9

Matrices and Matrix Applications

CHAPTER OUTLINE

CHAPTER CONNECTIONS

From pediatric and geriatric care, to the training of a modern athlete, dietetic applications have become increasingly effective. In the latter case, athletes generally need high levels of carbohydrates and protein, but only moderate levels of fat. Suppose a physical trainer wants to supply one of her clients with 24 g of fat, 244 g of "carbs," and 40 g of protein for the noontime meal. Knowing the amount of these nutrients contained in certain foods, the trainer can recommend a variety of foods and the amount of each that should be eaten. The matrix operations in this chapter demonstrate how to do this effectively. This application occurs as Exercise 78 in Section 9.3.

Check out these other real-world connections:

- Calculating Contract Totals for Home Improvement Jobs (Section 9.2, Exercise 61)
- Calculating Appropriate Resource Allocation (Section 9.3, Exercise 71)
- Applying the Mean Value Principle of Physics and Thermal Conductivity (Section 9.3, Exercise 73)
- Calculating Area of Norman Windows (Section 9.4, Exercise 49)

9.1 Solving Linear Systems Using Matrices and Row Operations

Learning Objectives

In Section 9.1 you will learn how to:

- ❑ **A.** State the size of a matrix and identify its entries
- ❑ **B.** Form the augmented matrix of a system of equations
- ❑ **C.** Solve a system of equations using row operations
- ❑ **D.** Recognize inconsistent and dependent systems
- ❑ **E.** Solve applications using linear systems

Just as synthetic division streamlines the process of polynomial division, matrices and row operations streamline the process of solving systems using elimination. With the equations of the system in standard form, the location of the variable terms and constant terms are set, and we simply apply the elimination process on the coefficients and constants.

A. Introduction to Matrices

In general terms, a **matrix** is simply a rectangular arrangement of numbers, called the **entries** of the matrix. **Matrices** (plural of matrix) are denoted by enclosing the entries between a left and right bracket, and named using a capital letter, such as $A = \begin{bmatrix} 1 & -3 & 2 \\ 5 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & -2 \\ 1 & 0 & -1 \end{bmatrix}$. They occur in many different sizes as defined by the number of **rows** and **columns** each has, with the number of rows always given first. Matrix A is said to be a 2×3 (two by three) matrix, since it has two rows and three columns. Matrix B is a 3×3 (three by three) matrix.

EXAMPLE 1A ► **Identifying the Size and Entries of a Matrix**

Determine the size of each matrix and identify the entry located in the second row and first column.

a. $C = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$ **b.** $D = \begin{bmatrix} 0.2 & -0.5 & 0.7 & 3.3 \\ -0.4 & 0.3 & 1 & 2 \\ 2.1 & -0.1 & 0.6 & 4.1 \end{bmatrix}$

Solution ► **a.** Matrix C is 3×2. The row 2, column 1 entry is 1.

b. Matrix D is 3×4. The row 2, column 1 entry is -0.4.

If a matrix has the same number of rows and columns, it's called a **square matrix.** Matrix B above is a square matrix, while matrix A is not. For square matrices, the values on a diagonal line *from the upper left to the lower right* are called the **diagonal entries** and are said to be **on the diagonal** of the matrix. When solving systems using matrices, much of our focus is on these diagonal entries.

EXAMPLE 1B ► **Identifying the Diagonal Entries of a Square Matrix**

Name the diagonal entries of each matrix.

a. $E = \begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}$ **b.** $F = \begin{bmatrix} 0.2 & -0.5 & 0.7 \\ -0.4 & 0.3 & 1 \\ 2.1 & -0.1 & 0.6 \end{bmatrix}$

Solution ► **a.** The diagonal entries of matrix E are 1 and -3.

b. For matrix F, the diagonal entries are 0.2, 0.3, and 0.6.

Now try Exercises 7 through 9 ►

☑ **A.** You've just learned how to state the size of a matrix and identify its entries

B. The Augmented Matrix of a System of Equations

A system of equations can be written in matrix form by augmenting or joining the **coefficient matrix,** formed by the variable coefficients, with the **matrix of constants.**

The coefficient matrix for the system $\begin{cases} 2x + 3y - z = 1 \\ x + z = 2 \\ x - 3y + 4z = 5 \end{cases}$ is $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \\ 1 & -3 & 4 \end{bmatrix}$ with column 1 for the coefficients of x, column 2 for the coefficients of y, and so on. The matrix of constants is $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$. These two are joined to form the **augmented matrix,** with a dotted line often used to separate the two as shown here: $\left[\begin{array}{ccc|c} 2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & -3 & 4 & 5 \end{array}\right]$. It's important to note the use of a zero placeholder for the y-variable in the second row of the matrix, signifying there is no y-variable in the corresponding equation.

EXAMPLE 2 ► Forming Augmented Matrices

Form the augmented matrix for each system, and name the diagonal entries of each coefficient matrix.

a. $\begin{cases} 2x + y = 11 \\ -x + 3y = -2 \end{cases}$ **b.** $\begin{cases} x + 4y - z = -10 \\ 2x + 5y + 8z = 4 \\ x - 2y - 3z = -7 \end{cases}$ **c.** $\begin{cases} \frac{1}{2}x + y = -7 \\ x + \frac{2}{3}y + \frac{5}{6}z = \frac{11}{12} \\ -2y - z = -3 \end{cases}$

Solution ► **a.** $\begin{cases} 2x + y = 11 \\ -x + 3y = -2 \end{cases} \longrightarrow \left[\begin{array}{cc|c} 2 & 1 & 11 \\ -1 & 3 & -2 \end{array}\right]$

Diagonal entries: 2 and 3.

b. $\begin{cases} x + 4y - z = -10 \\ 2x + 5y + 8z = 4 \\ x - 2y - 3z = -7 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} 1 & 4 & -1 & -10 \\ 2 & 5 & 8 & 4 \\ 1 & -2 & -3 & -7 \end{array}\right]$

Diagonal entries: 1, 5, and -3.

c. $\begin{cases} \frac{1}{2}x + y = -7 \\ x + \frac{2}{3}y + \frac{5}{6}z = \frac{11}{12} \\ -2y - z = -3 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} \frac{1}{2} & 1 & 0 & -7 \\ 1 & \frac{2}{3} & \frac{5}{6} & \frac{11}{12} \\ 0 & -2 & -1 & -3 \end{array}\right]$

Diagonal entries: $\frac{1}{2}$, $\frac{2}{3}$, and -1.

Now try Exercises 10 through 12 ►

This process can easily be reversed to write a system of equations from a given augmented matrix.

EXAMPLE 3 ► Writing the System Corresponding to an Augmented Matrix

Write the system of equations corresponding to each matrix.

a. $\left[\begin{array}{cc|c} 3 & -5 & -14 \\ 0 & 1 & 4 \end{array}\right]$ **b.** $\left[\begin{array}{ccc|c} 1 & 4 & -1 & -10 \\ 0 & -3 & 10 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$

Solution ► **a.** $\left[\begin{array}{cc|c} 3 & -5 & -14 \\ 0 & 1 & 4 \end{array}\right] \longrightarrow \begin{cases} 3x - 5y = -14 \\ 0x + 1y = 4 \end{cases}$

b. $\left[\begin{array}{ccc|c} 1 & 4 & -1 & -10 \\ 0 & -3 & 10 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{cases} 1x + 4y - 1z = -10 \\ 0x - 3y + 10z = 7 \\ 0x + 0y + 1z = 1 \end{cases}$

☑ **B.** You've just learned how to form the augmented matrix of a system of equations

Now try Exercises 13 through 18 ►

C. Solving a System Using Matrices

When a system of equations is written in augmented matrix form, we can solve the system by applying the same operations to each row of the matrix, that would be applied to the equations in the system. In this context, the operations are referred to as **elementary row operations.**

Elementary Row Operations

1. Any two rows in a matrix can be interchanged.
2. The elements of any row can be multiplied by a nonzero constant.
3. Any two rows can be added together, and the sum used to replace one of the rows.

In this section, we'll use these operations to **triangularize the augmented matrix,** employing a solution method known as **Gaussian elimination.** A matrix is said to be in **triangular form** when all of the entries below the diagonal are zero. For example, the matrix $\left[\begin{array}{ccc|c} 1 & 4 & -1 & -10 \\ 0 & -3 & 10 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$ is in triangular form: $\left[\begin{array}{ccc|c} 1 & 4 & -1 & -10 \\ 0 & -3 & 10 & 7 \\ 0 & 0 & 1 & 1 \end{array}\right]$.

In system form we have $\begin{cases} x + 4y - z = -10 \\ \quad -3y + 10z = 7 \\ \qquad\qquad z = 1 \end{cases}$, meaning a matrix written in triangular form can be used to solve the system using back-substitution. We'll illustrate by solving $\begin{cases} 1x + 4y - 1z = 4 \\ 2x + 5y + 8z = 15 \\ 1x + 3y - 3z = 1 \end{cases}$, using elimination to the left, and *row operations on the augmented matrix* to the right. As before, R1 represents the first equation in the system and the first row of the matrix, R2 represents equation 2 and row 2, and so on. The calculations involved are shown for the first stage only and are designed to offer a careful comparison. In actual practice, the format shown in Example 4 is used.

Elimination (System of Equations)	**Row Operations (Augmented Matrix)**
$\begin{cases} 1x + 4y - 1z = 4 \\ 2x + 5y + 8z = 15 \\ 1x + 3y - 3z = 1 \end{cases}$	$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 4 \\ 2 & 5 & 8 & 15 \\ 1 & 3 & -3 & 1 \end{array}\right]$

To eliminate the x-term in R2, we use $-2\text{R1} + \text{R2} \rightarrow \text{R2}$. For R3 the operations would be $-1\text{R1} + \text{R3} \rightarrow \text{R3}$. Identical operations are performed on the matrix, which begins the process of triangularizing the matrix.

System Form		**Matrix Form**				
−2R1	$-2x - 8y + 2z = -8$	−2R1	−2	−8	2	−8
+		+				
R2	$2x + 5y + 8z = 15$	R2	2	5	8	15
New R2	$-3y + 10z = 7$	New R2	0	−3	10	7
−1R1	$-1x - 4y + 1z = -4$	−1R1	−1	−4	1	−4
+		+				
R3	$1x + 3y - 3z = 1$	R3	1	3	−3	1
New R3	$-1y - 2z = -3$	New R3	0	−1	−2	−3

As always, we should look for opportunities to simplify any equation in the system (and any row in the matrix). Note that $-1R3$ will make the coefficients and related matrix entries positive. Here is the new system and matrix.

New System

$$\begin{cases} 1x + 4y - 1z = 4 \\ \quad -3y + 10z = 7 \\ \quad\quad 1y + 2z = 3 \end{cases}$$

New Matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 4 \\ 0 & -3 & 10 & 7 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

On the left, we would finish by solving the 2×2 subsystem using $R2 + 3R3 \to R3$. In matrix form, we eliminate the corresponding entry (third row, second column) to triangularize the matrix.

$$\left[\begin{array}{ccc|c} 1 & 4 & -1 & 4 \\ 0 & -3 & 10 & 7 \\ 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow{R2 + 3R3 \to R3} \left[\begin{array}{ccc|c} 1 & 4 & -1 & 4 \\ 0 & -3 & 10 & 7 \\ 0 & 0 & 16 & 16 \end{array}\right]$$

Dividing R3 by 16 gives $z = 1$ in the system, and entries of 0 0 1 1 in the augmented matrix. Completing the solution by back-substitution in the system gives the ordered triple (1, 1, 1). **See Exercises 19 through 27.**

The general solution process is summarized here.

WORTHY OF NOTE

The procedure outlined for solving systems using matrices is virtually identical to that for solving systems by elimination. Using a 3×3 system for illustration, the "zeroes below the first diagonal entry" indicates we've eliminated the x-term from R2 and R3, the "zeroes below the second entry" indicates we've eliminated the y-term from the subsystem, and the division "to obtain a '1' in the final entry" indicates we have just solved for z.

Solving Systems by Triangularizing the Augmented Matrix

1. Write the system as an augmented matrix.
2. Use row operations to obtain zeroes below the first diagonal entry.
3. Use row operations to obtain zeroes below the second diagonal entry.
4. Continue until the matrix is triangularized (entries below diagonal are zero).
5. Divide to obtain a "1" in the last diagonal entry (if it is nonzero), then convert to equation form and solve using back-substitution.

