $\begin{cases} 2w + 2h = P_1 \\ 2l + 2w = P_2 \\ 2l + 2h = P_3 \end{cases}$



Using the formula shown, the dimensions of a rectangular solid can be found if the perimeters of the three distinct faces are known. Find the dimensions of the solid shown.

53. Distance from a point (x, y, z) to the plane $Ax + By + Cz = D$: $\left|\dfrac{Ax + By + Cz - D}{\sqrt{A^2 + B^2 + C^2}}\right|$

The perpendicular distance from a given point (x, y, z) to the plane defined by $Ax + By + Cz = D$ is given by the formula shown. Consider the plane given in Figure 8.2 ($x + y + z = 6$). What is the distance from this plane to the point (3, 4, 5)?

▶ APPLICATIONS

Solve the following applications by setting up and solving a system of three equations in three variables. Note that some equations may have only two of the three variables used to create the system.

Investment/Finance and Simple Interest Problems

54. Investing the winnings: After winning \$280,000 in the lottery, Maurika decided to place the money in three different investments: a certificate of deposit paying 4%, a money market certificate paying 5%, and some Aa bonds paying 7%. After 1 yr she earned \$15,400 in interest. Find how much was invested at each rate if \$20,000 more was invested at 7% than at 5%.

55. Purchase at auction: At an auction, a wealthy collector paid \$7,000,000 for three paintings: a Monet, a Picasso, and a van Gogh. The Monet cost \$800,000 more than the Picasso. The price of the van Gogh was \$200,000 more than twice the price of the Monet. What was the price of each painting?

Descriptive Translation

56. Major wars: The United States has fought three major wars in modern times: World War II, the Korean War, and the Vietnam War. If you sum the years that each conflict ended, the result is 5871. The Vietnam War ended 20 years after the Korean War and 28 years after World War II. In what year did each end?

57. Animal gestation periods: The average gestation period (in days) of an elephant, rhinoceros, and camel sum to 1520 days. The gestation period of a rhino is 58 days longer than that of a camel. Twice the camel's gestation period decreased by 162 gives the gestation period of an elephant. What is the gestation period of each?

58. Moments in U.S. history: If you sum the year the Declaration of Independence was signed, the year the 13th Amendment to the Constitution abolished slavery, and the year the Civil Rights Act was signed, the total would be 5605. Ninety-nine years separate the 13th Amendment and the Civil Rights Act. The Civil Rights Act was signed 188 years after the Declaration of Independence. What year was each signed?

59. Aviary wingspan: If you combine the wingspan of the California Condor, the Wandering Albatross (see photo), and the prehistoric Quetzalcoatlus, you get an astonishing 18.6 m (over 60 ft). If the wingspan of the Quetzalcoatlus is equal to five times that of the Wandering Albatross minus twice that of the California Condor, and six times the wingspan of the Condor is equal to five times the wingspan of the Albatross, what is the wingspan of each?

Mixtures

60. Chemical mixtures: A chemist mixes three different solutions with concentrations of 20%, 30%, and 45% glucose to obtain 10 L of a 38% glucose solution. If the amount of 30% solution used is 1 L more than twice the amount of 20% solution used, find the amount of each solution used.

61. Value of gold coins: As part of a promotion, a local bank invites its customers to view a large sack full of \$5, \$10, and \$20 gold pieces, promising to give the sack to the first person able to state the number of coins for each denomination. Customers are told there are exactly 250 coins, with a total face value of \$1875. If there are also seven times as many \$5 gold pieces as \$20 gold pieces, how many of each denomination are there?

62. Rewriting a rational function: It can be shown that the rational function $V(x) = \dfrac{3x + 11}{x^3 - 3x^2 + x - 3}$ can be written as a sum of the terms $\dfrac{A}{x - 3} + \dfrac{Bx + C}{x^2 + 1}$, where the coefficients A, B, and C are solutions to $\begin{cases} A + B = 0 \\ -3B + C = 3 \\ A - 3C = 11 \end{cases}$. Find the missing coefficients and verify your answer by adding the terms.

63. Rewriting a rational function: It can be shown that the rational function $V(x) = \dfrac{x - 9}{x^3 - 6x^2 + 9x}$ can be written as a sum of the terms $\dfrac{A}{x} + \dfrac{B}{x - 3} + \dfrac{C}{(x - 3)^2}$, where the coefficients A, B, and C are solutions to $\begin{cases} A + B = 0 \\ -6A - 3B + C = 1 \\ 9A = -9 \end{cases}$. Find the missing coefficients and verify your answer by adding the terms.

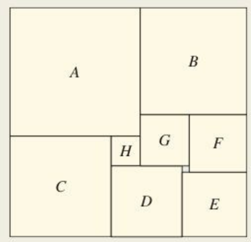
▶ EXTENDING THE CONCEPT

64. The system $\begin{cases} x - 2y - z = 2 \\ x - 2y + kz = 5 \\ 2x - 4y + 4z = 10 \end{cases}$ is inconsistent if $k =$ ______, and dependent if $k =$ ____.

65. One form of the equation of a circle is $x^2 + y^2 + Dx + Ey + F = 0$. Use a system to find the equation of the circle through the points $(2, -1)$, $(4, -3)$, and $(2, -5)$.

66. The lengths of each side of the squares A, B, C, D, E, F, G, H, and I (the smallest square) shown are whole numbers. Square B has sides of 15 cm and square G has sides of 7 cm. What are the dimensions of square D?

a. 9 cm **b.** 10 cm **c.** 11 cm
d. 12 cm **e.** 13 cm



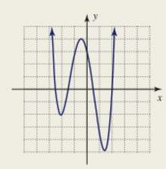
▶ MAINTAINING YOUR SKILLS

67. **(7.3)** Given $\mathbf{u} = \langle 1, -7\rangle$ and $\mathbf{v} = \langle -3, \frac{1}{2}\rangle$, compute $\mathbf{u} + 4\mathbf{v}$ and $3\mathbf{u} - \mathbf{v}$.

68. **(5.2)** Given $\cot A = 1.6831$, use a calculator to find the acute angle A to the nearest tenth of a degree.

69. **(4.4)** Solve the logarithmic equation: $\log(x + 2) + \log x = \log 3$

70. **(2.5)** Analyze the graph of g shown. Clearly state the domain and range, the zeroes of g, intervals where $g(x) > 0$, intervals where $g(x) < 0$, local maximums or minimums, and intervals where the function is increasing or decreasing. Assume each tick mark is one unit and estimate endpoints to the nearest tenths.



MID-CHAPTER CHECK

1. Solve using the substitution method. State whether the system is consistent, inconsistent, or dependent.
$\begin{cases} x - 3y = -2 \\ 2x + y = 3 \end{cases}$

2. Solve the system using elimination. State whether the system is consistent, inconsistent, or dependent.
$\begin{cases} x - 3y = -4 \\ 2x + y = 13 \end{cases}$

3. Solve using a system of linear equations and any method you choose: How many ounces of a 40% acid, should be mixed with 10 oz of a 64% acid, to obtain a 48% acid solution?

4. Determine whether the ordered triple is a solution to the system.

$$\begin{cases} 5x + 2y - 4z = 22 \\ 2x - 3y + z = -1 \\ 3x - 6y + z = 2 \end{cases} \quad (2, 0, -3)$$

5. The system given is a dependent system. Without solving, state why.

$$\begin{cases} x + 2y - 3z = 3 \\ 2x + 4y - 6z = 6 \\ x - 2y + 5z = -1 \end{cases}$$

6. Solve the system of equations:

$$\begin{cases} x + 2y - 3z = -4 \\ 2y + z = 7 \\ 5y - 2z = 4 \end{cases}$$

7. Solve using elimination:

$$\begin{cases} 2x + 3y - 4z = -4 \\ x - 2y + z = 0 \\ -3x - 2y + 2z = -1 \end{cases}$$

8. Solve the following system and write the solution as an ordered triple in terms of the parameter p.

$$\begin{cases} 2x - y + z = 1 \\ -5x + 2y - 3z = 2 \end{cases}$$

9. If you add Mozart's age when he wrote his first symphony, with the age of American chess player Paul Morphy when he began dominating the international chess scene, and the age of Blaise Pascal when he formulated his well-known *Essai pour les coniques* (Essay on Conics), the sum is 37. At the time of each event, Paul Morphy's age was 3 yr less than twice Mozart's, and Pascal was 3 yr older than Morphy. Set up a system of equations and find the age of each.

10. The *William Tell Overture* (Gioachino Rossini, 1829) is one of the most famous, and best-loved overtures known. It is played in four movements: a prelude, the storm (often used in animations with great clashes of thunder and a driving rain), the sunrise (actually, *A call to the dairy cows* . . .), and the finale (better known as the Lone Ranger theme song). The prelude takes 2.75 min. Depending on how fast the finale is played, the total playing time is about 11 min. The playing time for the prelude and finale is 1 min longer than the playing time of the storm and the sunrise. Also, the playtime of the storm plus twice the playtime of the sunrise is 1 min longer than twice the finale. Find the playtime for each movement.

REINFORCING BASIC CONCEPTS

Window Size and Graphing Technology

Since most substantial applications involve noninteger values, technology can play an important role in applying mathematical models. However, with its use comes a heavy responsibility to use it carefully. A very real effort must be made to determine the best approach and to secure a reasonable estimate. This is the only way to guard against (the inevitable) keystroke errors, or ensure a window size that properly displays the results.

Rationale

On October 1, 1999, the newspaper *USA TODAY* ran an article titled, "Bad Math added up to Doomed Mars Craft." The article told of how a $125,000,000.00 spacecraft was lost, apparently because the team of scientists that *plotted the course* for the craft used U.S. units of measurement, while the team of scientists *guiding* the craft were using metric units. NASA's space chief was later quoted, "The problem here was not the error, it was the failure of . . . the checks and balances in our process to detect the error."

No matter how powerful the technology, always try to begin your problem-solving efforts with an estimate. Begin by exploring the **context** of the problem, asking questions about the range of possibilities: How fast can a human run? How much does a new car cost? What is a reasonable price for a ticket? What is the total available to invest? There is no calculating involved in these estimates, they simply rely on "horse sense" and human experience. In many applied problems, the input and output values must be positive — which means the solution will appear in the first quadrant, narrowing the possibilities considerably.

This information will be used to set the viewing window of your graphing calculator, in preparation for solving the problem using a system and graphing technology.

Illustration 1 ▶ Erin just filled both her boat and Blazer with gas, at a total cost of $211.14. She purchased 35.7 gallons of premium for her boat and 15.3 gal of regular for her Blazer. Premium gasoline cost $0.10 per gallon more than regular. What was the cost per gallon of each grade of gasoline?

Solution ▶ Asking how much *you* paid for gas the last time you filled up should serve as a fair estimate. Certainly (in 2008) a cost of $6.00 or more per gallon in the United States is too high, and a cost of $1.50 per gallon or less would be too low. Also, we can estimate a solution by assuming that both kinds of gasoline cost the same. This would mean 51 gal were purchased for about $211, and a quick division would place the estimate at near $\frac{211}{51} \approx \$4.14$ per gallon. A good viewing window would be restricted to the first quadrant (since cost > 0) with maximum values of Xmax = 6 and Ymax = 6.

```
WINDOW
 Xmin=0
 Xmax=6
 Xscl=.5
 Ymin=0
 Ymax=6
 Yscl=.5
 Xres=1
```

Exercise 1: Solve Illustration 1 using graphing technology.

Exercise 2: Re-solve Exercises 63 and 64 from Section 8.1 using graphing technology. Verify results are identical.

8.3 Nonlinear Systems of Equations and Inequalities

Learning Objectives

In Section 8.3 you will learn how to:

- **A.** Visualize possible solutions
- **B.** Solve nonlinear systems using substitution
- **C.** Solve nonlinear systems using elimination
- **D.** Solve nonlinear systems of inequalities
- **E.** Solve applications of nonlinear systems

Equations where the variables have exponents other than 1 or that are transcendental (like logarithmic and exponential equations) are all nonlinear equations. A nonlinear system of equations has at least one nonlinear equation, and these occur in a great variety.

A. Possible Solutions for a Nonlinear System

When solving nonlinear systems, it is often helpful to *visualize* the graphs of each equation in the system. This can help determine the number of possible intersections and further assist the solution process.

EXAMPLE 1 ► Sketching Graphs to Visualize the Number of Possible Solutions

Identify each equation in the system as the equation of a line, parabola, circle, or one of the toolbox functions. Then determine the number of solutions possible by considering the different ways the graphs might intersect: $\begin{cases} x^2 + y^2 = 25 \\ x - y = 1 \end{cases}$.

Finally, solve the system by graphing.

Solution ► The first equation contains a sum of second-degree terms with equal coefficients, which we recognize as the equation of a circle. The second equation is obviously linear. This means the system may have no solution, one solution, or two solutions, as shown in Figure 8.11. The graph of the system is shown in Figure 8.12 and the two points of intersection appear to be $(-3, -4)$ and $(4, 3)$. After checking these in the original equations we find that both are solutions to the system.

A. You've just learned how to visualize possible solutions

Figure 8.11

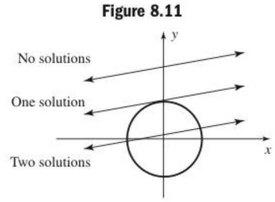
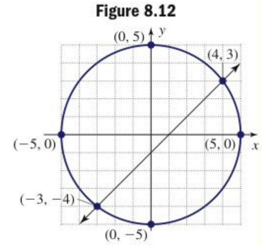


Figure 8.12



Now try Exercises 7 through 12 ►

B. Solving Nonlinear Systems by Substitution

Since graphical methods at best offer an estimate for the solution (points of intersection may not have integer values), we more often turn to the algebraic methods just developed. Recall the substitution method involves solving one of the equations for a variable or expression that can be substituted in the other equation to eliminate one of the variables.

EXAMPLE 2 ► **Solving a Nonlinear System Using Substitution**

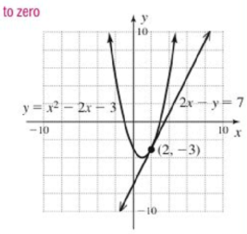
Solve the system using substitution: $\begin{cases} y = x^2 - 2x - 3 \\ 2x - y = 7 \end{cases}$.

Solution ► The first equation is the equation of a parabola. The second equation is linear. Since the first equation is already written with y in terms of x, we can substitute $x^2 - 2x - 3$ for y in the second equation to solve.

$$\begin{aligned} 2x - y &= 7 && \text{second equation} \\ 2x - (x^2 - 2x - 3) &= 7 && \text{substitute } x^2 - 2x - 3 \text{ for } y \\ 2x - x^2 + 2x + 3 &= 7 && \text{distribute} \\ -x^2 + 4x + 3 &= 7 && \text{simplify} \\ x^2 - 4x + 4 &= 0 && \text{set equal to zero} \\ (x - 2)^2 &= 0 && \text{factor} \end{aligned}$$

We find that $x = 2$ is a repeated root.

Since the second equation is simpler than the first, we substitute 2 for x in this equation and find $y = -3$. The system has only one (repeated) solution at $(2, -3)$, as shown in the figure.



☑ **B** You've just learned how to solve nonlinear systems using substitution

Now try Exercises 13 through 18 ►

C. Solving Nonlinear Systems by Elimination

When both equations in the system have second-degree terms with like variables, it is generally easier to use the elimination method, rather than substitution. Remember to watch for systems that have no solutions.

EXAMPLE 3 ► **Solving a Nonlinear System Using Elimination**

Solve the system using elimination: $\begin{cases} y - \frac{1}{2}x^2 = -3 \\ x^2 + y^2 = 41 \end{cases}$.

Solution ► The first equation can be rewritten as $y = \frac{1}{2}x^2 - 3$ and is a parabola opening upward with vertex $(0, -3)$. The second equation represents a circle with center at $(0, 0)$ and radius $r = \sqrt{41} \approx 6.4$. Mentally visualizing these graphs indicates there will be two solutions (see figure). After writing the system with x- and y-terms in the same order, we find that using 2R1 + R2 will eliminate the variable x:

WORTHY OF NOTE

Note that the x-terms sum to zero, and the y-terms cannot be combined as they are not like terms.

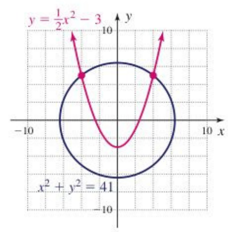
$$\begin{aligned} &2R1 \quad \begin{cases} -x^2 + 2y = -6 \\ x^2 + y^2 = 41 \end{cases} && \text{rewrite first equation; multiply by 2} \\ &+ R2 && \text{second equation} \\ &\quad y^2 + 2y = 35 && \text{add} \end{aligned}$$

To find solutions, we set the equation equal to zero and factor or use the quadratic formula if needed.

$$y^2 + 2y - 35 = 0 \quad \text{standard form}$$
$$(y + 7)(y - 5) = 0 \quad \text{factored form}$$
$$y = -7 \text{ or } y = 5 \quad \text{result}$$

The solution $y = -7$ is extraneous, due to the radius of the circle. Using $y = 5$ in the second equation gives the following:

$$x^2 + y^2 = 41 \quad \text{equation 2}$$
$$x^2 + (5)^2 = 41 \quad \text{substitute 5 for } y$$
$$x^2 + 25 = 41 \quad 5^2 = 25$$
$$x^2 = 16 \quad \text{subtract 25}$$
$$x = \pm 4 \quad \text{square root property}$$



The solutions are $(-4, 5)$ and $(4, 5)$, which is supported by the graph shown.

Now try Exercises 19 through 24 ►

Nonlinear systems may involve other relations as well, including power, polynomial, logarithmic, or exponential functions. These are solved using the same methods.

EXAMPLE 4 ► Solving a System of Logarithmic Equations

Solve the system using the method of your choice: $\begin{cases} y = -\log(x + 7) + 2 \\ y = \log(x + 4) + 1 \end{cases}$.

Solution ► Since both equations have y written in terms of x, substitution appears to be the better choice. The result is a logarithmic equation, which we can solve using the techniques from Chapter 4.

$$\log(x + 4) + 1 = -\log(x + 7) + 2 \quad \text{substitute } \log(x + 4) + 1 \text{ for } y \text{ in first equation}$$
$$\log(x + 4) + \log(x + 7) = 1 \quad \text{add } \log(x + 7)\text{; subtract 1}$$
$$\log(x + 4)(x + 7) = 1 \quad \text{product property of logarithms}$$
$$(x + 4)(x + 7) = 10^1 \quad \text{exponential form}$$
$$x^2 + 11x + 18 = 0 \quad \text{eliminate parentheses and set equal to zero}$$
$$(x + 9)(x + 2) = 0 \quad \text{factor}$$
$$x + 9 = 0 \text{ or } x + 2 = 0 \quad \text{zero factor theorem}$$
$$x = -9 \text{ or } x = -2 \quad \text{possible solutions}$$

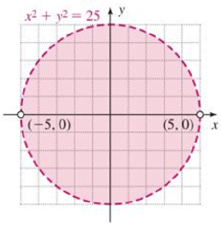
By inspection, we see that $x = -9$ is not a solution, since $\log(-9 + 4)$ and $-\log(-9 + 7)$ are not real numbers. Substituting -2 for x in the second equation we find one form of the (exact) solution is $(-2, \log 2 + 1)$. If we substitute -2 for x in the first equation the exact solution is $(-2, -\log 5 + 2)$. Use a calculator to verify the answers are equivalent and approximately $(-2, 1.3)$.

Now try Exercises 25 through 36 ►

C. You've just learned how to solve nonlinear systems using elimination

For practice solving more complex systems using a graphing calculator, see **Exercises 37 to 42.**

Figure 8.13



D. Solving Systems of Nonlinear Inequalities

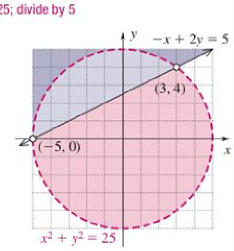
Nonlinear inequalities can be solved by graphing the boundary given by the related equation, and checking the regions that result using a test point. For example, the inequality $x^2 + y^2 < 25$ is solved by first graphing $x^2 + y^2 = 25$, a circle with radius 5, and deciding if the boundary is included or excluded (in this case it is not). We then use a test point from either "outside" or "inside" the region formed. The test point (0, 0) results in a true statement since $(0)^2 + (0)^2 < 25$, so the inside of the circle is shaded to indicate the solution region (Figure 8.13). For a ***system*** of nonlinear inequalities, we identify regions where the solution set for both inequalities overlap, paying special attention to points of intersection.

EXAMPLE 5 ► **Solving Systems of Nonlinear Inequalities**

Solve the system: $\begin{cases} x^2 + y^2 < 25 \\ 2y - x \geq 5 \end{cases}$.

Solution ► We recognize the first inequality from Figure 8.13, a circle with radius 5, and a solution region in the interior. The second inequality is linear and after solving for x we'll use a substitution to find points of intersection (if they exist). From $2y - x = 5$, we obtain $x = 2y - 5$.

$$\begin{aligned} x^2 + y^2 &= 25 && \text{given} \\ (2y - 5)^2 + y^2 &= 25 && \text{substitute } 2y - 5 \text{ for } x \\ 5y^2 - 20y + 25 &= 25 && \text{expand and simplify} \\ y^2 - 4y &= 0 && \text{subtract 25; divide by 5} \\ y(y - 4) &= 0 && \text{factor} \\ y = 0 \quad \text{or} \quad y &= 4 && \text{result} \end{aligned}$$



Back-substitution shows the graphs intersect at (−5, 0) and (3, 4). Graphing a line through these points and using (0, 0) as a test point shows the upper half plane is the solution region for the linear inequality [$2(0) - 0 \geq 5$ is *false*]. The overlapping (solution) region for *both* inequalities is the circular section shown. Note the points of intersection are graphed using "open dots" (see figure), since points on the graph of the circle are excluded from the solution set.

D You've just learned how to solve nonlinear systems of inequalities

Now try Exercises 43 through 50 ►

E. Applications of Nonlinear Systems

In the business world, a fast growing company can often reduce the average price of its products using what are called the **economies of scale.** These would include the ability to buy necessary materials in larger quantities, integrating new technology into the production process, and other means. However, there are also countering forces called the **diseconomies of scale,** which may include the need to hire additional employees, rent more production space, and the like.

EXAMPLE 6 ► **Solving an Application of Nonlinear Systems**

Suppose the cost to produce a new and inexpensive shoe made from molded plastic is modeled by the function $C(x) = x^2 - 5x + 18$, where $C(x)$ represents the cost to produce x thousand of these shoes. The revenue from the sales of these shoes is modeled by $R(x) = -x^2 + 10x - 4$. Use a break-even analysis to find the quantity of sales that will cause the company to break even.

Solution ► Essentially we are asked to solve the system formed by the two equations: $\begin{cases} C(x) = x^2 - 5x + 18 \\ R(x) = -x^2 + 10x - 4 \end{cases}$. Since we want to know the point where the company breaks even, we set $C(x) = R(x)$ and solve.

$$C(x) = R(x)$$

$$x^2 - 5x + 18 = -x^2 + 10x - 4 \quad \text{substitute for } C(x) \text{ and } R(x)$$

$$2x^2 - 15x + 22 = 0 \quad \text{set equal to zero}$$

$$(2x - 11)(x - 2) = 0 \quad \text{factored form}$$

$$x = \frac{11}{2} \text{ or } x = 2 \quad \text{result}$$

With x in thousands, it appears the company will break even if either 2000 shoes or 5500 shoes are made and sold.

E You've just learned how to solve applications of nonlinear systems

Now try Exercises 53 and 54 ►

8.3 EXERCISES

► CONCEPTS AND VOCABULARY

1. Draw sketches showing the different ways each pair of relations can intersect and give one, two, three, and/or four points of intersection. If a given number of intersections is not possible, so state.
   a. circle and line
   b. parabola and line
   c. circle and parabola
   d. circle and absolute value function
   e. absolute value function and line
   f. absolute value function and parabola

2. By inspection only, identify the systems having *no solutions* and justify your choices.
   a. $\begin{cases} y = |x| - 6 \\ x^2 + y^2 = 9 \end{cases}$ b. $\begin{cases} y = x^2 + 4 \\ x^2 + y^2 = 4 \end{cases}$
   c. $\begin{cases} y = x + 1 \\ x^2 + y^2 = 12 \end{cases}$

3. The solution to a system of nonlinear inequalities is a(n) ______ of the plane where the ______ for each individual inequality overlap.

4. When both equations in the system have at least one ______ -degree term, it is generally easier to use the ______ method to find a solution.

5. Suppose a nonlinear system contained a central hyperbola and an exponential function. Are three solutions possible? Are four solutions possible? Explain/Discuss.

6. Solve the system twice, once using elimination, then again using substitution. Compare/contrast each process and comment on which is more efficient in this case: $\begin{cases} x^2 + y^2 = 25 \\ x^2 + y = 5 \end{cases}$.

► DEVELOPING YOUR SKILLS

Identify each equation in the system as that of a line, parabola, circle, or absolute value function, then solve the system by graphing.

7. $\begin{cases} x^2 + y = 6 \\ x + y = 4 \end{cases}$

8. $\begin{cases} -x + y = 4 \\ x^2 + y^2 = 16 \end{cases}$

9. $\begin{cases} y^2 + x^2 = 100 \\ y = |x - 2| \end{cases}$

10. $\begin{cases} x^2 + y^2 = 25 \\ x^2 + y = 13 \end{cases}$

11. $\begin{cases} -(x - 1)^2 + 2 = y \\ y - x^2 = -3 \end{cases}$

12. $\begin{cases} y - 4 = -x^2 \\ y = -|x - 1| + 3 \end{cases}$

Solve using substitution. In Exercises 17 and 18, solve for x^2 or y^2 and use the result as a substitution.

13. $\begin{cases} x^2 + y^2 = 25 \\ y - x = 1 \end{cases}$ **14.** $\begin{cases} x + 7y = 50 \\ x^2 + y^2 = 100 \end{cases}$

15. $\begin{cases} x^2 + y = 9 \\ -2x + y = 1 \end{cases}$ **16.** $\begin{cases} x^2 - y = 8 \\ x + y = 4 \end{cases}$

17. $\begin{cases} x^2 + y = 13 \\ x^2 + y^2 = 25 \end{cases}$ **18.** $\begin{cases} y^2 + (x - 3)^2 = 25 \\ y^2 + (x + 1)^2 = 9 \end{cases}$

Solve each system.