Note: At each stage, look for opportunities to simplify row entries using multiplication or division. Also, to begin the process any equation with an x-coefficient of 1 can be made R1 by interchanging the equations.

EXAMPLE 4 ► **Solving Systems Using the Augmented Matrix**

Solve by triangularizing the augmented matrix: $\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ \quad -2y - z = -3 \end{cases}$

Solution ►

$$\begin{cases} 2x + y - 2z = -7 \\ x + y + z = -1 \\ \quad -2y - z = -3 \end{cases} \xrightarrow{\text{matrix form}} \left[\begin{array}{ccc|c} 2 & 1 & -2 & -7 \\ 1 & 1 & 1 & -1 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & -2 & -7 \\ 0 & -2 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 2 & 1 & -2 & -7 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{-2R1 + R2 \to R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & -1 & -4 & -5 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{-1R2 \to R2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & -2 & -1 & -3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & -2 & -1 & -3 \end{array}\right] \xrightarrow{2R2 + R3 \to R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 7 & 7 \end{array}\right] \xrightarrow{\frac{R3}{7} \to R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Converting the augmented matrix back into equation form yields $\begin{cases} x + y + z = -1 \\ y + 4z = 5 \\ z = 1 \end{cases}$.

Back-substitution shows the solution is $(-3, 1, 1)$.

Now try Exercises 28 through 32 ►

The process used in Example 4 is also called **Gaussian elimination** (Carl Friedrich Gauss, 1777–1855), with the last matrix written in **row-echelon form.** It's possible to solve a system entirely using only the augmented matrix, by continuing to use row operations until the diagonal entries are 1's, with 0's for all other entries: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{array}\right]$. The process is then called **Gauss-Jordan elimination** (Wilhelm Jordan, 1842–1899), with the final matrix written in **reduced row-echelon form** (see Appendix II).

Note that with Gauss-Jordan elimination, our *initial* focus is less on getting 1's along the diagonal, and more on obtaining zeroes for all entries *other than* the diagonal entries. This will enable us to work with integer values in the solution process.

☑ C. You've just learned how to solve a system of equations using row operations

EXAMPLE 5 ► Solving a System Using Gauss-Jordan Elimination

Solve using Gauss-Jordan elimination $\begin{cases} 2x + 5z - 15 = 2y \\ 2x + 3y = -1 + z \\ 4y + z = -7 \end{cases}$.

Solution ►

$$\overset{\text{standard form}}{\begin{cases} 2x - 2y + 5z = 15 \\ 2x + 3y - 1z = -1 \\ 0x + 4y + 1z = -7 \end{cases}} \xrightarrow{\text{matrix form}} \left[\begin{array}{ccc|c} 2 & -2 & 5 & 15 \\ 2 & 3 & -1 & -1 \\ 0 & 4 & 1 & -7 \end{array}\right] \xrightarrow{-R1 + R2 \to R2} \left[\begin{array}{ccc|c} 2 & -2 & 5 & 15 \\ 0 & 5 & -6 & -16 \\ 0 & 4 & 1 & -7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 5 & 15 \\ 0 & 5 & -6 & -16 \\ 0 & 4 & 1 & -7 \end{array}\right] \begin{array}{c} 2R2 + 5R1 \to R1 \\ \\ -4R2 + 5R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 10 & 0 & 13 & 43 \\ 0 & 5 & -6 & -16 \\ 0 & 0 & 29 & 29 \end{array}\right] \xrightarrow{\frac{R3}{29} \to R3} \left[\begin{array}{ccc|c} 10 & 0 & 13 & 43 \\ 0 & 5 & -6 & -16 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 10 & 0 & 13 & 43 \\ 0 & 5 & -6 & -16 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} -13R3 + R1 \to R1 \\ \\ 6R3 + R2 \to R2 \end{array} \left[\begin{array}{ccc|c} 10 & 0 & 0 & 30 \\ 0 & 5 & 0 & -10 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} \frac{R1}{10} \to R1 \\ \\ \frac{R2}{5} \to R2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The final matrix shows the solution is $(3, -2, 1)$.

Now try Exercises 33 through 36 ►

D. Inconsistent and Dependent Systems

Due to the strong link between a linear system and its augmented matrix, inconsistent and dependent systems can be recognized just as in Sections 5.1 and 5.2. An inconsistent system will yield an inconsistent or contradictory statement such as $0 = -12$, meaning all entries in a row of the matrix of coefficients are zero, but the constant is not. A linearly dependent system will yield an identity statement such as $0 = 0$, meaning all entries in one row of the matrix are zero. If the system has coincident dependence, there will be only one nonzero row of coefficients.

EXAMPLE 6 ► Solving a Dependent System

Solve the system using Gauss-Jordan elimination: $\begin{cases} x + y - 5z = 3 \\ -x + 2z = -1 \\ 2x - y - z = 0 \end{cases}$

Solution ►

$$\begin{cases} x + y - 5z = 3 \\ -x + 2z = -1 \\ 2x - y - z = 0 \end{cases} \xrightarrow{\text{standard form}} \begin{cases} x + y - 5z = 3 \\ -x + 0y + 2z = -1 \\ 2x - y - z = 0 \end{cases} \xrightarrow{\text{matrix form}} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ -1 & 0 & 2 & -1 \\ 2 & -1 & -1 & 0 \end{array}\right] \begin{array}{c} R1 + R2 \to R2 \\ \\ -2R1 + R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right] \begin{array}{c} -1R2 + R1 \to R1 \\ \\ 3R2 + R3 \to R3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since all entries in the last row are zeroes and it's the only row of zeroes, we conclude the system is linearly dependent and equivalent to $\begin{cases} x - 2z = 1 \\ y - 3z = 2 \end{cases}$. As in Chapter 5, we demonstrate this dependence by writing the (x, y, z) solution in terms of a parameter. Solving for y in R2 gives y in terms of z: $y = 3z + 2$. Solving for x in R1 gives x in terms of z: $x = 2z + 1$. As written, the solutions all depend on z: $x = 2z + 1$, $y = 3z + 2$, and $z = z$. Selecting p as the parameter (or some other "neutral" variable), we write the solution as $(2p + 1, 3p + 2, p)$. Two of the infinite number of solutions would be $(1, 2, 0)$ for $p = 0$, and $(-1, -1, -1)$ for $p = -1$. Test these triples in the original equations.

Now try Exercises 37 through 45 ►

D. You've just learned how to recognize inconsistent and dependent systems

E. Solving Applications Using Matrices

As in other areas, solving applications using systems relies heavily on the ability to mathematically model information given verbally or in context. As you work through the exercises, read each problem carefully. Look for relationships that yield a system of two equations in two variables, three equations in three variables and so on.

EXAMPLE 7 ► Determining the Original Value of Collector's Items

A museum purchases a famous painting, a ruby tiara, and a rare coin for its collection, spending a total of \$30,000. One year later, the painting has tripled in value, while the tiara and the coin have doubled in value. The items now have a total value of \$75,000. Find the purchase price of each if the original price of the painting was \$1000 more than twice the coin.

Solution ► Let P represent the price of the painting, T the tiara, and C the coin.

Total spent was \$30,000: $\to P + T + C = 30{,}000$

One year later: $\to 3P + 2T + 2C = 75{,}000$

Value of painting versus coin: $\to P = 2C + 1000$

$$\begin{cases} P + T + C = 30000 \\ 3P + 2T + 2C = 75000 \\ P = 2C + 1000 \end{cases} \xrightarrow{\text{standard form}} \begin{cases} 1P + 1T + 1C = 30000 \\ 3P + 2T + 2C = 75000 \\ 1P + 0T - 2C = 1000 \end{cases} \xrightarrow{\text{matrix form}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 3 & 2 & 2 & 75000 \\ 1 & 0 & -2 & 1000 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 3 & 2 & 2 & 75000 \\ 1 & 0 & -2 & 1000 \end{array}\right] \begin{array}{c} -3R1 + R2 \rightarrow R2 \\ \\ -1R1 + R3 \rightarrow R3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & -1 & -1 & -15000 \\ 0 & -1 & -3 & -29000 \end{array}\right] \begin{array}{c} -1R2 \rightarrow R2 \\ \\ -1R3 \rightarrow R3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 1 & 1 & 15000 \\ 0 & 1 & 3 & 29000 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 1 & 1 & 15000 \\ 0 & 1 & 3 & 29000 \end{array}\right] \xrightarrow{-1R2 + R3 \rightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 1 & 1 & 15000 \\ 0 & 0 & 2 & 14000 \end{array}\right] \xrightarrow{\frac{R3}{2} \rightarrow R3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 30000 \\ 0 & 1 & 1 & 15000 \\ 0 & 0 & 1 & 7000 \end{array}\right]$$

From R3 of the triangularized form, $C = \$7000$ directly. Since R2 represents $T + C = 15{,}000$, we find the tiara was purchased for $T = \$8000$. Substituting these values into the first equation shows the painting was purchased for $15,000. The solution is (15,000, 8000, 7000).

Now try Exercises 48 through 55 ►

☑ **E.** You've just learned how to solve applications using linear systems

TECHNOLOGY HIGHLIGHT

Solving Systems Using Matrices and Calculating Technology

Graphing calculators offer a very efficient way to solve systems using matrices. Once the system has been written in matrix form, it can easily be entered and solved by asking the calculator to instantly perform the row operations needed. Pressing 2nd X^{-1} **(MATRIX)** gives a screen similar to the one shown in Figure 9.1, where we begin by selecting the **EDIT** option (push the right arrow ► twice). Pressing ENTER places you on a screen where you can EDIT matrix *A*, changing the size as needed. Using the 3×4 matrix from Example 4, we press 3 and ENTER, then 4 and ENTER, giving the screen shown in Figure 9.2. The dash marks to the right indicate that there is a fourth column that cannot be seen, but that comes into view as you enter the elements of the matrix. Begin entering the first row of the matrix resulting from $R_1 \leftrightarrow R_2$, which has entries {1, 1, 1, −1}. Press ENTER after each entry and the cursor automatically goes to the next position in the matrix (note that the TI-84 Plus automatically shifts left and right to allow all four columns to be entered). After entering the second row {2, 1, −2, −7} and the third row {0, −2, −1, −3}, the completed matrix should look like the one shown in Figure 9.3 (the matrix is currently shifted to the right, showing the fourth column). To write this matrix in reduced row-echelon form **(rref)** we return to the home screen by pressing 2nd MODE **(QUIT)**. Press the CLEAR key for a clean home screen. To access the **rref** function, press 2nd X^{-1} **(MATRIX)** and select the **MATH** option, then scroll upward (or downward) until you get to **B:rref.** Pressing ENTER places this function on the home screen, where we must tell it to perform the **rref** operation on matrix [A]. Press 2nd X^{-1} **(MATRIX)** to select a matrix (notice that matrix **NAMES** is automatically highlighted. Press ENTER to select matrix

Figure 9.1

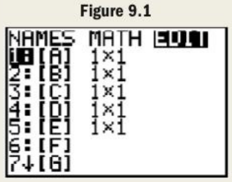


Figure 9.2

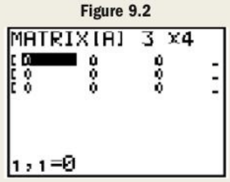
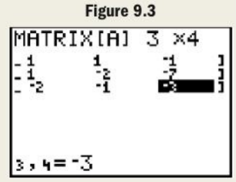


Figure 9.3



[A] as the object of the **rref** function. After pressing ENTER the calculator quickly computes the reduced row-echelon form and displays it on the screen as in Figure 9.4. The solution is easily read as $x = -3$, $y = 1$, and $z = 1$, as we found in Example 4. Use these ideas to complete the following.

Figure 9.4

Exercise 1: Use this method to solve the 2×2 system from Exercise 30.

Exercise 2: Use this method to solve the 3×3 system from Exercise 32.

9.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. A matrix with the same number of rows and columns is called a(n) _____ matrix.

2. When the coefficient matrix is used with the matrix of constants, the result is a(n) _______ matrix.

3. Matrix $A = \begin{bmatrix} 2 & 4 & -3 \\ 1 & -2 & 1 \end{bmatrix}$ is a ___ by ___ matrix. The entry in the second row and third column is ___.

4. Given matrix B shown here, the diagonal entries are ___, ___, and ___.

$$B = \begin{bmatrix} 1 & 4 & 3 \\ -1 & 5 & 2 \\ 3 & -2 & 1 \end{bmatrix}$$

5. The notation $-2\text{R1} + \text{R2} \rightarrow \text{R2}$ indicates that an equivalent matrix is formed by performing what operations/replacements?

6. Describe how to tell an inconsistent system apart from a dependent system when solving using matrix methods (row reduction).

▶ DEVELOPING YOUR SKILLS

Determine the size (order) of each matrix and identify the third row and second column entry. If the matrix given is a square matrix, identify the diagonal entries.

7. $\begin{bmatrix} 1 & 0 \\ 2.1 & 1 \\ -3 & 5.8 \end{bmatrix}$

8. $\begin{bmatrix} 1 & 0 & 4 \\ 1 & 3 & -7 \\ 5 & -1 & 2 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 4 \\ 1 & 3 & -7 \\ 5 & -1 & 2 \\ 2 & -3 & 9 \end{bmatrix}$

Form the augmented matrix, then name the diagonal entries of the coefficient matrix.

10. $\begin{cases} 2x - 3y - 2z = 7 \\ x - y + 2z = -5 \\ 3x + 2y - z = 11 \end{cases}$

11. $\begin{cases} x + 2y - z = 1 \\ x + z = 3 \\ 2x - y + z = 3 \end{cases}$

12. $\begin{cases} 2x + 3y + z = 5 \\ 2y - z = 7 \\ x - y - 2z = 5 \end{cases}$

Write the system of equations for each matrix. Then use back-substitution to find its solution.

13. $\left[\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & \frac{1}{2} \end{array}\right]$

14. $\left[\begin{array}{cc|c} 1 & -5 & -15 \\ 0 & -1 & -2 \end{array}\right]$

15. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$

16. $\left[\begin{array}{ccc|c} 1 & 0 & 7 & -5 \\ 0 & 1 & -5 & 15 \\ 0 & 0 & 1 & -26 \end{array}\right]$

17. $\left[\begin{array}{ccc|c} 1 & 3 & -4 & 29 \\ 0 & 1 & -\frac{3}{2} & \frac{21}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$

18. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & \frac{1}{6} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{22}{7} \end{array}\right]$

Perform the indicated row operation(s) and write the new matrix.

19. $\left[\begin{array}{cc|c} \frac{1}{2} & -3 & -1 \\ -5 & 2 & 4 \end{array}\right]$ $2R1 \to R1,$ $5R1 + R2 \to R2$

20. $\left[\begin{array}{cc|c} 7 & 4 & 3 \\ 4 & -8 & 12 \end{array}\right]$ $\frac{1}{4}R2 \to R2,$ $R1 \leftrightarrow R2$

21. $\left[\begin{array}{ccc|c} -2 & 1 & 0 & 4 \\ 5 & 8 & 3 & -5 \\ 1 & -3 & 3 & 2 \end{array}\right]$ $R1 \leftrightarrow R3,$ $-5R1 + R2 \to R2$

22. $\left[\begin{array}{ccc|c} -3 & 2 & 0 & 0 \\ 1 & 1 & 2 & 6 \\ 4 & 1 & -3 & 2 \end{array}\right]$ $R1 \leftrightarrow R2,$ $-4R1 + R3 \to R3$

23. $\left[\begin{array}{ccc|c} 3 & 1 & 1 & 8 \\ 6 & -1 & -1 & 10 \\ 4 & -2 & -3 & 22 \end{array}\right]$ $-2R1 + R2 \to R2,$ $-4R1 + 3R3 \to R3$

24. $\left[\begin{array}{ccc|c} 2 & 1 & -1 & -3 \\ 3 & 1 & 1 & 0 \\ 4 & 3 & 2 & 3 \end{array}\right]$ $-3R1 + 2R2 \to R2,$ $-2R1 + R3 \to R3$

What row operations would produce zeroes beneath the first entry in the diagonal?

25. $\left[\begin{array}{ccc|c} 1 & 3 & 0 & 2 \\ -2 & 4 & 1 & 1 \\ 3 & -1 & -2 & 9 \end{array}\right]$

26. $\left[\begin{array}{ccc|c} 1 & 1 & -4 & -3 \\ 3 & 0 & 1 & 5 \\ -5 & 3 & 2 & 3 \end{array}\right]$

27. $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 10 \\ 5 & 1 & 2 & 6 \\ -4 & 3 & -3 & 2 \end{array}\right]$

Solve each system by triangularizing the augmented matrix and using back-substitution. Simplify by clearing fractions or decimals before beginning.

28. $\begin{cases} 2y = 5x + 4 \\ -5x = 2 - 4y \end{cases}$

29. $\begin{cases} 0.15g - 0.35h = -0.5 \\ -0.12g + 0.25h = 0.1 \end{cases}$

30. $\begin{cases} -\frac{1}{5}u + \frac{1}{4}v = 1 \\ \frac{1}{10}u + \frac{1}{2}v = 7 \end{cases}$

31. $\begin{cases} x - 2y + 2z = 7 \\ 2x + 2y - z = 5 \\ 3x - y + z = 6 \end{cases}$

32. $\begin{cases} 2x - 3y - 2z = 7 \\ x - y + 2z = -5 \\ 3x + 2y - z = 11 \end{cases}$

33. $\begin{cases} x + 2y - z = 1 \\ x + z = 3 \\ 2x - y + z = 3 \end{cases}$

34. $\begin{cases} 2x + 3y + z = 5 \\ 2y - z = 7 \\ x - y - 2z = 5 \end{cases}$

35. $\begin{cases} -x + y + 2z = 2 \\ x + y - z = 1 \\ 2x + y + z = 4 \end{cases}$

36. $\begin{cases} x + y - 2z = -1 \\ 4x - y + 3z = 3 \\ 3x + 2y - z = 4 \end{cases}$

Solve each system by triangularizing the augmented matrix and using back-substitution. If the system is linearly dependent, give the solution in terms of a parameter. If the system has coincident dependence, answer in set notation as in Chapter 5.

37. $\begin{cases} 4x - 8y + 8z = 24 \\ 2x - 6y + 3z = 13 \\ 3x + 4y - z = -11 \end{cases}$

38. $\begin{cases} 3x + y + z = -2 \\ x - 2y + 3z = 1 \\ 2x - 3y + 5z = 3 \end{cases}$

39. $\begin{cases} x + 3y + 5z = 20 \\ 2x + 3y + 4z = 16 \\ x + 2y + 3z = 12 \end{cases}$

40. $\begin{cases} -x + 2y + 3z = -6 \\ x - y + 2z = -4 \\ 3x - 6y - 9z = 18 \end{cases}$

41. $\begin{cases} 3x - 4y + 2z = -2 \\ \frac{3}{2}x - 2y + z = -1 \\ -6x + 8y - 4z = 4 \end{cases}$

42. $\begin{cases} 2x - y + 3z = 1 \\ 4x - 2y + 6z = 2 \\ 10x - 5y + 15z = 5 \end{cases}$

43. $\begin{cases} 2x - y + 3z = 1 \\ 2y + 6z = 2 \\ x - \frac{1}{2}y + \frac{3}{2}z = 5 \end{cases}$

44. $\begin{cases} x + 2y + z = 4 \\ 3x - 4y + z = 4 \\ 6x - 8y + 2z = 8 \end{cases}$

45. $\begin{cases} -2x + 4y - 3z = 4 \\ 5x - 6y + 7z = -12 \\ x + 2y + z = -4 \end{cases}$

▶ WORKING WITH FORMULAS

Area of a triangle in the plane:

$$A = \pm\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

The area of a triangle in the plane is given by the formula shown, where the vertices of the triangle are located at the points (x_1, y_1), (x_2, y_2), and (x_3, y_3), and the sign is chosen to ensure a positive value.

46. Find the area of a triangle whose vertices are $(-1, -3)$, $(5, 2)$, and $(1, 8)$.

47. Find the area of a triangle whose vertices are $(6, -2)$, $(-5, 4)$, and $(-1, 7)$.

▶ APPLICATIONS

Model each problem using a system of linear equations. Then solve using the augmented matrix.

Descriptive Translation

48. The distance (via air travel) from Los Angeles (LA), California, to Saint Louis (STL), Missouri, to Cincinnati (CIN), Ohio, to New York City (NYC), New York, is approximately 2480 mi. Find the distances between each city if the distance from LA to STL is 50 mi more than five times the distance between STL and CIN and 110 mi less than three times the distance from CIN to NYC.

49. In the 2006 NBA Championship Series, Dwayne Wade of the Miami Heat carried his team to the title after the first two games were lost to the Dallas Mavericks. If 187 points were scored in the title game and the Heat won by 3 points, what was the final score?

50. Moe is lecturing Larry and Curly once again (Moe, Larry, and Curly of *The Three Stooges* fame) claiming he is twice as smart as Larry and three times as smart as Curly. If he is correct and the sum of their IQs is 165, what is the IQ of each stooge?

51. A collector of rare books buys a handwritten, autographed copy of Edgar Allan Poe's *Annabel Lee,* an original advance copy of L. Frank Baum's *The Wonderful Wizard of Oz,* and a first print copy of *The Caine Mutiny* by Herman Wouk, paying a total of $100,000. Find the cost of each one, given that the cost of *Annabel Lee* and twice the cost of *The Caine Mutiny* sum to the price paid for *The Wonderful Wizard of Oz,* and *The Caine Mutiny* cost twice as much as *Annabel Lee.*

Geometry

52. A right triangle has a hypotenuse of 39 m. If the perimeter is 90 m, and the longer leg is 6 m longer than twice the shorter leg, find the dimensions of the triangle.

53. In triangle ABC, the sum of angles A and C is equal to three times angle B. Angle C is 10 degrees more than twice angle B. Find the measure of each angle.

Investment and Finance

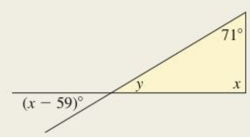
54. Suppose \$10,000 is invested in three different investment vehicles paying 5%, 7%, and 9% annual interest. Find the amount invested at each rate if the interest earned after 1 yr is \$760 and the amount invested at 9% is equal to the sum of the amounts invested at 5% and 7%.

55. The trustee of a union's pension fund has invested the funds in three ways: a savings fund paying 4% annual interest, a money market fund paying 7%, and government bonds paying 8%. Find the amount invested in each if the interest earned after one year is \$0.178 million and the amount in government bonds is \$0.3 million more than twice the amount in money market funds. The total amount invested is \$2.5 million dollars.

▶ EXTENDING THE CONCEPT

56. In previous sections, we noted that one condition for a 3×3 system to be dependent was for the third equation to be a linear combination of the other two. To test this, write any two (different) equations using the same three variables, then form a third equation by performing some combination of elementary row operations. Solve the resulting 3×3 system. What do you notice?

57. Given the drawing shown, use a system of equations and the matrix method to find the measure of the angles labeled as x and y. Recall that vertical angles are equal and that the sum of the angles in a triangle is 180°.



58. The system given here has a solution of $(1, -2, 3)$. Find the value of a and b.

$$\left[\begin{array}{ccc|c} 1 & a & b & 1 \\ 2b & 2a & 5 & 13 \\ 2a & 7 & 3b & -8 \end{array}\right]$$

▶ MAINTAINING YOUR SKILLS

59. **(7.5)** **a.** Convert $z_1 = -1 - 3i$ to trigonometric form.

b. Convert $z_2 = 5 \operatorname{cis}\left(\frac{2\pi}{3}\right)$ to rectangular form.

60. **(5.4)** State the exact value of the following trig functions:

a. $\sin\left(-\frac{\pi}{6}\right)$ **b.** $\cos\left(\frac{5\pi}{4}\right)$

c. $\tan\left(\frac{2\pi}{3}\right)$ **d.** $\csc\left(\frac{3\pi}{2}\right)$

61. **(4.4)** Since 2005, cable installations for an Internet company have been modeled by the function $C(t) = 15 \ln(t + 1)$, where $C(t)$ represents cable installations in thousands, t yr after 2005. In what year will the number of installations be greater than 30,000?