19. $\begin{cases} x^2 + y^2 = 25 \\ \frac{1}{4}x^2 + y = 1 \end{cases}$ **20.** $\begin{cases} y - \frac{1}{2}x^2 = -1 \\ x^2 + y^2 = 65 \end{cases}$

21. $\begin{cases} x^2 + y^2 = 4 \\ y + x^2 = 5 \end{cases}$ **22.** $\begin{cases} y + x^2 = 6x \\ y - 11 = (x - 3)^2 \end{cases}$

23. $\begin{cases} x^2 + y^2 = 65 \\ y = 3x + 25 \end{cases}$ **24.** $\begin{cases} y - 2x = 5 \\ x^2 + y^2 = 85 \end{cases}$

Solve using the method of your choice.

25. $\begin{cases} y - 5 = \log x \\ y = 6 - \log(x - 3) \end{cases}$

26. $\begin{cases} y = \log(x + 4) + 1 \\ y - 2 = -\log(x + 7) \end{cases}$

27. $\begin{cases} y = \ln(x^2) + 1 \\ y - 1 = \ln(x + 12) \end{cases}$

28. $\begin{cases} \log(x + 1.1) = y + 3 \\ y + 4 = \log(x^2) \end{cases}$ **29.** $\begin{cases} y - 9 = e^{2x} \\ 3 = y - 7e^x \end{cases}$

30. $\begin{cases} y - 2e^{2x} = 5 \\ y - 1 = 6e^x \end{cases}$ **31.** $\begin{cases} y = 4^{x+3} \\ y - 2^{x^2+3x} = 0 \end{cases}$

32. $\begin{cases} y - 3^{x^2+2x} = 0 \\ y = 9^{x+2} \end{cases}$ **33.** $\begin{cases} x^3 - y = 2x \\ y - 5x = -6 \end{cases}$

34. $\begin{cases} y - x^3 = -2 \\ y + 4 = 3x \end{cases}$ **35.** $\begin{cases} x^2 - 6x = y - 4 \\ y - 2x = -8 \end{cases}$

36. $\begin{cases} y + x = -2 \\ y + 4x = x^2 \end{cases}$

Solve each system using a graphing calculator. Round solutions to hundredths (as needed).

37. $\begin{cases} x^2 + y^2 = 34 \\ y^2 + (x - 3)^2 = 25 \end{cases}$ **38.** $\begin{cases} 5x^2 + 5y^2 = 40 \\ y + 2x = x^2 - 6 \end{cases}$

39. $\begin{cases} y = 2^x - 3 \\ y + 2x^2 = 9 \end{cases}$ **40.** $\begin{cases} y = -2\log(x + 8) \\ y + x^3 = 4x - 2 \end{cases}$

41. $\begin{cases} y = \dfrac{1}{(x - 3)^2} + 2 \\ (x - 3)^2 + y^2 = 10 \end{cases}$ **42.** $\begin{cases} y^2 + x^2 = 5 \\ y = \dfrac{1}{x - 1} - 2 \end{cases}$

Solve each system of inequalities.

43. $\begin{cases} y - x^2 \ge 1 \\ x + y \le 3 \end{cases}$ **44.** $\begin{cases} x^2 + y^2 \le 25 \\ x + 2y \le 5 \end{cases}$

45. $\begin{cases} x^2 + y^2 > 16 \\ x^2 + y^2 \le 64 \end{cases}$ **46.** $\begin{cases} y + 4 \ge x^2 \\ x^2 + y^2 \le 34 \end{cases}$

47. $\begin{cases} y - x^2 \le -16 \\ y^2 + x^2 < 9 \end{cases}$ **48.** $\begin{cases} x^2 + y^2 \le 16 \\ x + 2y > 10 \end{cases}$

49. $\begin{cases} y^2 + x^2 \le 25 \\ |x| - 1 > -y \end{cases}$ **50.** $\begin{cases} y^2 + x^2 \le 4 \\ x + y < 4 \end{cases}$

▶ WORKING WITH FORMULAS

51. Tunnel clearance: $h = \sqrt{r^2 - d^2}$

The maximum rectangular clearance allowed by a circular tunnel can be found using the formula shown, where $x^2 + y^2 = r^2$ models the tunnel's circular cross section and h is the height of the tunnel at a distance d from the center. If $r = 50$ ft, find the maximum clearance at distances of $d = 20$, 30, and 40 ft from center.

52. Manufacturing cylindrical vents: $\begin{cases} A = 2\pi rh \\ V = \pi r^2 h \end{cases}$

In the manufacture of cylindrical vents, a rectangular piece of sheet metal is rolled, riveted, and sealed to form the vent. The radius and height required to form a vent with a specified volume, using a piece of sheet metal with a given area, can be found by solving the system shown. Use the system to find the radius and height if the volume required is 4071 cm^3 and the area of the rectangular piece is 2714 cm^2.

▶ APPLICATIONS

Market equilibrium: In a free-enterprise (supply and demand) economy, the amount buyers are willing to pay for an item and the number of these items manufacturers are willing to produce depend on the price of the item. As the price increases, demand for the item decreases since buyers are less willing to pay the higher price. On the other hand, an increase in price increases the supply of the item since manufacturers are now more willing to supply it. When the supply and demand curves are graphed, their point of intersection is called the market equilibrium for the item.

Solve the following applications of economies of scale.

53. World's most inexpensive car: Early in 2008, the Tata Company (India) unveiled the new Tata Nano, the world's most inexpensive car. With its low price and 54 miles per gallon, the car may prove to be very popular. *Assume* the cost to produce these cars is modeled by the function $C(x) = 2.5x^2 - 120x + 3500$, where $C(x)$ represents the cost to produce x-thousand cars. Suppose the revenue from the sale of these cars is modeled by $R(x) = -2x^2 + 180x - 500$. Use a break-even analysis to find the quantity of sales (to the nearest hundred) that will cause the company to break even.

54. Document reproduction: In a world of technology, document reproduction has become a billion dollar business. With very stiff competition, the price of a single black and white copy has varied greatly in recent years. Suppose the cost to produce these copies is modeled by the function $C(x) = 0.1x^2 - 1.2x + 7$, where $C(x)$ represents the cost to produce x hundred thousand copies. If the revenue from the sale of these copies is modeled by $R(x) = -0.1x^2 + 1.8x - 2$, use a break-even analysis to find the quantity of copies that will cause the company to break even.

55. Suppose the monthly market demand D (in ten-thousands of gallons) for a new synthetic oil is related to the price P in dollars by the equation $10P^2 + 6D = 144$. For the market price P, assume the amount D that manufacturers are willing to supply is modeled by $8P^2 - 8P - 4D = 12$. (a) What is the minimum price at which manufacturers are willing to begin supplying the oil? (b) Use this information to create a system of nonlinear equations, then solve the system to find the market equilibrium price (per gallon) and the quantity of oil supplied and sold at this price.

56. The weekly demand D for organically grown carrots (in thousands of pounds) is related to the price per pound P by the equation $8P^2 + 4D = 84$. At this market price, the amount that growers are willing to supply is modeled by the equation $8P^2 + 6P - 2D = 48$. (a) What is the minimum price at which growers are willing to supply the organically grown carrots? (b) Use this information to create a system of nonlinear equations, then solve the system to find the market equilibrium price (per pound) and the quantity of carrots supplied and sold at this price.

Solve by setting up and solving a system of nonlinear equations.

57. Dimensions of a flag: A large American flag has an area of 85 m^2 and a perimeter of 37 m. Find the dimensions of the flag.

58. Dimensions of a sail: The sail on a boat is a right triangle with a perimeter of 36 ft and a hypotenuse of 15 ft. Find the height and width of the sail.

59. Dimensions of a tract: The area of a rectangular tract of land is 45 km^2. The length of a diagonal is $\sqrt{106}$ km. Find the dimensions of the tract.

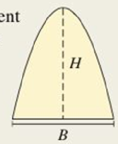
60. Dimensions of a deck: A rectangular deck has an area of 192 ft^2 and the length of the diagonal is 20 ft. Find the dimensions of the deck.

61. Dimensions of a trailer: The surface area of a rectangular trailer with square ends is 928 ft^2. If the sum of all edges of the trailer is 164 ft, find its dimensions.

62. Dimensions of a cylindrical tank: The surface area of a closed cylindrical tank is 192π m^2. Find the dimensions of the tank if the volume is 320π m^3 and the radius is as small as possible.

▶ EXTENDING THE CONCEPT

63. The area of a vertical parabolic segment is given by $A = \frac{2}{3}BH$, where B is the length of the horizontal base of the segment and H is the height from the base to the vertex. Investigate how this formula can be used to find the *area* of the solution region for the general system of inequalities shown.



$$\begin{cases} y \geq x^2 - bx + c \\ y \leq c + bx - x^2 \end{cases}$$

(*Hint:* Begin by investigating with $b = 6$ and $c = 8$, then use other values and try to generalize what you find.)

64. Find the area of the trapezoid formed by joining the points where the parabola $y = \frac{1}{2}x^2 - 26$ and the circle $x^2 + y^2 = 100$ intersect.

65. A rectangular fish tank has a bottom and four sides made out of glass. Use a system of equations to help find the dimensions of the tank if the height is 18 in., surface area is 4806 in^2, the tank must hold 108 gal (1 gal = 231 in^3), and all three dimensions are integers.

▶ MAINTAINING YOUR SKILLS

66. **(1.5)** Solve by factoring:
- **a.** $2x^2 + 5x - 63 = 0$
- **b.** $4x^2 - 121 = 0$
- **c.** $2x^3 - 3x^2 - 8x + 12 = 0$

67. **(6.3)** Find the exact value of $\cos\left(\frac{5\pi}{12}\right)$ using a sum or difference identity.

68. **(5.2)** Solve using any method. As an investment for retirement, Donovan bought three properties for a total of \$250,000. Ten years later, the first property had doubled in value, the second property had tripled in value, and the third property was worth \$10,000 less than when he bought it, for a current value of \$485,000. Find the original purchase price if he paid \$20,000 more for the first property than he did for the second.

69. **(7.4)** Kiaro is using a leash to drag his stubborn dog Maya out of the kitchen. (She is a shameless beggar of table scraps.) He is pulling her with a constant horizontal force of 20 lb, with the leash making an angle of 37° with the floor. How much work is done as Maya is dragged the 12 ft out of the kitchen? Round your answer to the nearest tenth.

8.4 Systems of Inequalities and Linear Programming

Learning Objectives

In Section 8.4 you will learn how to:

- ❑ **A.** Solve a linear inequality in two variables
- ❑ **B.** Solve a system of linear inequalities
- ❑ **C.** Solve applications using a system of linear inequalities
- ❑ **D.** Solve applications using linear programming

In this section, we'll build on many of the ideas from Section 8.3, with a more direct focus on systems of linear inequalities. While systems of linear *equations* have an unlimited number of applications, there are many situations that can only be modeled using linear *inequalities*. For example, many decisions in business and industry are based on a large number of limitations or constraints, with many different ways these constraints can be satisfied.

A. Linear Inequalities in Two Variables

A linear equation in two variables is any equation that can be written in the form $Ax + By = C$, where A and B are real numbers, not simultaneously equal to zero. A **linear inequality** in two variables is similarly defined, with the " $=$ " sign replaced by the " $<$," " $>$," " $\leq$," or " $\geq$ " symbol:

$$Ax + By < C \qquad Ax + By > C$$
$$Ax + By \leq C \qquad Ax + By \geq C$$

Solving a linear inequality in two variables has many similarities with the one variable case. For one variable, we graph the *boundary point* on a number line, decide whether the endpoint is *included* or *excluded,* and *shade the appropriate half line.* For $x + 1 \le 3$, we have the solution $x \le 2$ with the endpoint included and the line shaded to the left (Figure 8.14):

Figure 8.14

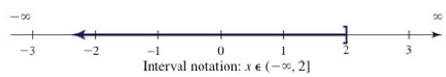
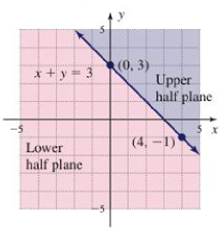


Figure 8.15



For linear inequalities in two variables, we graph a *boundary line,* decide whether the boundary line is *included* or *excluded,* and *shade the appropriate half plane.* For $x + y \le 3$, the boundary line $x + y = 3$ is graphed in Figure 8.15. Note it divides the coordinate plane into two regions called **half planes,** and it forms the **boundary** between the two regions. If the boundary is **included** in the solution set, we graph it using a *solid line.* If the boundary is **excluded,** a *dashed line* is used. Recall that solutions to a linear equation are ordered pairs that make the equation true. We use a similar idea to find or verify solutions to linear inequalities. If any one point in a half plane makes the inequality true, all points in that half plane will satisfy the inequality.

EXAMPLE 1 ▶ Checking Solutions to an Inequality in Two Variables

Determine whether the given ordered pairs are solutions to $-x + 2y < 2$:

a. $(4, -3)$ **b.** $(-2, 1)$

Solution ▶

a. Substitute 4 for x and -3 for y: $-(4) + 2(-3) < 2$ substitute 4 for x, -3 for y

$-10 < 2$ true

$(4, -3)$ is a solution.

b. Substitute -2 for x and 1 for y: $-(-2) + 2(1) < 2$ substitute -2 for x, 1 for y

$4 < 2$ false

$(-2, 1)$ is not a solution.

Now try Exercises 7 through 10 ▶

WORTHY OF NOTE

This relationship is often called the **trichotomy axiom** or the "*three-part truth.*" Given any two quantities, they are either equal to each other, or the first is less than the second, or the first is greater than the second.

Earlier we graphed linear equations by plotting a small number of ordered pairs or by solving for y and using the slope-intercept method. The line represented all ordered pairs that made the equation true, meaning *the left-hand expression was equal to the right-hand expression.* To graph linear inequalities, we reason that if the line represents all ordered pairs that make the expressions *equal,* then any point *not on that line* must make the expressions *unequal*—either greater than or less than. These ordered pair solutions must lie in one of the half planes formed by the line, which we shade to indicate the **solution region.** Note this implies the boundary line for any inequality *is determined by the related equation,* temporarily replacing the inequality symbol with an "=" sign.

EXAMPLE 2 ▶ Solving an Inequality in Two Variables

Solve the inequality $-x + 2y \le 2$.

Solution ▶ The related equation and boundary line is $-x + 2y = 2$. Since the inequality is inclusive (less than *or equal to*), we graph a solid line. Using the intercepts, we graph the line through (0, 1) and (−2, 0) shown in Figure 8.16. To determine the solution region and which side to shade, we select (0, 0) as a test point, which results in a true statement: $-(0) + 2(0) \le 2$✓. Since (0, 0) is in the "lower" half plane, we shade this side of the boundary (see Figure 8.17).

Figure 8.16

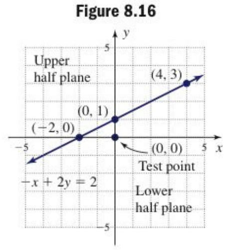
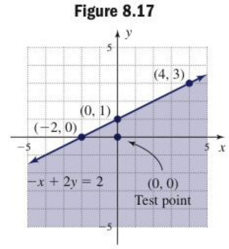


Figure 8.17



Now try Exercises 11 through 14 ►

A. You've just learned how to solve a linear inequality in two variables

The same solution would be obtained if we first solve for y and graph the boundary line using the slope-intercept method. However, using the slope-intercept method offers a distinct advantage—test points are no longer necessary since solutions to "less than" inequalities will always appear *below* the boundary line and solutions to "greater than" inequalities appear *above* the line. Written in slope-intercept form, the inequality from Example 2 is $y \leq \frac{1}{2}x + 1$. Note that (0, 0) still results in a true statement, but the "less than or equal to" symbol now indicates directly that solutions will be found in the lower half plane. This observation leads to our general approach for solving linear inequalities:

Solving a Linear Inequality

1. Graph the boundary line by solving for y and using the slope-intercept form.
   - Use a solid line if the boundary is included in the solution set.
   - Use a dashed line if the boundary is excluded from the solution set.
2. For "greater than" inequalities shade the upper half plane. For "less than" inequalities shade the lower half plane.

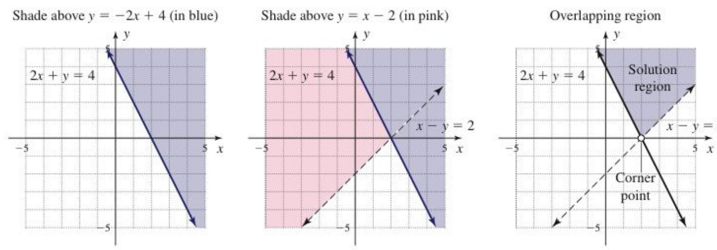
B. Solving Systems of Linear Inequalities

To solve a **system of inequalities,** we apply the procedure outlined above to all inequalities in the system, and note the ordered pairs that satisfy *all inequalities simultaneously.* In other words, we find *the intersection of all solution regions* (where they overlap), which then represents the solution for the system. In the case of vertical boundary lines, the designations *"above"* or *"below" the line* cannot be applied, and instead we simply note that for any vertical line $x = k$, points with x-coordinates larger than k will occur to the right.

EXAMPLE 3 ► Solving a System of Linear Inequalities

Solve the system of inequalities: $\begin{cases} 2x + y \geq 4 \\ x - y < 2 \end{cases}$.

Solution ► Solving for y, we obtain $y \geq -2x + 4$ and $y > x - 2$. The line $y = -2x + 4$ will be a solid boundary line (included), while $y = x - 2$ will be dashed (not included). Both inequalities are "greater than" and so we shade the upper half plane for each. The regions overlap and form the solution region (the lavender region shown). This sequence of events is illustrated here:



The solutions are all ordered pairs found in this region and its included boundaries. To verify the result, test the point (2, 3) from inside the region, (5, −2) from outside the region (the point (2, 0) is not a solution since $x - y < 2$).

Now try Exercises 15 through 42 ►

B. You've just learned how to solve a system of linear inequalities

For further reference, the point of intersection (2, 0) is called a **corner point** or **vertex** of the solution region. If the point of intersection is not easily found from the graph, we can find it by solving a linear system using the two lines. For Example 3, the system is

$$\begin{cases} 2x + y = 4 \\ x - y = 2 \end{cases}$$

and solving by elimination gives $3x = 6$, $x = 2$, and (2, 0) as the point of intersection.

C. Applications of Systems of Linear Inequalities

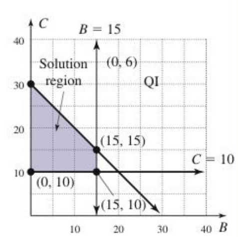
Systems of inequalities give us a way to model the decision-making process when certain ***constraints*** must be satisfied. A constraint is a fact or consideration that somehow limits or governs possible solutions, like the number of acres a farmer plants — which may be limited by time, size of land, government regulation, and so on.

EXAMPLE 4 ► Solving Applications of Linear Inequalities

As part of their retirement planning, James and Lily decide to invest up to $30,000 in two separate investment vehicles. The first is a bond issue paying 9% and the second is a money market certificate paying 5%. A financial adviser suggests they invest at least $10,000 in the certificate and not more than $15,000 in bonds. What various amounts can be invested in each?

Solution ► Consider the ordered pairs (B, C) where B represents the money invested in bonds and C the money invested in the certificate. Since they plan to invest no more than $30,000, the investment constraint would be $B + C \leq 30$ (in thousands). Following the adviser's recommendations, the constraints on each investment would be $B \leq 15$ and $C \geq 10$. Since they cannot invest less than zero dollars, the last two constraints are $B \geq 0$ and $C \geq 0$.

$$\begin{cases} B + C \leq 30 \\ B \leq 15 \\ C \geq 10 \\ B \geq 0 \\ C \geq 0 \end{cases}$$



The resulting system is shown in the figure, and indicates solutions will be in the first quadrant.

There is a vertical boundary line at $B = 15$ with shading to the left (less than) and a horizontal boundary line at $C = 10$ with shading above (greater than). After graphing $C = 30 - B$, we see the solution region is a quadrilateral with vertices at (0, 10), (0, 30), (15, 10), and (15, 15), as shown.

C. You've just learned how to solve applications using a system of linear inequalities

Now try Exercises 53 and 54 ►

D. Linear Programming

To become as profitable as possible, corporations look for ways to maximize their revenue and minimize their costs, while keeping up with delivery schedules and product demand. To operate at peak efficiency, plant managers must find ways to maximize productivity, while minimizing related costs and considering employee welfare, union agreements, and other factors. Problems where the goal is to **maximize** or **minimize** the value of a given quantity under certain **constraints** or restrictions are called programming problems. The quantity we seek to maximize or minimize is called the **objective function.** For situations where *linear* programming is used, the objective function is given as a linear function in two variables and is denoted $f(x, y)$. A function in two variables is evaluated in much the same way as a single variable function. To evaluate $f(x, y) = 2x + 3y$ at the point (4, 5), we substitute 4 for x and 5 for y: $f(4, 5) = 2(4) + 3(5) = 23$.

EXAMPLE 5 ► Determining Maximum Values

Determine which of the following ordered pairs maximizes the value of $f(x, y) = 5x + 4y$: (0, 6), (5, 0), (0, 0), or (4, 2).

Solution ► Organizing our work in table form gives

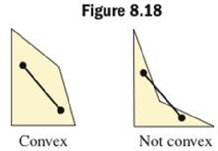
Given Point	Evaluate $f(x, y) = 5x + 4y$
(0, 6)	$f(0, 6) = 5(0) + 4(6) = 24$
(5, 0)	$f(5, 0) = 5(5) + 4(0) = 25$
(0, 0)	$f(0, 0) = 5(0) + 4(0) = 0$
(4, 2)	$f(4, 2) = 5(4) + 4(2) = 28$

The function $f(x, y) = 5x + 4y$ is maximized at (4, 2).

Now try Exercises 43 through 46 ►

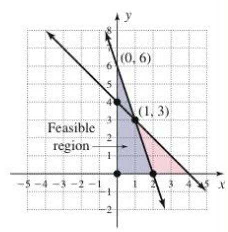
When the objective is stated as a linear function in two variables and the constraints are expressed as a system of linear inequalities, we have what is called a **linear programming** problem. The systems of inequalities solved earlier produced a solution region that was either **bounded** (as in Example 4) or **unbounded** (as in Example 3). We interpret the word *bounded* to mean we can enclose the solution region within a circle of appropriate size. If we cannot draw a circle around the region because it extends indefinitely in some direction, the region is said to be *unbounded.* In this study, we will consider only situations that produce a bounded solution region, meaning the region will have three or more vertices. The regions we study will also be **convex,** meaning that for any two points in the feasible region, the line segment between them is also in the region (Figure 8.18). Under these conditions, it can be shown that the optimal solution(s) *must occur at one of the corner points of the solution region,* also called the **feasible region.**

Figure 8.18



EXAMPLE 6 ▶ Finding the Maximum of an Objective Function

Find the maximum value of the objective function $f(x, y) = 2x + y$ given the constraints shown: $\begin{cases} x + y \leq 4 \\ 3x + y \leq 6 \\ x \geq 0 \\ y \geq 0 \end{cases}$.



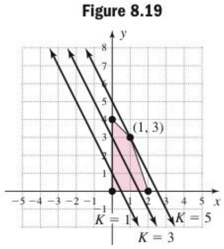
Solution ▶ Begin by noting that the solutions must be in QI, since $x \geq 0$ and $y \geq 0$. Graph the boundary lines $y = -x + 4$ and $y = -3x + 6$, shading the lower half plane in each case since they are "less than" inequalities. This produces the feasible region shown in lavender. There are four corner points to this region: (0, 0), (0, 4), (2, 0), and (1, 3). Three of these points are intercepts and can be found quickly. The point (1, 3) was found by solving the system $\begin{cases} x + y = 4 \\ 3x + y = 6 \end{cases}$. Knowing that the objective function will be maximized at one of the corner points, we test them in the objective function, using a table to organize our work.

Corner Point	Objective Function $f(x, y) = 2x + y$
(0, 0)	$f(0, 0) = 2(0) + (0) = 0$
(0, 4)	$f(0, 4) = 2(0) + (4) = 4$
(2, 0)	$f(2, 0) = 2(2) + (0) = 4$
(1, 3)	$f(1, 3) = 2(1) + (3) = 5$

The objective function $f(x, y) = 2x + y$ is maximized at (1, 3).

Now try Exercises 47 through 50 ▶

Figure 8.19



To help understand why solutions must occur at a vertex, note the objective function $f(x, y)$ is maximized using only (x, y) ordered pairs from the feasible region. If we let K represent this maximum value, the function from Example 6 becomes $K = 2x + y$ or $y = -2x + K$, which is a line with slope -2 and y-intercept K. The table in Example 6 suggests that K should range from 0 to 5 and graphing $y = -2x + K$ for $K = 1$, $K = 3$, and $K = 5$ produces the family of parallel lines shown in Figure 8.19. Note that values of K larger than 5 will cause the line to miss the solution region, and the maximum value of 5 occurs where the line intersects the feasible region at the vertex (1, 3). These observations lead to the following principles, which we offer without a formal proof.

Linear Programming Solutions

1. If the feasible region is convex and bounded, a maximum and a minimum value exist.
2. If a unique solution exists, it will occur at a vertex of the feasible region.
3. If more than one solution exists, at least one of them occurs at a vertex of the feasible region with others on a boundary line.
4. If the feasible region is unbounded, a linear programming problem may have no solutions.

Solving linear programming problems depends in large part on two things: (1) identifying the **objective** and the **decision variables** (what each variable represents in context), and (2) using the decision variables to write the *objective function* and **constraint inequalities.** This brings us to our five-step approach for solving linear programming applications.

Solving Linear Programming Applications

1. Identify the main objective and the decision variables (descriptive variables may help) and write the objective function in terms of these variables.
2. Organize all information in a table, with the *decision variables* and *constraints* heading up the columns, and their *components* leading each row.
3. Complete the table using the information given, and write the constraint inequalities using the decision variables, constraints, and the domain.
4. Graph the constraint inequalities, determine the feasible region, and identify all corner points.
5. Test these points in the objective function to determine the optimal solution(s).