62. **(4.5)** If a set amount of money p is deposited regularly (daily, weekly, monthly, etc.) n times per year at a fixed interest rate r, the amount of money A accumulated in t years is given by the formula shown. If a parent deposits \$250 per month for 18 yr at 4.6% beginning when her first child was born, how much has been accumulated to help pay for college expenses?

$$A = \frac{p\left[\left(1 + \frac{r}{n}\right)^{nt} - 1\right]}{\frac{r}{n}}$$

9.2 The Algebra of Matrices

Learning Objectives

In Section 9.2 you will learn how to:

- ☐ **A.** Determine if two matrices are equal
- ☐ **B.** Add and subtract matrices
- ☐ **C.** Compute the product of two matrices

Matrices serve a much wider purpose than just a convenient method for solving systems. To understand their broader application, we need to know more about matrix theory, the various ways matrices can be combined, and some of their more practical uses. The common operations of addition, subtraction, multiplication, and division are all defined for matrices, as are other operations. Practical applications of matrix theory can be found in the social sciences, inventory management, genetics, operations research, engineering, and many other fields.

A. Equality of Matrices

To effectively study matrix algebra, we first give matrices a more general definition. For the *general* matrix A, all entries will be denoted using the lowercase letter "a," with their position in the matrix designated by the dual subscript a_{ij}. The letter "i" gives the *row* and the letter "j" gives the *column* of the entry's location. The general $m \times n$ matrix A is written

$$\begin{array}{r} \\ \text{row } 1 \rightarrow \\ \text{row } 2 \rightarrow \\ \text{row } 3 \rightarrow \\ \\ \text{row } i \rightarrow \\ \\ \text{row } m \rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccc} \text{col } 1 & \text{col } 2 & \text{col } 3 & & \text{col } j & & \text{col } n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3j} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \mathbf{a_{ij}} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

a_{ij} is a general matrix element

The size of a matrix is also referred to as its **order,** and we say the order of general matrix A is $m \times n$. Note that diagonal entries have the same row and column number, a_{ij}, where $i = j$. Also, where the general entry of matrix A is a_{ij}, the general entry of matrix B is b_{ij}, of matrix C is c_{ij}, and so on.

EXAMPLE 1 ▶ Identifying the Order and Entries of a Matrix

State the order of each matrix and name the entries corresponding to a_{22}, a_{31}; b_{22}, b_{31}; and c_{22}, c_{31}.

a. $A = \begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}$ **b.** $B = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$ **c.** $C = \begin{bmatrix} 0.2 & -0.5 & 0.7 \\ -1 & 0.3 & 1 \\ 2.1 & -0.1 & 0.6 \end{bmatrix}$

Solution ▶

a. matrix A: order 2×2. Entry $a_{22} = -3$ (the row 2, column 2 entry is -3). There is no a_{31} entry (A is only 2×2).

b. matrix B: order 3×2. Entry $b_{22} = 5$, entry $b_{31} = -4$.

c. matrix C: order 3×3. Entry $c_{22} = 0.3$, entry $c_{31} = 2.1$.

Now try Exercises 7 through 12 ▶

Equality of Matrices

Two matrices are equal if they have the same order and their corresponding entries are equal. In symbols, this means that $A = B$ if $a_{ij} = b_{ij}$ for all i and j.

EXAMPLE 2 ► Determining If Two Matrices Are Equal

Determine whether the following statements are true, false, or conditional. If false, explain why. If conditional, find values that will make the statement true.

a. $\begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -4 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} a-2 & 2b \\ c & -3 \end{bmatrix}$

Solution ► **a.** $\begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}$ is false. The matrices have the same order and entries, but corresponding entries are not equal.

b. $\begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ 5 & -4 & 3 \end{bmatrix}$ is false. Their orders are not equal.

c. $\begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} a-2 & 2b \\ c & -3 \end{bmatrix}$ is conditional. The statement is true when $a - 2 = 1$ $(a = 3)$, $2b = 4$ $(b = 2)$, $c = -2$, and is false otherwise.

A. You've just learned how to determine if two matrices are equal

Now try Exercises 13 through 16 ►

B. Addition and Subtraction of Matrices

A sum or difference of matrices is found by combining the corresponding entries. This limits the operations to matrices of like orders, so that every entry in one matrix has a "corresponding entry" in the other. This also means the result is a new matrix of *like order,* whose entries are the corresponding sums or differences. Note that since a_{ij} represents a general entry of matrix A, $[a_{ij}]$ represents the entire matrix.

Addition and Subtraction of Matrices

Given matrices A, B, and C having like orders.

The sum $A + B = C$, where $[a_{ij} + b_{ij}] = [c_{ij}]$. The difference $A - B = D$, where $[a_{ij} - b_{ij}] = [d_{ij}]$.

EXAMPLE 3 ► Adding and Subtracting Matrices

Compute the sum or difference of the matrices indicated.

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 2 & -1 \\ -5 & 4 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$$

a. $A + C$ **b.** $A + B$ **c.** $C - A$

Solution ► **a.** $A + C = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix}$ sum of A and C

$$= \begin{bmatrix} 2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & 5 \\ -3 & 0 \end{bmatrix}$$ add corresponding entries

b. $A + B = \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 2 & -1 \\ -5 & 4 & 3 \end{bmatrix}$ Addition and subtraction are not defined for matrices of unlike order.

c. $C - A = \begin{bmatrix} 3 & -2 \\ 1 & 5 \\ -4 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 6 \\ 1 & 0 \\ 1 & -3 \end{bmatrix}$ difference of C and A

$= \begin{bmatrix} 3-2 & -2-6 \\ 1-1 & 5-0 \\ -4-1 & 3-(-3) \end{bmatrix} = \begin{bmatrix} 1 & -8 \\ 0 & 5 \\ -5 & 6 \end{bmatrix}$ subtract corresponding entries

☑ **B.** You've just learned how to add and subtract matrices

Now try Exercises 17 through 20 ►

Since the addition of two matrices is defined as the sum of corresponding entries, we find the properties of matrix addition closely resemble those of real number addition.

Properties of Matrix Addition

Given matrices A, B, C, and Z are $m \times n$ matrices, with Z the zero matrix. Then,

I.	$A + B = B + A$	matrix addition is commutative
II.	$(A + B) + C = A + (B + C)$	matrix addition is associative
III.	$A + Z = Z + A = A$	Z is the additive identity
IV.	$A + (-A) = (-A) + A = Z$	$-A$ is the additive inverse of A

C. Matrices and Multiplication

The algebraic terms $2a$ and ab have counterparts in matrix algebra. The product $2A$ represents a constant times a matrix and is called **scalar multiplication.** The product AB represents the product of two matrices.

Scalar Multiplication

Scalar multiplication is defined by taking the product of the constant with *each entry* in the matrix, forming a new matrix of like size. In symbols, for any real number k and matrix A, $kA = [ka_{ij}]$. Similar to standard algebraic properties, $-A$ represents the scalar product $-1 \cdot A$ and any subtraction can be rewritten as an algebraic sum: $A - B = A + (-B)$. As noted in the properties box, for any matrix A, the sum $A + (-A)$ will yield the **zero matrix Z,** a matrix of like size whose entries are all zeroes. Also note that matrix $-A$ is the **additive inverse** for A, while Z is the **additive identity.**

EXAMPLE 4 ► **Computing Operations on Matrices**

Given $A = \begin{bmatrix} 4 & 3 \\ \frac{1}{2} & 1 \\ 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -2 \\ 0 & 6 \\ -4 & 0.4 \end{bmatrix}$, compute the following:

a. $\frac{1}{2}B$ **b.** $-4A - \frac{1}{2}B$

Solution ► **a.** $\frac{1}{2}B = \left(\frac{1}{2}\right)\begin{bmatrix} 3 & -2 \\ 0 & 6 \\ -4 & 0.4 \end{bmatrix}$

$= \begin{bmatrix} (\frac{1}{2})(3) & (\frac{1}{2})(-2) \\ (\frac{1}{2})(0) & (\frac{1}{2})(6) \\ (\frac{1}{2})(-4) & (\frac{1}{2})(0.4) \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -1 \\ 0 & 3 \\ -2 & 0.2 \end{bmatrix}$

b. $-4A - \frac{1}{2}B = -4A + \left(-\frac{1}{2}\right)B$ rewrite using algebraic addition

$$= \begin{bmatrix} (-4)(4) & (-4)(3) \\ (-4)(\frac{1}{2}) & (-4)(1) \\ (-4)(0) & (-4)(-3) \end{bmatrix} + \begin{bmatrix} (-\frac{1}{2})(3) & (-\frac{1}{2})(-2) \\ (-\frac{1}{2})(0) & (-\frac{1}{2})(6) \\ (-\frac{1}{2})(-4) & (-\frac{1}{2})(0.4) \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -12 \\ -2 & -4 \\ 0 & 12 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} & 1 \\ 0 & -3 \\ 2 & -0.2 \end{bmatrix} \quad \text{simplify}$$

$$= \begin{bmatrix} -16 + (-\frac{3}{2}) & -12 + 1 \\ -2 + 0 & -4 + (-3) \\ 0 + 2 & 12 + (-0.2) \end{bmatrix} = \begin{bmatrix} -\frac{35}{2} & -11 \\ -2 & -7 \\ 2 & 11.8 \end{bmatrix} \quad \text{result}$$

Now try Exercises 21 through 24 ►

Matrix Multiplication

Consider a cable company offering three different levels of Internet service: Bronze—fast, Silver—very fast, and Gold—lightning fast. Table 9.1 shows the number and types of programs sold to households and businesses for the week. Each program has an incentive package consisting of a rebate and a certain number of free weeks, as shown in Table 9.2.

Table 9.1 Matrix A

	Bronze	Silver	Gold
Homes	40	20	**25**
Businesses	10	15	45

Table 9.2 Matrix B

	Rebate	Free Weeks
Bronze	$15	2
Silver	$25	4
Gold	**$35**	6

To compute the amount of rebate money the cable company paid to households for the week, we would take the first row (R1) in Table 9.1 and multiply by the corresponding entries (bronze with bronze, silver with silver, and so on) in the first column (C1) of Table 9.2 and add these products. In matrix form, we have $[40\ \ 20\ \ \mathbf{25}] \cdot \begin{bmatrix} 15 \\ 25 \\ \mathbf{35} \end{bmatrix} = 40 \cdot 15 + 20 \cdot 25 + \mathbf{25 \cdot 35} = \mathbf{\$1975}$. Using R1 of Table 9.1 with C2 from Table 9.2 gives the number of free weeks awarded to homes: $[40\ \ 20\ \ \mathbf{25}] \cdot \begin{bmatrix} 2 \\ 4 \\ \mathbf{6} \end{bmatrix} = 40 \cdot 2 + 20 \cdot 4 + \mathbf{25 \cdot 6} = 310$. Using the second row (R2) of Table 9.1 with the two columns from Table 9.2 will give the amount of rebate money and the number of free weeks, respectively, awarded to business customers. When all computations are complete, the result is a product matrix P with order 2×2. This is because the product of R1 from matrix A, with C1 from matrix B, *gives the entry in position P_{11} of the product matrix:* $\begin{bmatrix} 40 & 20 & 25 \\ 10 & 15 & 45 \end{bmatrix} \cdot \begin{bmatrix} 15 & 2 \\ 25 & 4 \\ 35 & 6 \end{bmatrix} = \begin{bmatrix} 1975 & 310 \\ 2100 & 350 \end{bmatrix}$. Likewise, the product R1 · C2 will give entry P_{12} (310), the product of R2 with C1 will give P_{21} (2100), and so on. This "row × column" multiplication can be generalized, and leads to the following. Given $m \times n$ matrix A and $s \times t$ matrix B,

A B	A B
$(m \times n)$ $(s \times t)$	$(m \times n)$ $(s \times t)$
matrix multiplication is possible only when $n = s$	result will be an $m \times t$ matrix

In more formal terms, we have the following definition of matrix multiplication.

Matrix Multiplication

Given the $m \times n$ matrix $A = [a_{ij}]$ and the $s \times t$ matrix $B = [b_{ij}]$. If $n = s$, then matrix multiplication is possible and the product AB is an $m \times t$ matrix $P = [p_{ij}]$, where p_{ij} is product of the ith row of A with the jth column of B.

In less formal terms, matrix multiplication involves multiplying the row entries of the first matrix with the corresponding column entries of the second, and adding them together.

In Example 5, two of the matrix products [parts (a) and (b)] are shown in full detail, with the first entry of the product matrix color-coded.