EXAMPLE 7 ► Solving an Application of Linear Programming

The owner of a snack food business wants to create two nut mixes for the holiday season. The regular mix will have 14 oz of peanuts and 4 oz of cashews, while the deluxe mix will have 12 oz of peanuts and 6 oz of cashews. The owner estimates he will make a profit of \$3 on the regular mixes and \$4 on the deluxe mixes. How many of each should be made in order to maximize profit, if only 840 oz of peanuts and 348 oz of cashews are available?

Solution ► Our *objective* is to maximize profit, and the *decision variables* could be r to represent the regular mixes sold, and d for the number of deluxe mixes. This gives $P(r, d) = \$3r + \$4d$ as our *objective function.* The information is organized in Table 8.1, using the variables r, d, and the constraints to head each column. Since the mixes are composed of peanuts and cashews, these lead the rows in the table.

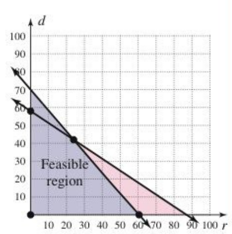
Table 8.1

$P(r, d) = \$3r + \$4d$

	Regular r	Deluxe d	Constraints: Total Ounces Available
Peanuts	14	12	840
Cashews	4	6	348

After filling in the appropriate values, reading the table from left to right along the "peanut" row and the "cashew" row, gives the constraint inequalities $14r + 12d \leq 840$ and $4r + 6d \leq 348$. Realizing we won't be making a negative number of mixes, the remaining constraints are $r \geq 0$ and $d \geq 0$. The complete system is

$$\begin{cases} 14r + 12d \leq 840 \\ 4r + 6d \leq 348 \\ r \geq 0 \\ d \geq 0 \end{cases}$$



Note once again that the solutions must be in QI, since $r \geq 0$ and $d \geq 0$. Graphing the first two inequalities using slope-intercept form gives $d \leq -\frac{7}{6}r + 70$ and $d \leq -\frac{2}{3}r + 58$ producing the feasible region shown in lavender. The four corner points are (0, 0), (60, 0), (0, 58), and (24, 42). Three of these points are intercepts and can be read from a table of values or the graph itself. The point (24, 42) was found by solving the system $\begin{cases} 14r + 12d = 840 \\ 4r + \;\;6d = 348 \end{cases}$.

Knowing the objective function will be maximized at one of these points, we test them in the objective function (Table 8.2).

Table 8.2

Corner Point	Objective Function $P(r, d) = \$3r + \$4d$
(0, 0)	$P(0, 0) = \$3(0) + \$4(0) = 0$
(60, 0)	$P(60, 0) = \$3(60) + \$4(0) = \$180$
(0, 58)	$P(0, 58) = \$3(0) + \$4(58) = \$232$
(24, 42)	$P(24, 42) = \$3(24) + \$4(42) = \$240$

Profit will be maximized if 24 boxes of the regular mix and 42 boxes of the deluxe mix are made and sold.

Now try Exercises 55 through 60 ►

Linear programming can also be used to minimize an objective function, as in Example 8.

EXAMPLE 8 ► Minimizing Costs Using Linear Programming

A beverage producer needs to minimize shipping costs from its two primary plants in Kansas City (KC) and St. Louis (STL). All wholesale orders within the state are shipped from one of these plants. An outlet in Macon orders 200 cases of soft drinks on the same day an order for 240 cases comes from Springfield. The plant in KC has 300 cases ready to ship and the plant in STL has 200 cases. The cost of shipping each case to Macon is \$0.50 from KC, and \$0.70 from STL. The cost of shipping each case to Springfield is \$0.60 from KC, and \$0.65 from STL. How many cases should be shipped from each warehouse to minimize costs?

Solution ► Our *objective* is to minimize costs, which depends on the number of cases shipped from each plant. To begin we use the following assignments:

$A \rightarrow$ cases shipped from KC to Macon
$B \rightarrow$ cases shipped from KC to Springfield
$C \rightarrow$ cases shipped from STL to Macon
$D \rightarrow$ cases shipped from STL to Springfield

From this information, the equation for total cost T is

$$T = 0.5A + 0.6B + 0.7C + 0.65D,$$

an equation in *four* variables. To make the cost equation more manageable, note since Macon ordered 200 cases, $A + C = 200$. Similarly, Springfield ordered 240 cases, so $B + D = 240$. After solving for C and D, respectively, these equations enable us to substitute for C and D, resulting in an equation with just two variables. For $C = 200 - A$ and $D = 240 - B$ we have

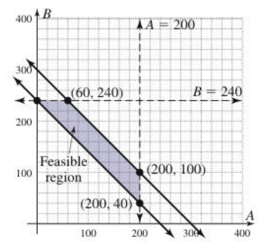
$$\begin{aligned} T(A, B) &= 0.5A + 0.6B + 0.7(200 - A) + 0.65(240 - B) \\ &= 0.5A + 0.6B + 140 - 0.7A + 156 - 0.65B \\ &= 296 - 0.2A - 0.05B \end{aligned}$$

The constraints involving the KC plant are $A + B \leq 300$ with $A \geq 0, B \geq 0$. The constraints for the STL plant are $C + D \leq 200$ with $C \geq 0, D \geq 0$. Since we want a system in terms of A and B only, we again substitute $C = 200 - A$ and $D = 240 - B$ in all the STL inequalities:

$$C + D \le 200 \quad \text{STL inequalities} \qquad C \ge 0 \qquad D \ge 0$$
$$(200 - A) + (240 - B) \le 200 \quad \text{substitute } 200 - A \text{ for } C,\ 240 - B \text{ for } D \qquad 200 - A \ge 0 \qquad 240 - B \ge 0$$
$$200 \ge A \qquad 240 \ge B$$
$$440 - A - B \le 200 \quad \text{simplify}$$
$$240 \le A + B \quad \text{result}$$

Combining the new STL constraints with those from KC produces the following system and solution. All points of intersection were read from the graph or located using the related system of equations.

$$\begin{cases} A + B \le 300 \\ A + B \ge 240 \\ A \le 200 \\ B \le 240 \\ A \ge 0 \\ B \ge 0 \end{cases}$$



To find the minimum cost, we check each vertex in the objective function.

Vertices	Objective Function $T(A, B) = 296 - 0.2A - 0.05B$
(0, 240)	$P(0, 240) = 296 - 0.2(0) - 0.05(240) = \$284$
(60, 240)	$P(60, 240) = 296 - 0.2(60) - 0.05(240) = \$272$
(200, 100)	$P(200, 100) = 296 - 0.2(200) - 0.05(100) = \$251$
(200, 40)	$P(200, 40) = 296 - 0.2(200) - 0.05(40) = \$254$

The minimum cost occurs when $A = 200$ and $B = 100$, meaning the producer should ship the following quantities:

$A \rightarrow$ cases shipped from KC to Macon $= 200$
$B \rightarrow$ cases shipped from KC to Springfield $= 100$
$C \rightarrow$ cases shipped from STL to Macon $= 0$
$D \rightarrow$ cases shipped from STL to Springfield $= 140$

☑ **D.** You've just learned how to solve applications using linear programming

Now try Exercises 61 and 62 ►

TECHNOLOGY HIGHLIGHT

Systems of Linear Inequalities

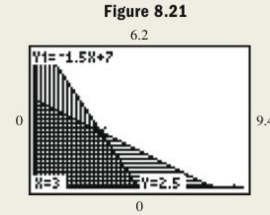
Solving systems of linear inequalities on the TI-84 Plus involves three steps, which are performed on both equations: (1) enter the related equations in Y_1 and Y_2 (solve for y in each equation) to create the boundary lines, (2) graph both lines and test the resulting half planes, and (3) shade the appropriate half plane. Since many real-world applications of linear inequalities preclude the use of negative numbers, we **set Xmin = 0 and Ymin = 0 for the WINDOW size.** Xmax and Ymax will depend on the equations given. We illustrate by solving the system $\begin{cases} 3x + 2y < 14 \\ x + 2y < 8 \end{cases}$.

—continued

1. *Enter the related equations*. For $3x + 2y = 14$, we have $y = -1.5 + 7$. For $x + 2y = 8$, we have $y = -0.5x + 4$. Enter these as Y_1 and Y_2 on the Y= screen.
2. *Graph the boundary lines*. Note the *x*- and *y*-intercepts of both lines are less than 10, so we can graph them using a friendly window where $x \in [0, 9.4]$ and $y \in [0, 6.2]$. After setting the window, press GRAPH to graph the lines.
3. *Shade the appropriate half plane*. Since both equations are in slope-intercept form, we shade *below* both lines for the less than inequalities, using the "◣" feature located to the far left of Y_1 and Y_2. Simply overlay the diagonal line and press ENTER repeatedly until the symbol appears (Figure 8.20). After pressing the GRAPH key, the calculator draws both lines and shades the appropriate regions (Figure 8.21). Note the calculator uses two different kinds of shading. This makes it easy to identify the solution region—it will be the "checker-board area" where the horizontal and vertical lines cross. As a final check, you could navigate the position marker into the solution region and test a few points in both equations.

Figure 8.20

Figure 8.21



Use these ideas to solve the following systems of linear inequalities. Assume all solutions lie in Quadrant I.

Exercise 1: $\begin{cases} y + 2x < 8 \\ y + x < 6 \end{cases}$ **Exercise 2:** $\begin{cases} 3x + y < 8 \\ x + y < 4 \end{cases}$ **Exercise 3:** $\begin{cases} -4x - y > -9 \\ -3x - y > -7 \end{cases}$

8.4 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Any line $y = mx + b$ drawn in the coordinate plane divides the plane into two regions called ______ ______.

2. For the line $y = mx + b$ drawn in the coordinate plane, solutions to $y > mx + b$ are found in the region ______ the line.

3. The overlapping region of two or more linear inequalities in a system is called the ______ region.

4. If a linear programming problem has a unique solution (x, y), it must be a ______ of the feasible region.

5. Suppose two boundary lines in a system of linear inequalities intersect, but the point of intersection is not a vertex of the feasible region. Describe how this is possible.

6. Describe the conditions necessary for a linear programming problem to have multiple solutions. (*Hint:* Consider the diagram in Figure 8.19, and the slope of the line from the objective function.)

► DEVELOPING YOUR SKILLS

Determine whether the ordered pairs given are solutions.

7. $2x + y > 3$; (0, 0), (3, −5), (−3, −4), (−3, 9)

8. $3x - y > 5$; (0, 0), (4, −1), (−1, −5), (1, −2)

9. $4x - 2y \le -8$; (0, 0), (−3, 5), (−3, −2), (−1, 1)

10. $3x + 5y \ge 15$; (0, 0), (3, 5), (−1, 6), (7, −3)

Solve the linear inequalities by shading the appropriate half plane.

11. $x + 2y < 8$

12. $x - 3y > 6$

13. $2x - 3y \ge 9$

14. $4x + 5y \ge 15$

Determine whether the ordered pairs given are solutions to the accompanying system.

15. $\begin{cases} 5y - x \ge 10 \\ 5y + 2x \le -5 \end{cases}$; $(-2, 1), (-5, -4), (-6, 2), (-8, 2.2)$

16. $\begin{cases} 8y + 7x \ge 56 \\ 3y - 4x \ge -12 \\ y \ge 4 \end{cases}$; $(1, 5), (4, 6), (8, 5), (5, 3)$

Solve each system of inequalities by graphing the solution region. Verify the solution using a test point.

17. $\begin{cases} x + 2y \ge 1 \\ 2x - y \le -2 \end{cases}$

18. $\begin{cases} -x + 5y < 5 \\ x + 2y \ge 1 \end{cases}$

19. $\begin{cases} 3x + y > 4 \\ x > 2y \end{cases}$

20. $\begin{cases} 3x \le 2y \\ y \ge 4x + 3 \end{cases}$

21. $\begin{cases} 2x + y < 4 \\ 2y > 3x + 6 \end{cases}$

22. $\begin{cases} x - 2y < -7 \\ 2x + y > 5 \end{cases}$

23. $\begin{cases} x > -3y - 2 \\ x + 3y \le 6 \end{cases}$

24. $\begin{cases} 2x - 5y < 15 \\ 3x - 2y > 6 \end{cases}$

25. $\begin{cases} 5x + 4y \ge 20 \\ x - 1 \ge y \end{cases}$

26. $\begin{cases} 10x - 4y \le 20 \\ 5x - 2y > -1 \end{cases}$

27. $\begin{cases} 0.2x > -0.3y - 1 \\ 0.3x + 0.5y \le 0.6 \end{cases}$

28. $\begin{cases} x > -0.4y - 2.2 \\ x + 0.9y \le -1.2 \end{cases}$

29. $\begin{cases} y \le \frac{3}{2}x \\ 4y \ge 6x - 12 \end{cases}$

30. $\begin{cases} 3x + 4y > 12 \\ y < \frac{2}{3}x \end{cases}$

31. $\begin{cases} \frac{-2}{3}x + \frac{3}{4}y \le 1 \\ \frac{1}{2}x + 2y \ge 3 \end{cases}$

32. $\begin{cases} \frac{1}{2}x + \frac{2}{5}y \le 5 \\ \frac{5}{6}x - 2y \ge -5 \end{cases}$

33. $\begin{cases} x - y \ge -4 \\ 2x + y \le 4 \\ x \ge 1, y \ge 0 \end{cases}$

34. $\begin{cases} 2x - y \le 5 \\ x + 3y \le 6 \\ x \ge 1 \end{cases}$

35. $\begin{cases} y \le x + 3 \\ x + 2y \le 4 \\ y \ge 0 \end{cases}$

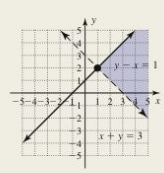
36. $\begin{cases} 4y < 3x + 12 \\ x \ge 0 \\ y \le x + 1 \end{cases}$

37. $\begin{cases} 2x + 3y \le 18 \\ x \ge 0 \\ y \ge 0 \end{cases}$

38. $\begin{cases} 8x + 5y \le 40 \\ x \ge 0 \\ y \ge 0 \end{cases}$

Use the equations given to write the system of linear inequalities represented by each graph.