EXAMPLE 5 ▶ Multiplying Matrices

Given the matrices A through E shown here, compute the following products:

a. AB **b.** CD **c.** DC **d.** AE **e.** EA

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix} \quad C = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 5 & 1 \\ 4 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix} \quad E = \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$$

Solution ▶ a. $AB = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 36 & 13 \end{bmatrix}$

Computation: $\begin{bmatrix} (-2)(4) + (1)(6) & (-2)(3) + (1)(1) \\ (3)\,4 + (4)(6) & (3)(3) + (4)(1) \end{bmatrix}$

A B	A B
(2×2) (2×2)	(2×2) (2×2)
multiplication is possible since $2 = 2$	result will be a 2×2 matrix

b. $CD = \begin{bmatrix} -2 & 1 & \mathbf{3} \\ 1 & 0 & 2 \\ 4 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 5 & 1 \\ 4 & -1 & 1 \\ \mathbf{0} & 3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -7 \\ 2 & 11 & -3 \\ 12 & 16 & 7 \end{bmatrix}$

C D	C D
(3×3) (3×3)	(3×3) (3×3)
multiplication is possible since $3 = 3$	result will be a 3×3 matrix

Computation: $\begin{bmatrix} (-2)(2) + (1)(4) + (\mathbf{3})(\mathbf{0}) & (-2)(5) + (1)(-1) + (\mathbf{3})(3) & (-2)(1) + (1)(1) + (\mathbf{3})(-2) \\ (1) \quad + (0)(4) + (2)(\mathbf{0}) & (1)(5) + (0)(-1) + (2)(3) & (1)(1) + (0)(1) + (2)(-2) \\ (4) \quad + (1)(4) + (-1)(\mathbf{0}) & (4)(5) + (1)(-1) + (-1)(3) & (4)(1) + (1)(1) + (-1)(-2) \end{bmatrix}$

c. $DC = \begin{bmatrix} 2 & 5 & 1 \\ 4 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 5 & 3 & 15 \\ -5 & 5 & 9 \\ -5 & -2 & 8 \end{bmatrix}$

D C	D C
(3×3) (3×3)	(3×3) (3×3)
multiplication is possible since $3 = 3$	result will be a 3×3 matrix

d. $AE = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix}$

A (2 × 2) *E* (3 × 2)
multiplication is not possible since 2 ≠ 3

e. $EA = \begin{bmatrix} -2 & -1 \\ 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ -6 & 3 \\ 4 & 9 \end{bmatrix}$

E (3 × 2) *A* (2 × 2) — multiplication is possible since 2 = 2
E (3 × 2) *A* (2 × 2) — result will be a 3 × 2 matrix

Now try Exercises 25 through 36 ►

Example 5 shows that in general, matrix multiplication is not commutative. Parts (b) and (c) show $CD \neq DC$ since we get different results, and parts (d) and (e) show $AE \neq EA$, since AE is not defined while EA *is*.

Operations on matrices can be a laborious process for larger matrices and for matrices with noninteger or large entries. For these, we can turn to available technology for assistance.This shifts our focus from a meticulous computation of entries, to carefully entering each matrix into the calculator, double-checking each entry, and appraising results to see if they're reasonable.

EXAMPLE 6 ► **Using Technology for Matrix Operations**

Use a calculator to compute the difference $A - B$ for the matrices given.

$$A = \begin{bmatrix} \frac{2}{11} & -0.5 & \frac{6}{5} \\ 0.9 & \frac{3}{4} & -4 \\ 0 & 6 & -\frac{5}{12} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{6} & \frac{-7}{10} & 0.75 \\ \frac{11}{25} & 0 & -5 \\ -4 & \frac{-5}{9} & \frac{-5}{12} \end{bmatrix}$$

Solution ► The entries for matrix A are shown in Figure 9.5. After entering matrix B, exit to the home screen [2nd MODE **(QUIT)**], call up matrix A, press the − (subtract) key, then call up matrix B and press ENTER. The calculator quickly finds the difference and displays the results shown in Figure 9.6. The last line on the screen shows the result can be stored for future use in a new matrix C by pressing the STO→ key, calling up matrix C, and pressing ENTER.

Figure 9.5

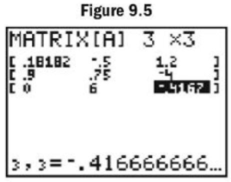
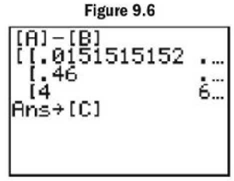
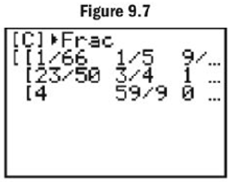


Figure 9.6



Now try Exercises 37 through 40 ►

Figure 9.7



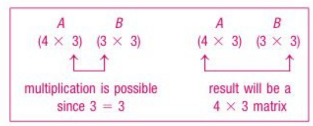
In Figure 9.6 the dots to the right on the calculator screen indicate there are additional digits or matrix columns that can't fit on the display, as often happens with larger matrices or decimal numbers. Sometimes, converting entries to fraction form will provide a display that's easier to read. Here, this is done by calling up the matrix C, and using the MATH **1: ►Frac** option. After pressing ENTER, all entries are converted to fractions in simplest form (where possible), as in Figure 9.7. The third column can be viewed by pressing the right arrow.

EXAMPLE 7 ▶ Using Technology for Matrix Operations

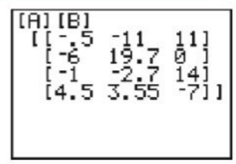
Use a calculator to compute the product AB.

$$A = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 5 & 4 \\ 6 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{2} & -0.7 & 1 \\ 0.5 & 3.2 & -3 \\ -2 & \frac{3}{4} & 4 \end{bmatrix}$$

Solution ▶ Carefully enter matrices A and B into the calculator, then press 2nd MODE **(QUIT)** to get to the home screen. Use [A][B] ENTER, and the calculator finds the product shown in the figure.



$$AB = \begin{bmatrix} 2 & -3 & 0 \\ -1 & 5 & 4 \\ 6 & 0 & 2 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -0.7 & 1 \\ 0.5 & 3.2 & -3 \\ -2 & \frac{3}{4} & 4 \end{bmatrix}$$



Now try Exercises 41 through 52 ▶

Properties of Matrix Multiplication

Earlier, Example 5 demonstrated that matrix multiplication is not commutative. Here is a group of properties that *do* hold for matrices. You are asked to check these properties in the exercise set using various matrices. See **Exercises 53 through 56.**

Properties of Matrix Multiplication

Given matrices A, B, and C for which the products are defined:

I. $A(BC) = (AB)C$	matrix multiplication is associative
II. $A(B + C) = AB + AC$	matrix multiplication is distributive from the left
III. $(B + C)A = BA + CA$	matrix multiplication is distributive from the right
IV. $k(A + B) = kA + kB$	a constant k can be distributed over addition

We close this section with an application of matrix multiplication. There are many other interesting applications in the exercise set.

EXAMPLE 8 ▶ Using Matrix Multiplication to Track Volunteer Enlistments

In a certain country, the number of males and females that will join the military depends on their age. This information is stored in matrix A (Table 9.3). The likelihood a volunteer will join a *particular branch* of the military also depends on their age, with this information stored in matrix B (Table 9.4). (a) Compute the product $P = AB$ and discuss/interpret what is indicated by the entries P_{11}, P_{13}, and P_{24} of the product matrix. (b) How many males are expected to join the Navy this year?

Table 9.3 Matrix *A*

A	Age Groups		
Sex	18–19	20–21	22–23
Female	1000	1500	500
Male	2500	3000	2000

Table 9.4 Matrix *B*

B	Likelihood of Joining			
Age Group	Army	Navy	Air Force	Marines
18–19	0.42	0.28	0.17	0.13
20–21	0.38	0.26	0.27	0.09
22–23	0.33	0.25	0.35	0.07

Solution ► **a.** Matrix A has order 2×3 and matrix B has order 3×4. The product matrix P can be found and is a 2×4 matrix. Carefully enter the matrices in your calculator. Figure 9.8 shows the entries of matrix B. Using $[A][B]$ ENTER, the calculator finds the product matrix shown in Figure 9.9. Pressing the right arrow shows the complete product matrix is

$$P = \begin{bmatrix} 1155 & 795 & 750 & 300 \\ 2850 & 1980 & 1935 & 735 \end{bmatrix}.$$

The entry P_{11} is the product of R1 from A and C1 from B, and indicates that for the year, 1155 females are projected to join the Army. In like manner, entry P_{13} shows that 750 females are projected to join the Air Force. Entry P_{24} indicates that 735 males are projected to join the Marines.

C. You've just learned how to compute the product of two matrices

Figure 9.8

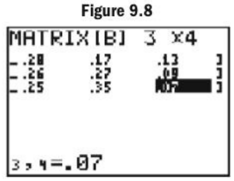
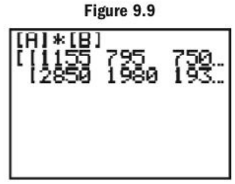


Figure 9.9



b. The product R2 (males) · C2 (Navy) gives $P_{22} = 1980$, meaning 1980 males are expected to join the Navy.

Now try Exercise 59 through 66 ►

9.2 EXERCISES

CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. Two matrices are equal if they are like size and the corresponding entries are equal. In symbols, $A = B$ if _____ = _____.
2. The sum of two matrices (of like size) is found by adding the corresponding entries. In symbols, $A + B =$ _____.
3. The product of a constant times a matrix is called _____ multiplication.
4. The size of a matrix is also referred to as its _____. The order of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is _____.
5. Give two reasons why matrix multiplication is generally not commutative. Include several examples using matrices of various sizes.
6. Discuss the conditions under which matrix multiplication is defined. Include several examples using matrices of various sizes.

▶ DEVELOPING YOUR SKILLS

State the order of each matrix and name the entries in positions a_{12} and a_{23} if they exist. Then name the position a_{ij} of the 5 in each.

7. $\begin{bmatrix} 1 & -3 \\ \mathbf{5} & -7 \end{bmatrix}$

8. $\begin{bmatrix} 19 \\ -11 \\ \mathbf{5} \end{bmatrix}$

9. $\begin{bmatrix} 2 & -3 & 0.5 \\ 0 & \mathbf{5} & 6 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 0.4 \\ -0.1 & \mathbf{5} \\ 0.3 & -3 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 1 & -7 \\ 0 & 8 & 1 \\ \mathbf{5} & -1 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 89 & 55 & 34 & 21 \\ 13 & 8 & \mathbf{5} & 3 \\ 2 & 1 & 1 & 0 \end{bmatrix}$

Determine if the following statements are true, false, or conditional. If false, explain why. If conditional, find values of a, b, c, p, q, and r that will make the statement true.

13. $\begin{bmatrix} \sqrt{1} & \sqrt{4} & \sqrt{8} \\ \sqrt{16} & \sqrt{32} & \sqrt{64} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2\sqrt{2} \\ 4 & 4\sqrt{2} & 8 \end{bmatrix}$

14. $\begin{bmatrix} \frac{3}{2} & \frac{-7}{5} & \frac{13}{10} \\ \frac{-1}{2} & \frac{-2}{5} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1.5 & -1.4 & 1.3 \\ -0.5 & -0.4 & 0.\overline{3} \end{bmatrix}$

15. $\begin{bmatrix} -2 & 3 & a \\ 2b & -5 & 4 \\ 0 & -9 & 3c \end{bmatrix} = \begin{bmatrix} c & 3 & -4 \\ 6 & -5 & -a \\ 0 & -3b & -6 \end{bmatrix}$

16. $\begin{bmatrix} 2p+1 & -5 & 9 \\ 1 & 12 & 0 \\ q+5 & 9 & -2r \end{bmatrix} = \begin{bmatrix} 7 & -5 & 2-q \\ 1 & 3r & 0 \\ -2 & 3p & -8 \end{bmatrix}$

For matrices A through H as given, perform the indicated operation(s), if possible. Do not use a calculator. If an operation cannot be completed, state why.

$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 0.5 \\ 0.2 & 5 \\ -1 & 3 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$ $\quad F = \begin{bmatrix} 6 & -3 & 9 \\ 12 & 0 & -6 \end{bmatrix}$

$G = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & -2 \\ -4 & -3 & 6 \end{bmatrix}$ $\quad H = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$

17. $A + H$ **18.** $E + G$
19. $F + H$ **20.** $G + D$
21. $3H - 2A$ **22.** $2E + 3G$
23. $\frac{1}{2}E - 3D$ **24.** $F - \frac{2}{3}F$
25. ED **26.** DE
27. AH **28.** HA
29. FD **30.** FH
31. HF **32.** EB
33. H^2 **34.** F^2
35. FE **36.** EF

For matrices A through H as given, use a calculator to perform the indicated operation(s), if possible. If an operation cannot be completed, state why.