39.



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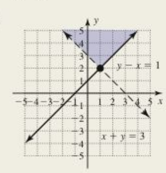

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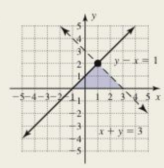
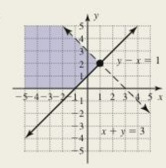


42.



Determine which of the ordered pairs given produces the maximum value of $f(x, y)$.

43. $f(x, y) = 12x + 10y$; $(0, 0), (0, 8.5), (7, 0), (5, 3)$

44. $f(x, y) = 50x + 45y$; $(0, 0), (0, 21), (15, 0), (7.5, 12.5)$

Determine which of the ordered pairs given produces the minimum value of $f(x, y)$.

45. $f(x, y) = 8x + 15y$; $(0, 20), (35, 0), (5, 15), (12, 11)$

46. $f(x, y) = 75x + 80y$; $(0, 9), (10, 0), (4, 5), (5, 4)$

For Exercises 47 and 48, find the *maximum* value of the objective function $f(x, y) = 8x + 5y$ given the constraints shown.

47. $\begin{cases} x + 2y \le 6 \\ 3x + y \le 8 \\ x \ge 0 \\ y \ge 0 \end{cases}$

48. $\begin{cases} 2x + y \le 7 \\ x + 2y \le 5 \\ x \ge 0 \\ y \ge 0 \end{cases}$

For Exercises 49 and 50, find the *minimum* value of the objective function $f(x, y) = 36x + 40y$ given the constraints shown.

49. $\begin{cases} 3x + 2y \ge 18 \\ 3x + 4y \ge 24 \\ x \ge 0 \\ y \ge 0 \end{cases}$

50. $\begin{cases} 2x + y \ge 10 \\ x + 4y \ge 3 \\ x \ge 2 \\ y \ge 0 \end{cases}$

▶ WORKING WITH FORMULAS

Area Formulas

51. The area of a triangle is usually given as $A = \frac{1}{2}BH$, where B and H represent the base and height, respectively. The area of a rectangle can be stated as $A = BH$. If the base of both the triangle and rectangle is equal to 20 in., what are the possible values for H if the triangle must have an area *greater than* 50 in^2 and the rectangle must have an area *less than* 200 in^2?

Volume Formulas

52. The volume of a cone is $V = \frac{1}{3}\pi r^2h$, where r is the radius of the base and h is the height. The volume of a cylinder is $V = \pi r^2h$. If the radius of both the cone and cylinder is equal to 10 cm, what are the possible values for h if the cone must have a volume *greater than* 200 cm^3 and the volume of the cylinder must be *less than* 850 cm^3?

▶ APPLICATIONS

Write a system of linear inequalities that models the information given, then solve.

53. Gifts to grandchildren: Grandpa Augustus is considering how to divide a $50,000 gift between his two grandchildren, Julius and Anthony. After weighing their respective positions in life and family responsibilities, he decides he must bequeath at least $20,000 to Julius, but no more than $25,000 to Anthony. Determine the possible ways that Grandpa can divide the $50,000.

54. Guns versus butter: Every year, governments around the world have to make the decision as to how much of their revenue must be spent on national defense and domestic improvements (guns versus butter). Suppose total revenue for these two needs was $120 billion, and a government decides they need to spend at least $42 billion on butter and no more than $80 billion on defense. Determine the possible amounts that can go toward each need.

Solve the following linear programming problems.

55. Land/crop allocation: A farmer has 500 acres of land to plant corn and soybeans. During the last few years, market prices have been stable and the farmer anticipates a profit of $900 per acre on the corn harvest and $800 per acre on the soybeans. The farmer must take into account the time it takes to plant and harvest each crop, which is 3 hr/acre for corn and 2 hr/acre for soybeans. If the farmer has at most 1300 hr to plant, care for, and harvest each crop, how many acres of each crop should be planted in order to maximize profits?

56. Coffee blends: The owner of a coffee shop has decided to introduce two new blends of coffee in order to attract new customers—a *Deluxe Blend* and a *Savory Blend.* Each pound of the deluxe blend contains 30% Colombian and 20% Arabian coffee, while each pound of the savory blend contains 35% Colombian and 15% Arabian coffee (the remainder of each is made up of cheap and plentiful domestic varieties). The profit on the deluxe blend will be $1.25 per pound, while the profit on the savory blend will be $1.40 per pound. How many pounds of each should the owner make in order to maximize profit, if only 455 lb of Colombian coffee and 250 lb of Arabian coffee are currently available?

57. Manufacturing screws: A machine shop manufactures two types of screws—sheet metal screws and wood screws, using three different machines. Machine Moe can make a sheet metal screw in 20 sec and a wood screw in 5 sec. Machine Larry can make a sheet metal screw in 5 sec and a wood screw in 20 sec. Machine Curly, the newest machine (nyuk, nyuk) can make a sheet metal screw in 15 sec and a wood screw in 15 sec. (Shemp couldn't get a job because he failed the math portion of the employment exam.) Each machine can operate for only 3 hr each day before shutting down for maintenance. If sheet metal screws sell for 10 cents and wood screws sell for 12 cents, how many of each type should the machines be programmed to make in order to maximize revenue? (*Hint:* Standardize time units.)

58. Hauling hazardous waste: A waste disposal company is contracted to haul away some hazardous waste material. A full container of liquid waste weighs 800 lb and has a volume of 20 ft^3. A full container of solid waste weighs 600 lb and has a volume of 30 ft^3. The trucks used can carry at most 10 tons (20,000 lb) and have a carrying volume of 800 ft^3. If the trucking company makes $300 for disposing of liquid waste and $400 for disposing of solid waste, what is the maximum revenue per truck that can be generated?

59. Maximizing profit—food service: P. Barrett & Justin, Inc., is starting up a fast-food restaurant specializing in peanut butter and jelly sandwiches. Some of the peanut butter varieties are smooth, crunchy, reduced fat, and reduced sugar. The jellies will include those expected and common, as well as some exotic varieties such as kiwi and mango. Independent research has determined the two most popular sandwiches will be the traditional P&J (smooth peanut butter and grape jelly), and the Double-T (three slices of bread). A traditional P&J uses 2 oz of peanut butter and 3 oz of jelly. The Double-T uses 4 oz of peanut butter and 5 oz of jelly. The traditional sandwich will be priced at \$2.00, and a Double-T at \$3.50. If the restaurant has 250 oz of smooth peanut butter and 345 oz of grape jelly on hand for opening day, how many of each should they make and sell to maximize revenue?

60. Maximizing profit—construction materials: Mooney and Sons produces and sells two varieties of concrete mixes. The mixes are packaged in 50-lb bags. Type A is appropriate for finish work, and contains 20 lb of cement and 30 lb of sand. Type B is appropriate for foundation and footing work, and contains 10 lb of cement and 20 lb of sand. The remaining weight comes from gravel aggregate. The profit on type A is \$1.20/bag, while the profit on type B is \$0.90/bag. How many bags of each should the company make to maximize profit, if 2750 lb of cement and 4500 lb of sand are currently available?

61. Minimizing shipping costs: An oil company is trying to minimize shipping costs from its two primary refineries in Tulsa, Oklahoma, and Houston, Texas. All orders within the region are shipped from one of these two refineries. An order for 220,000 gal comes in from a location in Colorado, and another for 250,000 gal from a location in Mississippi. The Tulsa refinery has 320,000 gal ready to ship, while the Houston refinery has 240,000 gal. The cost of transporting each gallon to Colorado is \$0.05 from Tulsa and \$0.075 from Houston. The cost of transporting each gallon to Mississippi is \$0.06 from Tulsa and \$0.065 from Houston. How many gallons should be distributed from each refinery to minimize the cost of filling both orders?

62. Minimizing transportation costs: Robert's Las Vegas Tours needs to drive 375 people and 19,450 lb of luggage from Salt Lake City, Utah, to Las Vegas, Nevada, and can charter buses from two companies. The buses from company X carry 45 passengers and 2750 lb of luggage at a cost of \$1250 per trip. Company Y offers buses that carry 60 passengers and 2800 lb of luggage at a cost of \$1350 per trip. How many buses should be chartered from each company in order for Robert to minimize the cost?

▶ EXTENDING THE CONCEPT

63. Graph the feasible region formed by the system $\begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 3 \\ x \leq 3 \end{cases}$. How would you describe this region?

Select random points within the region or on any boundary line and evaluate the objective function $f(x, y) = 4.5x + 7.2y$. At what point (x, y) will this function be maximized? How does this relate to optimal solutions to a linear programing problem?

64. Find the maximum value of the objective function $f(x, y) = 22x + 15y$ given the constraints $\begin{cases} 2x + 5y \leq 24 \\ 3x + 4y \leq 29 \\ x + 6y \leq 26 \\ x \geq 0 \\ y \geq 0 \end{cases}$.

▶ MAINTAINING YOUR SKILLS

65. (5.3) Given the point $(-3, 4)$ is on the terminal side of angle θ with θ in standard position, find

a. $\cos\theta$

b. $\csc\theta$

c. $\cot\theta$

66. (3.7) Solve the rational inequality. Write your answer in interval notation. $\frac{x+2}{x^2-9} > 0$

67. (3.8) The resistance to current flow in copper wire varies directly as its length and inversely as the square of its diameter. A wire 8 m long with a 0.004-m diameter has a resistance of 1500 Ω. Find the resistance in a wire of like material that is 2.7 m long with a 0.005-m diameter.

68. (6.4) Use a half-angle identity to find an exact value for $\cos\left(\frac{7\pi}{12}\right)$.

SUMMARY AND CONCEPT REVIEW

SECTION 8.1 Linear Systems in Two Variables with Applications

KEY CONCEPTS

- A *solution* to a linear system in two variables is an ordered pair (x, y) that makes all equations in the system true.
- Since every point on the graph of a line satisfies the equation of that line, a point where two lines intersect must satisfy both equations and is a solution of the system.
- A system with at least one solution is called a *consistent system.*
- If the lines have different slopes, there is a unique solution to the system (they intersect at a single point). The system is called a *consistent* and *independent system.*
- If the lines have equal slopes and the same y-intercept, they form identical or *coincident* lines. Since one line is right atop the other, they intersect at all points with an infinite number of solutions. The system is called a *consistent* and *dependent system.*
- If the lines have equal slopes but different y-intercepts, they will never intersect. The system has no solution and is called an *inconsistent system.*

EXERCISES

Solve each system by graphing. If the solution does not have integer values indicate your solution is an estimate. If the system is inconsistent or dependent, so state.

1. $\begin{cases} 3x - 2y = 4 \\ -x + 3y = 8 \end{cases}$ **2.** $\begin{cases} 0.2x + 0.5y = -1.4 \\ x - 0.3y = 1.4 \end{cases}$ **3.** $\begin{cases} 2x + y = 2 \\ x - 2y = 4 \end{cases}$

Solve using substitution. Indicate whether each system is consistent, inconsistent, or dependent. Write unique solutions as an ordered pair.

4. $\begin{cases} y = 5 - x \\ 2x + 2y = 13 \end{cases}$ **5.** $\begin{cases} x + y = 4 \\ 0.4x + 0.3y = 1.7 \end{cases}$ **6.** $\begin{cases} x - 2y = 3 \\ x - 4y = -1 \end{cases}$

Solve using elimination. Indicate whether each system is consistent, inconsistent, or dependent. Write unique solutions as an ordered pair.

7. $\begin{cases} 2x - 4y = 10 \\ 3x + 4y = 5 \end{cases}$ **8.** $\begin{cases} -x + 5y = 8 \\ x + 2y = 6 \end{cases}$ **9.** $\begin{cases} 2x = 3y + 6 \\ 2.4x + 3.6y = 6 \end{cases}$

10. When it was first constructed in 1968, the John Hancock building in Chicago, Illinois, was the tallest structure in the world. In 1985, the Sears Tower in Chicago became the world's tallest structure. The Sears Tower is 323 ft taller than the John Hancock Building, and the sum of their heights is 2577 ft. How tall is each structure?