$A = \begin{bmatrix} -5 & 4 \\ 3 & 9 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{3} \\ \sqrt{3} & 2\sqrt{3} \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$ $\quad F = \begin{bmatrix} -0.52 & 0.002 & 1.032 \\ 1.021 & -1.27 & 0.019 \end{bmatrix}$

$G = \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{-1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{-1}{4} & \frac{11}{16} & \frac{1}{16} \end{bmatrix}$ $\quad H = \begin{bmatrix} \frac{-3}{19} & \frac{4}{57} \\ \frac{1}{19} & \frac{5}{57} \end{bmatrix}$

37. $C + H$ **38.** $A - H$
39. $E + G$ **40.** $G - E$
41. AH **42.** HA
43. EG **44.** GE
45. HB **46.** BH
47. DG **48.** GD
49. C^2 **50.** E^2
51. FG **52.** AF

For Exercises 53 through 56, use a calculator and matrices *A*, *B*, and *C* to verify each statement.

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 2 & 7 & -1 \\ 4 & 0 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0.3 & -0.4 & 1.2 \\ -2.5 & 2 & 0.9 \\ 1 & -0.5 & 0.2 \end{bmatrix}$$

$$C = \begin{bmatrix} 45 & -1 & 3 \\ -6 & 10 & -15 \\ 21 & -28 & 36 \end{bmatrix}$$

53. Matrix multiplication is not generally commutative: (a) $AB \neq BA$, (b) $AC \neq CA$, and (c) $BC \neq CB$.

54. Matrix multiplication is distributive from the left: $A(B + C) = AB + AC$.

55. Matrix multiplication is distributive from the right: $(B + C)A = BA + CA$.

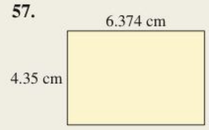
56. Matrix multiplication is associative: $(AB)C = A(BC)$.

▶ WORKING WITH FORMULAS

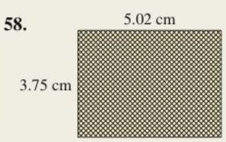
$$\begin{bmatrix} 2 & 2 \\ W & 0 \end{bmatrix} \cdot \begin{bmatrix} L \\ W \end{bmatrix} = \begin{bmatrix} \text{Perimeter} \\ \text{Area} \end{bmatrix}$$

The perimeter and area of a rectangle can be simultaneously calculated using the matrix formula shown, where *L* represents the length and *W* represents the width of the rectangle. Use the matrix formula and your calculator to find the perimeter and area of the rectangles shown, then check the results using $P = 2L + 2W$ and $A = LW$.

57.



58.



▶ APPLICATIONS

59. Custom T's designs and sells specialty T-shirts and sweatshirts, with plants in Verdi and Minsk. The company offers this apparel in three quality levels: standard, deluxe, and premium. Last fall the Verdi office produced 3820 standard, 2460 deluxe, and 1540 premium T-shirts, along with 1960 standard, 1240 deluxe, and 920 premium sweatshirts. The Minsk office produced 4220 standard, 2960 deluxe, and 1640 premium T-shirts, along with 2960 standard, 3240 deluxe, and 820 premium sweatshirts in the same time period.

- **a.** Write a 3×2 "production matrix" for each plant [$V \rightarrow$ Verdi, $M \rightarrow$ Minsk], with a *T-shirt* column, a *sweatshirt* column, and three rows showing how many of the different types of apparel were manufactured.
- **b.** Use the matrices from Part (a) to determine how many more or less articles of clothing were produced by Minsk than Verdi.
- **c.** Use scalar multiplication to find how many shirts of each type will be made at Verdi and Minsk next fall, if each is expecting a 4% increase in business.
- **d.** What will be Custom T's total production next fall (from both plants), for each type of apparel?

60. Terry's Tire Store sells automobile and truck tires through three retail outlets. Sales at the Cahokia store for the months of January, February, and March broke down as follows: 350, 420, and 530 auto tires and 220, 180, and 140 truck tires. The Shady Oak branch sold 430, 560, and 690 auto tires and 280, 320, and 220 truck tires during the same 3 months. Sales figures for the downtown store were 864, 980, and 1236 auto tires and 535, 542, and 332 truck tires.

- **a.** Write a 2×3 "sales matrix" for each store [$C \rightarrow$ Cahokia, $S \rightarrow$ Shady Oak, $D \rightarrow$ Downtown], with *January, February,* and *March* columns, and two rows showing the sales of auto and truck tires respectively.
- **b.** Use the matrices from Part (a) to determine how many more or fewer tires of each type the downtown store sold (each month) over the other two stores combined.
- **c.** Market trends indicate that for the same three months in the following year, the Cahokia store will likely experience a 10% increase in sales, the Shady Oak store a 3% decrease, with sales at the downtown store remaining level (no change). What will be the combined monthly sales from all three stores next year, for each type of tire?

61. Home improvements: Dream-Makers Home Improvements specializes in replacement windows, replacement doors, and new siding. During the peak season, the number of contracts that came from various parts of the city (North, South, East, and West) are shown in matrix *C*. The average profit per contract is shown in matrix P. Compute the product PC and discuss what each entry of the product matrix represents.

$$\begin{array}{r} \\ \text{Windows} \\ \text{Doors} \\ \text{Siding} \end{array}\begin{array}{c} \begin{array}{cccc} \text{N} & \text{S} & \text{E} & \text{W} \end{array} \\ \begin{bmatrix} 9 & 6 & 5 & 4 \\ 7 & 5 & 7 & 6 \\ 2 & 3 & 5 & 2 \end{bmatrix} \end{array} = C$$

$$\begin{array}{c} \begin{array}{ccc} \text{Windows} & \text{Doors} & \text{Siding} \end{array} \\ \begin{bmatrix} 1500 & 500 & 2500 \end{bmatrix} \end{array} = P$$

62. Classical music: Station 90.7—*The Home of Classical Music*—is having their annual fund drive. Being a loyal listener, Mitchell decides that for the next 3 days he will donate money according to his favorite composers, by the number of times their music comes on the air: \$3 for every piece by Mozart (M), \$2.50 for every piece by Beethoven (B), and \$2 for every piece by Vivaldi (V). This information is displayed in matrix D. The number of pieces he heard from each composer is displayed in matrix C. Compute the product DC and discuss what each entry of the product matrix represents.

$$\begin{array}{r} \\ M \\ B \\ V \end{array}\begin{array}{c} \begin{array}{ccc} \text{Mon.} & \text{Tue.} & \text{Wed.} \end{array} \\ \begin{bmatrix} 4 & 3 & 5 \\ 3 & 2 & 4 \\ 2 & 3 & 3 \end{bmatrix} \end{array} = C$$

$$\begin{array}{c} \begin{array}{ccc} \text{M} & \text{B} & \text{V} \end{array} \\ \begin{bmatrix} 3 & 2.5 & 2 \end{bmatrix} \end{array} = D$$

63. Pizza and salad: The science department and math department of a local college are at a pre-semester retreat, and decide to have pizza, salads, and soft drinks for lunch. The quantity of food ordered by each department is shown in matrix Q. The cost of the food item at each restaurant is shown in matrix C using the published prices from three popular restaurants: Pizza Home (PH), Papa Jeff's (PJ), and Dynamos (D).

a. What is the total cost to the math department if the food is ordered from Pizza Home?

b. What is the total cost to the science department if the food is ordered from Papa Jeff's?

c. Compute the product QC and discuss the meaning of each entry in the product matrix.

$$\begin{array}{r} \\ \text{Science} \\ \text{Math} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Pizza} & \text{Salad} & \text{Drink} \end{array} \\ \begin{bmatrix} 8 & 12 & 20 \\ 10 & 8 & 18 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{r} \\ \text{Pizza} \\ \text{Salad} \\ \text{Drink} \end{array}\begin{array}{c} \begin{array}{ccc} \text{PH} & \text{PJ} & \text{D} \end{array} \\ \begin{bmatrix} 8 & 7.5 & 10 \\ 1.5 & 1.75 & 2 \\ 0.90 & 1 & 0.75 \end{bmatrix} \end{array} = C$$

64. Manufacturing pool tables: Cue Ball Incorporated makes three types of pool tables, for homes, commercial use, and professional use. The amount of time required to pack, load, and install each is summarized in matrix T, with all times in hours. The cost of these components in dollars per hour, is summarized in matrix C for two of its warehouses, one on the west coast and the other in the midwest.

a. What is the cost to package, load, and install a commercial pool table from the coastal warehouse?

b. What is the cost to package, load, and install a commercial pool table from the warehouse in the midwest?

c. Compute the product TC and discuss the meaning of each entry in the product matrix.

$$\begin{array}{r} \\ \text{Home} \\ \text{Comm} \\ \text{Prof} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Pack} & \text{Load} & \text{Install} \end{array} \\ \begin{bmatrix} 1 & 0.2 & 1.5 \\ 1.5 & 0.5 & 2.2 \\ 1.75 & 0.75 & 2.5 \end{bmatrix} \end{array} = T$$

$$\begin{array}{r} \\ \text{Pack} \\ \text{Load} \\ \text{Install} \end{array}\begin{array}{c} \begin{array}{cc} \text{Coast} & \text{Midwest} \end{array} \\ \begin{bmatrix} 10 & 8 \\ 12 & 10.5 \\ 13.5 & 12.5 \end{bmatrix} \end{array} = C$$

65. Joining a club: Each school year, among the students planning to join a club, the likelihood a student joins a particular club depends on their class standing. This information is stored in matrix C. The number of males and females from each class that are projected to join a club each year is stored in matrix J. Compute the product JC and use the result to answer the following:

a. Approximately how many females joined the chess club?

b. Approximately how many males joined the writing club?

c. What does the entry P_{13} of the product matrix tells us?

$$\begin{array}{r} \\ \text{Female} \\ \text{Male} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Fresh} & \text{Soph} & \text{Junior} \end{array} \\ \begin{bmatrix} 25 & 18 & 21 \\ 22 & 19 & 18 \end{bmatrix} \end{array} = J$$

$$\begin{array}{r} \\ \text{Fresh} \\ \text{Soph} \\ \text{Junior} \end{array}\begin{array}{c} \begin{array}{ccc} \text{Spanish} & \text{Chess} & \text{Writing} \end{array} \\ \begin{bmatrix} 0.6 & 0.1 & 0.3 \\ 0.5 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.4 \end{bmatrix} \end{array} = C$$

66. Designer shirts: The SweatShirt Shoppe sells three types of designs on its products: stenciled (S), embossed (E), and applique (A). The quantity of each size sold is shown in matrix Q. The retail price of each sweatshirt depends on its size and whether it was finished by hand or machine. Retail prices are shown in matrix C. Assuming all stock is sold,

a. How much revenue was generated by the large sweatshirts?

b. How much revenue was generated by the extra-large sweatshirts?

c. What does the entry P_{11} of the product matrix QC tell us?

$$\begin{array}{r} \\ \text{med} \\ \text{large} \\ \text{x-large} \end{array}\begin{array}{c} \begin{array}{ccc} \text{S} & \text{E} & \text{A} \end{array} \\ \begin{bmatrix} 30 & 30 & 15 \\ 60 & 50 & 20 \\ 50 & 40 & 30 \end{bmatrix} \end{array} = Q$$

$$\begin{array}{r} \\ \text{S} \\ \text{E} \\ \text{A} \end{array}\begin{array}{c} \begin{array}{cc} \text{Hand} & \text{Machine} \end{array} \\ \begin{bmatrix} 40 & 25 \\ 60 & 40 \\ 90 & 60 \end{bmatrix} \end{array} = C$$

▶ EXTENDING THE CONCEPT

67. For the matrix A shown, use your calculator to compute A^2, A^3, A^4, and A^5. Do you notice a pattern? Try to write a "matrix formula" for A^n, where n is a positive integer, then use your formula to find A^6. Check results using a calculator.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

68. The matrix $M = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ has some very interesting properties. Compute the powers M^2, M^3, M^4, and M^5, then discuss what you find. Try to find/create another 2×2 matrix that has similar properties.

69. For the "matrix equation" $\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, use matrix multiplication and two systems of equations to find the entries a, b, c, and d that make the equation true.