SECTION 8.2 Linear Systems in Three Variables with Applications

KEY CONCEPTS

- The graph of a linear equation in three variables is a *plane.*
- Systems in three variables can be solved using substitution and elimination.
- A linear system in three variables has the following possible solution sets:
  - If the planes intersect at a point, the system has one *unique solution* (x, y, z).
  - If the planes intersect at a line, the system has *linear dependence* and the solution (x, y, z) can be written as linear combinations of a single variable *(a parameter)*.
  - If the planes are *coincident,* the equations in the system differ by a constant multiple, meaning they are all "disguised forms" of the *same equation.* The solutions have *coincident dependence,* and the solution set can be represented by any one of the equations.
  - In all other cases, the system has *no solutions* and is an inconsistent system.

EXERCISES

Solve using elimination. If a system is inconsistent or dependent, so state. For systems with linear dependence, give the answer as an ordered triple using a parameter.

11. $\begin{cases} x + y - 2z = -1 \\ 4x - y + 3z = 3 \\ 3x + 2y - z = 4 \end{cases}$

12. $\begin{cases} -x + y + 2z = 2 \\ x + y - z = 1 \\ 2x + y + z = 4 \end{cases}$

13. $\begin{cases} 3x + y + 2z = 3 \\ x - 2y + 3z = 1 \\ 4x - 8y + 12z = 7 \end{cases}$

Solve using a system of three equations in three variables.

14. In one version of the card game Gin Rummy, numbered cards (N) 2 through 9 are worth 5 points, the 10s and all face cards (F) are worth 10 points, and aces (A) are worth 20 points. At the moment his opponent said "Gin!" Kenan had 12 cards in his hand, worth a total value of 125 points. If the value of his aces and face cards was equal to four times the value of his numbered cards, how many aces, face cards, and numbered cards was he holding?

15. A vending machine accepts nickels, dimes, and quarters. At the end of a week, there is a total of $536 in the machine. The number of nickels and dimes combined is 360 more than the number of quarters. The number of quarters is 110 more than twice the number of nickels. How many of each type of coin are in the machine?

SECTION 8.3 Nonlinear Systems of Equations and Inequalities

KEY CONCEPTS

- Nonlinear systems of equations can be solved using substitution or elimination.
- First identify the graphs of the equations in the system to help determine the number of solutions possible.
- For nonlinear systems of inequalities, graph the related equation for each inequality given, then use a test point to decide what region to shade as the solution.
- The solution for the system is the overlapping region (if it exists) created by solutions to the individual inequalities.
- If the boundary is included, graph it using a solid line; if the boundary is not included use a dashed line.

EXERCISES

Solve Exercises 16–21 using substitution or elimination. Identify the graph of each relation before you begin.

16. $\begin{cases} x^2 + y^2 = 25 \\ y - x = -1 \end{cases}$

17. $\begin{cases} x = y^2 - 1 \\ x + 4y = -5 \end{cases}$

18. $\begin{cases} -x^2 + y = -1 \\ x^2 + y^2 = 7 \end{cases}$

19. $\begin{cases} x^2 + y^2 = 10 \\ y - 3x^2 = 0 \end{cases}$

20. $\begin{cases} y \le x^2 - 2 \\ x^2 + y^2 \le 16 \end{cases}$

21. $\begin{cases} x^2 + y^2 > 9 \\ x^2 + y \le -3 \end{cases}$

SECTION 8.4 Systems of Linear Inequalities and Linear Programming

KEY CONCEPTS

- As in Section 8.3, to solve a *system of linear inequalities,* we find the intersecting or *overlapping areas* of the solution regions from the individual inequalities. The common area is called the *feasible region.*
- The process known as *linear programming* seeks to *maximize* or *minimize* the value of a given quantity under certain *constraints* or restrictions.
- The quantity we attempt to maximize or minimize is called the *objective function.*
- The solution(s) to a linear programming problem *occur at one of the corner points of the feasible region.*
- The process of solving a linear programming application contains these six steps:
  - Identify the main objective and the decision variables.
  - Write the objective function in terms of these variables.
  - Organize all information in a table, using the decision variables and constraints.
  - Fill in the table with the information given and write the constraint inequalities.
  - Graph the constraint inequalities and determine the feasible region.
  - Identify all corner points of the feasible region and test these points in the objective function.

EXERCISES

Graph the solution region for each system of linear inequalities and verify the solution using a test point.

22. $\begin{cases} -x - y > -2 \\ -x + y < -4 \end{cases}$

23. $\begin{cases} x - 4y \le 5 \\ -x + 2y \le 0 \end{cases}$

24. $\begin{cases} x + 2y \ge 1 \\ 2x - y \le -2 \end{cases}$

25. Carefully graph the feasible region for the system of inequalities shown, then maximize the objective function: $f(x, y) = 30x + 45y$ $\begin{cases} x + y \le 7 \\ 2x + y \le 10 \\ 2x + 3y \le 18 \\ x \ge 0, y \ge 0 \end{cases}$

26. After retiring, Oliver and Lisa Douglas buy and work a small farm (near Hooterville) that consists mostly of milk cows and egg-laying chickens. Although the price of a commodity is rarely stable, suppose that milk sales bring in an average of \$85 per cow and egg sales an average of \$50 per chicken over a period of time. During this time period, the new ranchers estimate that care and feeding of the animals took about 3 hr per cow and 2 hr per chicken, while maintaining the related equipment took 2 hr per cow and 1 hr per chicken. How many animals of each type should be maintained in order to maximize profits, if at most 1000 hr can be spent on care and feeding, and at most 525 hr on equipment maintenance?

MIXED REVIEW

1. Write the equations in each system in slope-intercept form, then state whether the system is consistent/independent, consistent/dependent, or inconsistent. Do not solve.

a. $\begin{cases} -3x + 5y = 10 \\ 6x + 20 = 10y \end{cases}$ **b.** $\begin{cases} 4x - 3y = 9 \\ -2x + 5y = -10 \end{cases}$

c. $\begin{cases} x - 3y = 9 \\ -6y + 2x = 10 \end{cases}$

2. Solve by graphing.

$\begin{cases} x - 2y = 6 \\ -2x + y = -9 \end{cases}$

3. Solve using a substitution.

$\begin{cases} 2x + 3y = 5 \\ -x + 5y = 17 \end{cases}$

4. Solve using elimination.

$\begin{cases} 7x - 4y = -5 \\ 3x + 2y = 9 \end{cases}$

5. A burrito stand sells a veggie burrito for \$2.45 and a beef burrito for \$2.95. If the stand sold 54 burritos in one day, for a total revenue of \$148.80, how many of each did they sell?

Solve using elimination.

6. $\begin{cases} x + 2y - 3z = -4 \\ -3x + 4y + z = 1 \\ 2x - 6y + z = 1 \end{cases}$

7. $\begin{cases} 0.1x - 0.2y + z = 1.7 \\ 0.3x + y - 0.1z = 3.6 \\ -0.2x - 0.1y + 0.2z = -1.7 \end{cases}$

Solve using elimination. If the system has coincident dependence, state the solution set using set notation.

8. $\begin{cases} x - 2y + 3z = 4 \\ 2x + y - z = 1 \\ 5x + z = 2 \end{cases}$ **9.** $\begin{cases} x - 2y + 3z = 4 \\ 2x + y - z = 1 \\ 5x + z = 6 \end{cases}$

10. It's the end of another big day at the circus, and the clowns are putting away their riding equipment—a motley collection of unicycles, bicycles, and tricycles. As she loads them into the storage shed, Trixie counts 21 cycles in all with a total of 40 wheels. In addition, she notes the number of bicycles is one fewer than twice the number of tricycles. How many cycles of each type do the clowns use?

Solve each system of inequalities by graphing the solution region.

11. $\begin{cases} 2x + y \le 4 \\ x - 3y > 6 \end{cases}$ **12.** $\begin{cases} 2x + y < 3 \\ 2x + y > -3 \end{cases}$

13. $\begin{cases} x - 2y \ge 5 \\ x \le 2y \end{cases}$

14. Graph the solution region for the system of inequalities. $\begin{cases} 4x + 2y \le 14 \\ 2x + 3y \le 15 \\ y \ge 0 \\ x \ge 0 \end{cases}$

15. Maximize $P(x, y) = 2.5x + 3.75y$, given
$$\begin{cases} x + y \le 8 \\ x + 2y \le 14 \\ 4x + 3y \le 30 \\ x, y \ge 0 \end{cases}$$

16. Solve the system by substitution.
$$\begin{cases} x^2 + y^2 = 1 \\ x - y = -1 \end{cases}$$

17. Solve using elimination: $\begin{cases} 4x^2 - y^2 = -9 \\ x^2 + 3y^2 = 79 \end{cases}$

18. Solve using the method of your choice.
$$\begin{cases} y + 1 = x^2 \\ x^2 + y = 7 \end{cases}$$

Solve each system of inequalities.

19. $\begin{cases} x + y > 1 \\ x^2 + y^2 \ge 16 \end{cases}$

20. $\begin{cases} x^2 + y^2 < 4 \\ x^2 + y < 0 \end{cases}$

PRACTICE TEST

Solve each system and state whether the system is consistent, inconsistent, or dependent.

1. Solve graphically:
$$\begin{cases} 3x + 2y = 12 \\ -x + 4y = 10 \end{cases}$$

2. Solve using substitution:
$$\begin{cases} 3x - y = 2 \\ -7x + 4y = -6 \end{cases}$$

3. Solve using elimination:
$$\begin{cases} 5x + 8y = 1 \\ 3x + 7y = 5 \end{cases}$$

4. Solve using elimination:
$$\begin{cases} x + 2y - z = -4 \\ 2x - 3y + 5z = 27 \\ -5x + y - 4z = -27 \end{cases}$$

5. Solve using elimination:
$$\begin{cases} 2x - y + z = 4 \\ -x + 2z = 1 \\ x - 2y + 8z = 11 \end{cases}$$

6. Find values of a and b such that $(2, -1)$ is a solution of the system.
$$\begin{cases} ax - by = 12 \\ bx + ay = -1 \end{cases}$$

Create a system of equations to model each exercise, then solve using the method of your choice.

7. The perimeter of a "legal-size" paper is 114.3 cm. The length of the paper is 7.62 cm less than twice the width. Find the dimensions of a legal-size sheet of paper.

8. The island nations of Tahiti and Tonga have a combined land area of 692 mi^2. Tahiti's land area is 112 mi^2 more than Tonga's. What is the land area of each island group?

9. Many years ago, two cans of corn (C), 3 cans of green beans (B), and 1 can of peas (P) cost \$1.39. Three cans of C, 2 of B, and 2 of P cost \$1.73. One can of C, 4 of B, and 3 of P cost \$1.92. What was the price of a single can of C, B, and P?

10. After inheriting \$30,000 from a rich aunt, David decides to place the money in three different investments: a savings account paying 5%, a bond account paying 7%, and a stock account paying 9%. After 1 yr he earned \$2080 in interest. Find how much was invested at each rate if \$8000 less was invested at 9% than at 7%.

11. Solve the system of inequalities by graphing.
$$\begin{cases} x - y \le 2 \\ x + 2y \ge 8 \end{cases}$$

12. Maximize the objective function: $P = 50x - 12y$
$$\begin{cases} x + 2y \le 8 \\ 8x + 5y \ge 40 \\ x, \quad y \ge 0 \end{cases}$$

Solve the linear programming problem.

13. A company manufactures two types of T-shirts, a plain T-shirt and a deluxe monogrammed T-shirt. To produce a plain shirt requires 1 hr of working time on machine A and 2 hr on machine B. To produce a deluxe shirt requires 1 hr on machine A and 3 hr on machine B. Machine A is available for at most 50 hr/week, while machine B is available for at most 120 hr/week. If a plain shirt can be sold at a profit of \$4.25 each and a deluxe shirt can be sold at a profit of \$5.00 each, how many of each should be manufactured to maximize the profit?

Solve each nonlinear system using the technique of your choice.

14.
$$\begin{cases} x^2 + y^2 = 16 \\ y - x = 2 \end{cases}$$

15.
$$\begin{cases} 4y - x^2 = 1 \\ y^2 + x^2 = 4 \end{cases}$$

16. A support bracket on the frame of a large ship is a steel right triangle with a hypotenuse of 25 ft and a perimeter of 60 ft. Find the lengths of the other sides using a system of nonlinear equations.

17. Solve $\begin{cases} x^2 - y \le 2 \\ x - y^2 \ge -2 \end{cases}$.

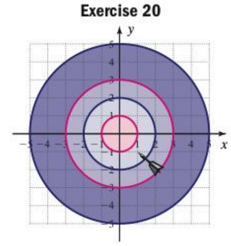
18. Write a system of inequalities that describes all the points with positive y-values that are less than 3 units away from the origin.

19. Solve the system of inequalities.

$$\begin{cases} 2x - y \le -1 \\ 3x + 2y \ge 2 \\ x - 3y \ge -3 \end{cases}$$

20. Write a system of four inequalities that describes the location of the dart on the dartboard shown.

Exercise 20



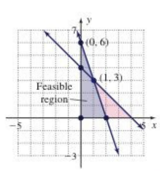
CALCULATOR EXPLORATION AND DISCOVERY

Optimal Solutions and Linear Programming

In this exercise, we'll use a graphing calculator to explore various areas of the feasible region, repeatedly evaluating the objective function to see where the maximal values (optimal solutions) seem to "congregate." If all goes as expected, ordered pairs nearest to a vertex should give relatively larger values. To demonstrate, we'll use Example 6 from Section 8.4, stated below.