▶ MAINTAINING YOUR SKILLS

70. (5.2) Solve the system using elimination.

$$\begin{cases} x + 2y - z = 3 \\ -2x - y + 3z = -5 \\ 5x + 3y - 2z = 2 \end{cases}$$

71. (6.5) Evaluate $\cos(\cos^{-1}0.3211)$.

72. (7.6) Solve $z^4 - 81i = 0$ using the nth roots theorem. Leave your answer in trigonometric form.

73. (3.2) Find the quotient using synthetic division, then check using multiplication.

$$\frac{x^3 - 9x + 10}{x - 2}$$

MID-CHAPTER CHECK

State the size of each matrix and identify the entry in second row, third column.

1. $A = \begin{bmatrix} 0.4 & 1.1 & 0.2 \\ -0.2 & 0.1 & -0.9 \\ 0.7 & 0.4 & 0.8 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 1 & \frac{1}{2} & 5 \\ 4 & \frac{3}{4} & 0 & -3 \end{bmatrix}$

Write each system in matrix form and solve using row operations to triangularize the matrix. If the system is linearly dependent, write the solution using a parameter.

3. $\begin{cases} 2x + 3y = -5 \\ -5x - 4y = 2 \end{cases}$

4. $\begin{cases} -x + y - 5z = 23 \\ 2x + 4y - z = 9 \\ 3x - 5y + z = 1 \end{cases}$

5. $\begin{cases} x + y - 3z = 11 \\ 4x - y - 2z = -4 \\ 3x - 2y + z = 7 \end{cases}$

6. For matrices A and B given, compute:

$$A = \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 10 & 15 \\ -30 & -5 \end{bmatrix}$$

a. $A - B$ **b.** $\frac{2}{5}B$ **c.** $5A + B$

7. For matrices C and D given, use a calculator to find:

$$C = \begin{bmatrix} -0.2 & 0 & 0.2 \\ 0.4 & 0.8 & 0 \\ 0.1 & -0.2 & -0.1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 2.5 & 10 \\ -2.5 & 0 & -5 \\ 10 & 2.5 & 10 \end{bmatrix}$$

a. $C + \frac{1}{5}D$ **b.** $-0.6D$ **c.** CD

8. For the matrices A, B, C, and D given, compute the products indicated (if possible):

$$A = \begin{bmatrix} 4 & -1 \\ 0 & -5 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -2 \\ 0 & 1 \\ 4 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 4 & -8 & -3 \\ -1 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 & -6 \\ -1 & -3 & 0 \\ 1 & 5 & -4 \end{bmatrix}$$

a. AC **b.** $-2CD$ **c.** BA **d.** $CB - 4A$

9. Create a system of equations to model this exercise, then write the system in matrix form and solve. The campus bookstore offers both new and used texts to students. In a recent biology class with 24 students, 14 bought used texts and 10 bought new texts, with the class as a whole paying $2370. Of the 6 premed students in class, 2 bought used texts, and 4 bought new texts, with the group paying a total of $660. How much does a used text cost? How much does a new text cost?

10. Matrix A shown gives the number and type of extended warranties sold to individual car owners and to business fleets. Matrix B shows the promotions offered to those making the purchase. Compute the product matrix $P = AB$, and state what each entry of the product matrix represents.

Matrix *A*

Extended Warranties	80,000 mi	100,000 mi	120,000 mi
Individuals	30	25	10
Businesses	20	12	5

Matrix *B*

Promotions	Rebate	Free AAA Membership
80,000 mi	$50	1 yr
100,000 mi	$75	2 yr
120,000 mi	$100	3 yr

REINFORCING BASIC CONCEPTS

More on Matrix Multiplication

To help understand and master the concept of matrix multiplication, it helps to take a closer look at the entries of the product matrix. Recall for the product $AB = P$, the entry P_{11} in the product matrix is the result of multiplying the 1st row of A with the 1st column of B, the entry P_{12} is the result of multiplying 1st row of A, with the 2nd column of B, and so on.

Exercise 1: The product of the 3rd row of A with the 2nd column of B, gives what entry in P?

Exercise 2: The entry P_{13} is the result of what product? The entry P_{22} is the result of what product?

Exercise 3: If P_{33} is the last entry of the product matrix, what are the possible sizes of A and B?

Exercise 4: Of the eight matrices shown here, only two produce the product matrix P shown. Use the ideas highlighted above to determine which two.

$$P = \begin{bmatrix} 1 & 2 \\ 6 & -2 \\ 7 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 & -6 \\ -1 & -3 & 0 \\ 1 & 5 & -4 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 4 \\ -1 & 0 \\ 3 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 3 \end{bmatrix}$$

9.3 Solving Linear Systems Using Matrix Equations

Learning Objectives

In Section 9.3 you will learn how to:

- ❑ **A.** Recognize the identity matrix for multiplication
- ❑ **B.** Find the inverse of a square matrix
- ❑ **C.** Solve systems using matrix equations
- ❑ **D.** Use determinants to find whether a matrix is invertible

While using matrices and row operations offers a degree of efficiency in solving systems, we are still required to solve for each variable *individually*. Using matrix multiplication we can actually rewrite a given system as a single *matrix equation,* in which the solutions are computed *simultaneously*. As with other kinds of equations, the use of identities and inverses are involved, which we now develop in the context of matrices.

A. Multiplication and Identity Matrices

From the properties of real numbers, 1 is the identity for multiplication since $n \cdot 1 = 1 \cdot n = n$. A similar identity exists for matrix multiplication. Consider the 2×2 matrix $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$. While matrix multiplication is not *generally* commutative, if we can find a matrix B where $AB = BA = A$, then B is a prime candidate for the identity matrix, which is denoted I. For the products AB and BA to be possible and have the same order as A, we note B must also be a 2×2 matrix. Using the arbitrary matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have the following.

EXAMPLE 1A ► Solving $AB = A$ to Find the Identity Matrix

For $\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$, use matrix multiplication, the equality of matrices, and systems of equations to find the value of a, b, c, and d.

Solution ► The product on the left gives $\begin{bmatrix} a + 4c & b + 4d \\ -2a + 3c & -2b + 3d \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$.

Since corresponding entries must be equal (shown by matching colors), we can find a, b, c, and d by solving the systems $\begin{cases} a + 4c = 1 \\ -2a + 3c = -2 \end{cases}$ and $\begin{cases} b + 4d = 4 \\ -2b + 3d = 3 \end{cases}$. For the first system, $2R1 + R2$ shows $a = 1$ and $c = 0$. Using $2R1 + R2$ for the second shows $b = 0$ and $d = 1$. It appears $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a candidate for the identity matrix.

►

Before we name B as the identity matrix, we must show that $AB = BA = A$.

EXAMPLE 1B ► Verifying $AB = BA = A$

Given $A = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, determine if $AB = A$ and $BA = A$.

Solution ►

$$AB = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1(1) + 4(0) & 1(0) + 4(1) \\ -2(1) + 3(0) & -2(0) + 3(1) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} = A \checkmark$$

$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1(1) + 0(-2) & 1(4) + 0(3) \\ 0(1) + 1(-2) & 0(4) + 1(3) \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 3 \end{bmatrix} = A \checkmark$$

Since $AB = A = BA$, B is the identity matrix I.

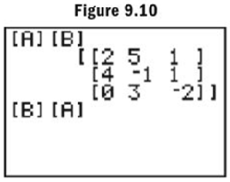
Now try Exercises 7 through 10 ►

By replacing the entries of $A = \begin{bmatrix} 1 & 4 \\ -2 & -3 \end{bmatrix}$ with those of the general matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we can show that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity for *all* 2×2 matrices. In considering the identity for larger matrices, we find that only *square matrices* have inverses, since $AI = IA$ is the primary requirement (the multiplication must be possible in both directions). This is commonly referred to as *multiplication from the right* and *multiplication from the left.* Using the same procedure as before we can show $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the identity for 3×3 matrices (denoted I_3). The $n \times n$ identity matrix I_n consists of 1's down the main diagonal and 0's for all other entries. Also, the identity I_n for a square matrix is unique.

A. You've just learned how to recognize the identity matrix for multiplication

As in Section 9.2, a graphing calculator can be used to investigate operations on matrices and matrix properties. For the 3×3 matrix $A = \begin{bmatrix} 2 & 5 & 1 \\ 4 & -1 & 1 \\ 0 & 3 & -2 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, a calculator will confirm that $AI_3 = A = I_3A$. Carefully enter A into your calculator as matrix A, and I_3 as matrix B. Figure 9.10 shows $AB = A$ and after pressing ENTER, the calculator will verify $BA = A$, although the screen cannot display the result without scrolling. **See Exercises 11 through 14.**

Figure 9.10



B. The Inverse of a Matrix

Again from the properties of real numbers, we know the multiplicative inverse for a is $a^{-1} = \frac{1}{a}$ $(a \neq 0)$, since the products $a \cdot a^{-1}$ and $a^{-1} \cdot a$ yield the identity 1. To show that a similar inverse exists for matrices, consider the square matrix $A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ and an arbitrary matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If we can find a matrix B, where $AB = BA = I$, then B is a prime candidate for the inverse matrix of A, which is denoted A^{-1}. Proceeding as in Examples 1A and 1B gives the result shown in Example 2.

EXAMPLE 2A ▶ Solving $AB = I$ to find A^{-1}

For $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, use matrix multiplication, the equality of matrices, and systems of equations to find the entries of B.

Solution ▶ The product on the left gives $\begin{bmatrix} \mathbf{6a + 5c} & 6b + 5d \\ 2a + 2c & 2b + 2d \end{bmatrix} = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 1 \end{bmatrix}$. Since corresponding entries must be equal (shown by matching colors), we find the values of a, b, c, and d by solving the systems $\begin{cases} \mathbf{6a + 5c = 1} \\ 2a + 2c = 0 \end{cases}$ and $\begin{cases} 6b + 5d = 0 \\ 2b + 2d = 1 \end{cases}$. Using $-3R2 + R1$ for the first system shows $a = 1$ and $c = -1$, while $-3R2 + R1$ for the second system shows $b = -2.5$ and $d = 3$. Matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}$ is the prime candidate for A^{-1}.

To determine if A^{-1} has truly been found, we check to see if multiplication from the right and multiplication from the left yields the matrix I: $AB = BA = I$.

EXAMPLE 2B ► Verifying $B = A^{-1}$

For the matrices $A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}$ from Example 2A, determine if $AB = BA = I$.

Solution ►

$$AB = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6(1) + 5(-1) & 6(-2.5) + 5(3) \\ 2(1) + 2(-1) & 2(-2.5) + 2(3) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

$$BA = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1(6) + (-2.5)(2) & 1(5) + (-2.5)(2) \\ -1(6) + 3(2) & -1(5) + 3(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Since $AB = BA = I$, we conclude $B = A^{-1}$.

Now try Exercises 15 through 22 ►

These observations guide us to the following definition of an inverse matrix.

The Inverse of a Matrix

Given an $n \times n$ matrix A, if there exists an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$, then A^{-1} is the inverse of matrix A.

We will soon discover that while only square matrices have inverses, not every square matrix has an inverse. If an inverse exists, the matrix is said to be **invertible.** For 2×2 matrices that are invertible, a simple formula exists for computing the inverse. The formula is derived in the *Strengthening Core Skills* feature at the end of Chapter 9.

The Inverse of a 2×2 Matrix

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad - bc \neq 0$

To "test" the formula, again consider the matrix $A = \begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$, where $a = 6$, $b = 5$, $c = 2$, and $d = 2$:

$$A^{-1} = \frac{1}{(6)(2) - (5)(2)}\begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -5 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix} \checkmark$$

B. You've just learned how to find the inverse of a square matrix

See Exercises 63 through 66 for more practice with this formula.

Almost without exception, real-world applications involve much larger matrices, with entries that are not integer-valued. Although the *equality of matrices* method from Example 2 can be extended to find the inverse of larger matrices, the process becomes very tedious and too time consuming to be useful. As an alternative, the **augmented matrix method** can be used. This process is discussed in the *Strengthening Core Skills* feature at the end Chapter 9 (see page 904). For practical reasons, we will rely on a calculator to produce these larger inverse matrices. This is done by (1) carefully entering a square matrix A into the calculator, (2) returning to the home screen and (3) calling up matrix A and pressing the x^{-1} key and ENTER to find A^{-1}. In the context of matrices, calculators are programmed to compute an inverse matrix, rather than to somehow find a reciprocal. **See Exercises 23 through 26.**

C. Solving Systems Using Matrix Equations

One reason matrix multiplication has its row × column definition is to assist in writing a linear system of equations as a single matrix equation. The equation consists of the matrix of constants B on the right, and a product of the coefficient matrix A with the matrix of variables X on the left: $AX = B$. For $\begin{cases} x + 4y - z = 10 \\ 2x + 5y - 3z = 7 \\ 8x + y - 2z = 11 \end{cases}$, the matrix equation is $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & -3 \\ 8 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 11 \end{bmatrix}$. Note that computing the product on the left will yield the original system.