Example 6 ▶ Find the maximum value of the objective function $f(x, y) = 2x + y$ given the constraints shown:

$$\begin{cases} x + y \le 4 \\ 3x + y \le 6 \\ x \ge 0 \\ y \ge 0 \end{cases}.$$



Solution ▶ The feasible region is shown in lavender. There are four corner points to this region: (0, 0), (0, 4), (2, 0), and (1, 3), and we found $f(x, y)$ was maximized at (1, 3): $f(1, 3) = 5$.

To explore this feasible region in terms of the objective function $f(x, y) = 2x + y$, enter the boundary lines $Y_1 = -x + 4$ and $Y_2 = -3x + 6$ on the Y= screen. However, instead of shading below the lines to show the feasible region (using the ◣ feature to the extreme left), we shade above both lines (using the ◥ feature) so that the feasible region remains clear. Setting the window size at $x \in [0, 3]$ and $y \in [-1.5, 4]$ produces Figure 8.22. Using YMin = −1.5 will leave a blank area just below QI that enables you to explore the feasible region as the *x*- and *y*-values are displayed. Next we place the calculator in "split-screen" mode so that we can view the graph and the home screen simultaneously. Press the MODE key and notice the second-to-last line reads **Full Horiz G-T.** The **Full** (screen) mode is the default operating mode. The **Horiz** mode splits the screen horizontally, placing the graph directly above a shorter home screen. Highlight **Horiz,** then press ENTER and GRAPH to have the calculator reset the screen in this mode. The TI-84 Plus has a free-moving cursor that is brought into view by pressing the left ◄ or right ► arrow (Figure 8.23). A useful feature of this cursor is that it automatically stores the current X value as the variable X (X,T,θ,n or ALPHA STO→) and the current Y value as the variable Y (ALPHA 1), which allows us to evaluate the objective function $f(x, y) = 2x + y$ right on the home screen. To access the graph and free-moving cursor you must press GRAPH each time, and to access the home screen you must press 2nd MODE (**QUIT**) each time. Begin by moving the cursor to the upper-left corner of the region, near the *y*-intercept [we stopped at (~0.0957, 3.26)]. Once you have the cursor "tucked up into the corner," press 2nd MODE (**QUIT**) to get to the home screen, then enter the objective function: 2X + Y. Pressing ENTER evaluates the function for the values indicated by the cursor's location

Figure 8.22

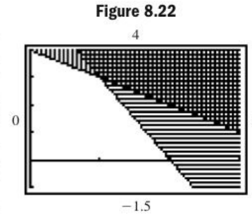
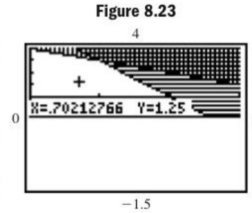


Figure 8.23



(Figure 8.24). It appears the value of the objective function for points (x, y) in this corner are close to 4, and it's no accident that at the corner point (0, 4) the maximum value is in fact 4. Repeating this procedure for the lower-right corner suggests the maximum value near (2, 0) is also 4. Finally, press GRAPH to explore the region in the upper-right corner, where the lines intersect. Move the cursor to this vicinity, locate it very near the point of intersection [we stopped at $(\sim 0.957, 2.71\overline{6})$] and return to the home screen and evaluate (Figure 8.25). The value of the objective function is near 5 in this corner of the region, and at the corner point (1, 3) the maximum value is 5.

Figure 8.24

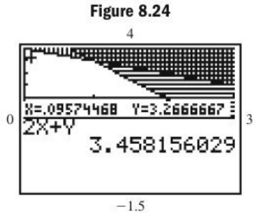
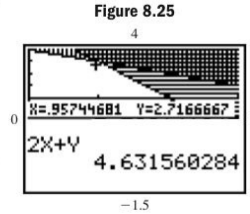


Figure 8.25



Exercise 1: The feasible region for the system given to the right has four corner points. Use the ideas here to explore the area near each corner point of the feasible region to determine which point is the likely candidate to produce the *minimum* value of the objective function $f(x, y) = 2x + 4y$. Then solve the linear programming problem to verify your guess.

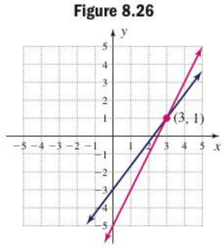
$$\begin{cases} 2x + 2y \le 15 \\ x + y \ge 6 \\ x + 4y \ge 9 \\ x, y \ge 0 \end{cases}$$

STRENGTHENING CORE SKILLS

Understanding Why Elimination and Substitution "Work"

When asked to solve a system of two equations in two variables, we first select an appropriate method. In Section 8.1, we learned three basic techniques: graphing, substitution, and elimination. In this feature, we'll explore how these methods are related using Example 2 from Section 8.1 where we were asked to solve the system $\begin{cases} 4x - 3y = 9 \\ -2x + y = -5 \end{cases}$ by graphing. The resulting graph, shown here in Figure 8.26, clearly indicates the solution is (3, 1).

Figure 8.26



As for the elimination method, either x or y can be easily eliminated. If the second equation is multiplied by 2, the x-coefficients will be additive inverses, and the sum results in an equation with y as the only unknown.

$$\begin{array}{ll} \text{R1} & 4x - 3y = 9 \\ + & \\ \underline{\text{2R2}} & \underline{-4x + 2y = -10} \\ \text{sum} & -y = -1 \end{array}$$

The result is $y = 1$ but remember, this is a system of *linear equations,* and $y = 1$ is still the equation of a (horizontal) line. Since the system $\begin{cases} 4x - 3y = 9 \\ y = 1 \end{cases}$ is equivalent to the original, it will have the same solution set. In Figure 8.27, we note the point of intersection for the new system is still (3, 1). If we eliminate the y-terms instead (using R1 + 3R2), the result is $x = 3$, which is also the equation of a (vertical) line. Creating another equivalent system using this line produces $\begin{cases} x = 3 \\ y = 1 \end{cases}$ and the graph shown in Figure 8.28, where the vertical and horizontal lines intersect at (3, 1), making the solution trivial.

Figure 8.27

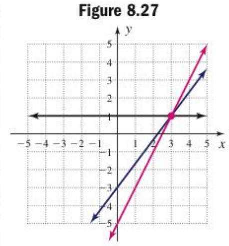
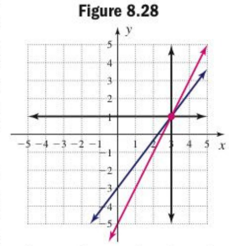


Figure 8.28



Note: Here we see a close connection to solving general equations, in that the goal is to write a series of equivalent yet simpler equations, continuing until the solution is obvious.

As for the substitution method, consider the second equation written as $y = 2x - 5$. This equation represents every point (x, y) on its graph, meaning the relationship for the ordered pair solutions can also be written $(x, 2x - 5)$. The same thing can be said for the line $4x - 3y = 9$, with its ordered pair solutions represented by $(x, \frac{4}{3}x - 3)$. At the point of intersection the y-coordinates must be identical, giving $2x - 5 = \frac{4}{3}x - 3$. In other words, we can substitute $2x - 5$ for y in the first equation, or $\frac{4}{3}x - 3$ for y in the second equation, with both yielding the correct solution. Substituting $2x - 5$ for y in the first equation gives

$$\begin{aligned} 4x - 3(2x - 5) &= 9 && \text{substitute } 2x - 5 \text{ for } y \\ 4x - 6x + 15 &= 9 && \text{expand} \\ -2x &= -6 && \text{simplify} \\ x &= 3 \end{aligned}$$

and the solution $(x, 2x - 5)$ becomes $(3, 2(3) - 5)$ or $(3, 1)$.

All three methods will produce the same solution, and the best method to use at the time often depends on the nature of the system given, or even personal preference.

Exercise 1: Solve the system by (a) graphing, (b) elimination, and (c) substitution. Which method was most efficient for solving this system?

$$\begin{cases} 2x + y = 2 \\ 4x + 3y = 8 \end{cases}$$

CUMULATIVE REVIEW CHAPTERS 1–8

Graph each of the following. Include x- and y-intercepts and other important features of each graph.

1. $y = \frac{2}{3}x + 2$

2. $f(x) = |x - 2| + 3$

3. $g(x) = \sqrt{x - 3} + 1$

4. $h(x) = \frac{1}{x - 1} + 2$

5. $g(x) = (x - 3)(x + 1)(x + 4)$

6. $y = 2^x + 3$

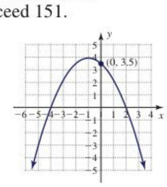
7. $y = 2\sin\left(x - \frac{\pi}{4}\right)$

8. $y = -\tan\left(2x + \frac{\pi}{2}\right)$

9. Graph $h(x) = \frac{9 - x^2}{x^2 - 4}$. Give the coordinates of all intercepts and the equation of all asymptotes.

10. Chance's skill at bowling is slowly improving with practice. In February his average score was 102, but by May he had raised his average to 126. Assuming the relationship is linear, (a) find the equation of the line, (b) explain what the slope of the line means in this context, and (c) predict the month when Chance's average score will exceed 151.

11. Determine the following for the graph shown to the right. Use interval notation as appropriate.



a. domain

b. range

c. interval(s) where $f(x)$ is increasing or decreasing

d. interval(s) where $f(x)$ is constant

e. location of any maximum or minimum value(s)

f. interval(s) where $f(x)$ is positive or negative

g. the average rate of change using $(-4, 0)$ and $(-2, 3.5)$.

12. Suppose the cost of making a rubber ball is given by $C(x) = 3x + 10$, where x is the number of balls in hundreds. If the revenue from the sale of these balls is given by $R(x) = -x^2 + 123x - 1990$, find the profit function (Profit = Revenue − Cost). How many balls should be produced and sold to obtain the maximum profit? What is this maximum profit?

13. Solve each equation.

a. $\sqrt{x} - 2 = \sqrt{3x + 4}$

b. $x^{\frac{3}{2}} + 8 = 0$

c. $2|n + 4| + 3 = 13$

d. $x^2 - 6x + 13 = 0$

e. $x^{-2} - 3x^{-1} - 40 = 0$

f. $4 \cdot 2^{x+1} = \frac{1}{8}$

g. $3^{x-2} = 7$

h. $\log_3 81 = x$

i. $\log_3 x + \log_3(x - 2) = 1$

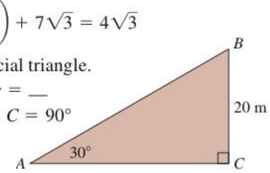
Given $f(x) = 2x - 5$ and $g(x) = 3x^2 + 2x$ find:

14. Solve each equation in $[0, 2\pi)$.

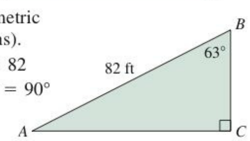
a. $2\sin(2x) + \sqrt{3} = 2\sqrt{3}$

b. $-3\tan\left(x - \frac{\pi}{4}\right) + 7\sqrt{3} = 4\sqrt{3}$

15. Solve using a special triangle.
$a = 20$, $b = \_\_$, $c = \_\_$
$A = 30°$, $B = \_\_$, $C = 90°$



16. Solve using trigonometric ratios (round to tenths).
$a = \_\_$, $b = \_\_$, $c = 82$
$A = \_\_$, $B = 63°$, $C = 90°$



17. State the three Pythagorean identities.

18. Find θ in Quadrant IV given $\theta_r = 32°$.

19. State the value of all six trig functions given $\tan\alpha = -\frac{3}{4}$ with $\cos\alpha > 0$.

20. Verify the following is an identity:
$$\tan^4\alpha = \frac{1 - \sec^2\alpha}{1 - \csc^2\alpha}$$

21. Find the average rate of change in the interval [1.1, 1.2] for $f(x) = x^2 - 3x$.

22. Use the rational roots theorem to factor the polynomial completely: $x^4 - 6x^3 - 13x^2 + 24x + 36$.

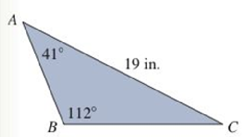
Solve each inequality. Write your answer using interval notation.

23. $x^2 - 3x - 10 < 0$

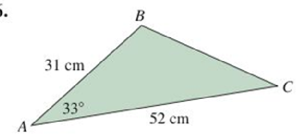
24. $\frac{x - 2}{x + 3} \le 3$

Solve each triangle using either the law of sines or the law of cosines, whichever is appropriate.

25.



26.

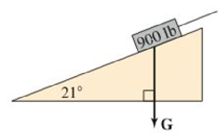


Solve each system using elimination.

27. $\begin{cases} 4x + 3y = 13 \\ -9x + 5y = 6 \end{cases}$

28. $\begin{cases} x + 2y - z = 0 \\ 2x - 5y + 4z = 6 \\ -x + 3y - 4z = -5 \end{cases}$

29. A 900-lb crate is sitting on a ramp that has a 28° incline. Find the force needed to hold the crate stationary.



30. A jet plane is flying at 750 mph on a heading of 30°. There is a strong, 50 mph wind blowing from due south (heading 0°). What is the true course and speed of the plane (relative to the ground)?