Once written as a matrix equation, the system can be solved using an inverse matrix and the following sequence. If A represents the matrix of coefficients, X the matrix of variables, B the matrix of constants, and I the appropriate identity, the sequence is

(1)	$AX = B$	matrix equation
(2)	$A^{-1}(AX) = A^{-1}B$	multiply from the left by the inverse of A
(3)	$(A^{-1}A)X = A^{-1}B$	associative property
(4)	$IX = A^{-1}B$	$A^{-1}A = I$
(5)	$X = A^{-1}B$	$IX = X$

Lines 1 through 5 illustrate the steps that make the method work. In actual practice, after carefully entering the matrices, only step 5 is used when solving matrix equations using technology. Once matrix A is entered, the calculator will automatically *find* and *use* A^{-1} as we enter $A^{-1}B$.

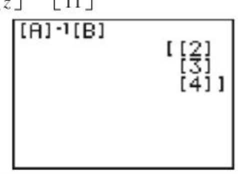
EXAMPLE 3 ► Using Technology to Solve a Matrix Equation

Use a calculator and a matrix equation to solve the system

$$\begin{cases} x + 4y - z = 10 \\ 2x + 5y - 3z = 7. \\ 8x + y - 2z = 11 \end{cases}$$

Solution ► As before, the matrix equation is $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 5 & -3 \\ 8 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 11 \end{bmatrix}$.

Carefully enter (and double-check) the matrix of coefficients as matrix A in your calculator, and the matrix of constants as matrix B. The product $A^{-1}B$ shows the solution is $x = 2$, $y = 3$, $z = 4$. Verify by substitution.



Now try Exercises 27 through 44 ►

Figure 9.11

The matrix equation method does have a few shortcomings. Consider the system whose corresponding matrix equation is $\begin{bmatrix} 4 & -10 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \end{bmatrix}$. After entering the matrix of coefficients A and matrix of constants B, attempting to compute $A^{-1}B$ results in the error message shown in Figure 9.11. The calculator is unable to return a solution due to something called a **"singular matrix."** To investigate further, we attempt to find A^{-1} for $\begin{bmatrix} 4 & -10 \\ -2 & 5 \end{bmatrix}$ using the formula for a 2×2 matrix. With $a = 4$, $b = -10$, $c = -2$, and $d = 5$, we have

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{(4)(5) - (-10)(-2)}\begin{bmatrix} 5 & 10 \\ 2 & 4 \end{bmatrix}$$
$$= \frac{1}{0}\begin{bmatrix} 5 & 10 \\ 2 & 4 \end{bmatrix}$$

C. You've just learned how to solve systems using matrix equations

Since division by zero is undefined, we conclude that matrix A has no inverse. A matrix having no inverse is said to be **singular** or **noninvertible.** Solving systems using matrix equations is only possible when the matrix of coefficients is **nonsingular.**

D. Determinants and Singular Matrices

As a practical matter, it becomes important to know ahead of time whether a particular matrix has an inverse. To help with this, we introduce one additional operation on a square matrix, that of calculating its **determinant.** For a 1×1 matrix the determinant is the entry itself. For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant of A, written as $\det(A)$ or denoted with vertical bars as $|A|$, is computed as *a difference of diagonal products* beginning with the upper-left entry:

2nd diagonal product

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

1st diagonal product

The Determinant of a 2 × 2 Matrix

Given any 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

$$\det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

EXAMPLE 4 ► Calculating Determinants

Compute the determinant of each matrix given.

a. $B = \begin{bmatrix} 3 & 2 \\ 1 & -6 \end{bmatrix}$

b. $C = \begin{bmatrix} 5 & 2 & 1 \\ -1 & -3 & 4 \end{bmatrix}$

c. $D = \begin{bmatrix} 4 & -10 \\ -2 & 5 \end{bmatrix}$

Solution ▶ **a.** $\det(B) = \begin{vmatrix} 3 & 2 \\ 1 & -6 \end{vmatrix} = (3)(-6) - (1)(2) = -20$

b. Determinants are only defined for square matrices.

c. $\det(D) = \begin{vmatrix} 4 & -10 \\ -2 & 5 \end{vmatrix} = (4)(5) - (-2)(-10) = 20 - 20 = 0$

Now try Exercises 45 through 48 ▶

Notice from Example 4c, the determinant of $\begin{bmatrix} 4 & -10 \\ -2 & 5 \end{bmatrix}$ was zero, and this is the same matrix we earlier found had no inverse. This observation can be extended to larger matrices and offers the connection we seek between a given matrix, its inverse, and matrix equations.

Singular Matrices

If A is a square matrix and $\det(A) = 0$, the inverse matrix *does not exist* and A is said to be *singular* or *noninvertible.*

In summary, inverses exist only for square matrices, but not every square matrix has an inverse. If the determinant of a square matrix is zero, an inverse does not exist and the method of matrix equations cannot be used to solve the system.

WORTHY OF NOTE

For the determinant of a general $n \times n$ matrix using **cofactors,** see Appendix II.

To use the determinant test for a 3×3 system, we need to compute a 3×3 determinant. At first glance, our experience with 2×2 determinants appears to be of little help. However, every entry in a 3×3 matrix is associated with a smaller 2×2 matrix, formed by *deleting the row and column* of that entry and using the entries that remain. These 2×2's are called the **associated minor matrices** or simply the **minors.** Using a general matrix of coefficients, we'll identify the minors associated with the entries in the first row.

$$\begin{bmatrix} \textcircled{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & \textcircled{a_{12}} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & \textcircled{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Entry: a_{11} associated minor $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$ **Entry: a_{12} associated minor** $\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$ **Entry: a_{13} associated minor** $\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

To illustrate, consider the system shown, and (1) form the matrix of coefficients, (2) identify the minor matrices associated with the entries in the first row, and (3) compute the determinant of each *minor.*

$$\begin{cases} 2x + 3y - z = 1 \\ x - 4y + 2z = -3 \\ 3x + y = -1 \end{cases} \quad \text{(1) Matrix of coefficients } \begin{bmatrix} 2 & 3 & -1 \\ 1 & -4 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$

$$(2) \begin{bmatrix} \textcircled{2} & 3 & -1 \\ 1 & -4 & 2 \\ 3 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & \textcircled{3} & -1 \\ 1 & -4 & 2 \\ 3 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & \textcircled{-1} \\ 1 & -4 & 2 \\ 3 & 1 & 0 \end{bmatrix}$$

Entry a_{11}: 2 associated minor $\begin{bmatrix} -4 & 2 \\ 1 & 0 \end{bmatrix}$ **Entry a_{12}: 3 associated minor** $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ **Entry a_{13}: −1 associated minor** $\begin{bmatrix} 1 & -4 \\ 3 & 1 \end{bmatrix}$

(3) **Determinant of minor**	**Determinant of minor**	**Determinant of minor**
$(-4)(0) - (1)(2) = -2$	$(1)(0) - (3)(2) = -6$	$(1)(1) - (3)(-4) = 13$

For computing a 3×3 determinant, we illustrate a technique called **expansion by minors.**

The Determinant of a 3×3 Matrix—Expansion by Minors

For the matrix M shown, det(M) is the unique number computed as follows:

1. Select any row or column and form the product of each entry with its minor matrix. The illustration here uses the entries in row 1:

$$\det(M) = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2. The *signs used between terms* of the expansion depends on the row or column chosen, according to the *sign chart* shown.

matrix M

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Sign Chart

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The determinant of a matrix is unique and *any* row or column can be used. For this reason, it's helpful to select the row or column having the most zero, positive, and/or smaller entries.

EXAMPLE 5 ► Calculating a 3×3 Determinant

Compute the determinant of $M = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$.

Solution ► Since the second row has the "smallest" entries as well as a zero entry, we compute the determinant using this row. According to the sign chart, the signs of the terms will be negative–positive–negative, giving

$$\begin{aligned} \det(M) &= -(1)\begin{vmatrix} 1 & -3 \\ 1 & 4 \end{vmatrix} + (-1)\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} - (0)\begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix} \\ &= -1(4 + 3) + (-1)(8 - 6) - (0)(2 + 2) \\ &= -7 + (-2) - 0 \\ &= -9 \rightarrow \text{The value of } \det(M) \text{ is } -9. \end{aligned}$$

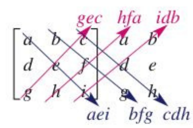
Now try Exercises 49 through 54 ►

Try computing the determinant of M two more times, using a different row or column each time. Since the determinant is unique, you should obtain the same result.

There are actually other alternatives for computing a 3×3 determinant. The first is called **determinants by column rotation,** and takes advantage of patterns generated from the expansion of minors. This method is applied to the matrix shown, which uses alphabetical entries for simplicity.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{aligned} &= a(ei - fh) - b(di - fg) + c(dh - eg) && \text{expansion using R1} \\ &= aei - afh - bdi + bfg + cdh - ceg && \text{distribute} \\ &= aei + bfg + cdh - afh - bdi - ceg && \text{rewrite result} \end{aligned}$$

Although history is unsure of who should be credited, notice that if you repeat the first two columns to the right of the given matrix ("rotation of columns"), identical products are obtained using the six diagonals formed—three in the downward direction using addition, three in the upward direction using subtraction.

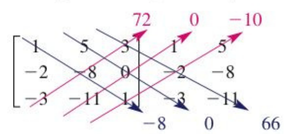


Adding the products in **blue** (regardless of sign) and subtracting the products in **red** (regardless of sign) gives the determinant. This method is more efficient than expansion by minors, *but can only be used for* 3 × 3 *matrices!*

EXAMPLE 6 ► Calculating det(*A*) Using Column Rotation

Use the column rotation method to find the determinant of $A = \begin{bmatrix} 1 & 5 & 3 \\ -2 & -8 & 0 \\ -3 & -11 & 1 \end{bmatrix}$.

Solution ► Rotate columns 1 and 2 to the right as above, and compute the diagonal products.



Adding the products in **blue** (regardless of sign) and subtracting the products in **red** (regardless of sign) shows det(A) = −4:

$$-8 + 0 + 66 - 72 - 0 - (-10) = -4.$$

Now try Exercises 55 through 58 ►

The final method is presented in the *Extending the Concept* feature of the Exercise Set, and shows that if certain conditions are met, the determinant of a matrix can be found using its triangularized form.

EXAMPLE 7 ► Solving a System after Verifying *A* is Invertible

Given the system shown here, (1) form the matrix equation $AX = B$; (2) compute the determinant of the coefficient matrix and determine if you can proceed; and (3) if so, solve the system using a matrix equation.

$$\begin{cases} 2x + 1y - 3z = 11 \\ 1x - 1y \qquad\; = 1 \\ -2x + 1y + 4z = -8 \end{cases}$$

Solution ► **1.** Form the matrix equation $AX = B$:

$$\begin{bmatrix} 2 & 1 & -3 \\ 1 & -1 & 0 \\ -2 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ -8 \end{bmatrix}$$

2. Since det(A) is nonzero (from Example 5), we can proceed.

3. For $X = A^{-1}B$, input $A^{-1}B$ on your calculator and press ENTER.

$$X = \begin{bmatrix} \frac{4}{9} & \frac{7}{9} & \frac{1}{3} \\ \frac{4}{9} & -\frac{2}{9} & \frac{1}{3} \\ \frac{1}{9} & \frac{4}{9} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 11 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

calculator computes and uses A^{-1} in one step

The solution is the ordered triple $(3, 2, -1)$

Now try Exercises 59 through 62 ►

We close this section with an application involving a 4×4 system.

EXAMPLE 8 ► Solving an Application Using Technology and Matrix Equations

A local theater sells four sizes of soft drinks: 32 oz @ \$2.25; 24 oz @ \$1.90; 16 oz @ \$1.50; and 12 oz @ \$1.20/each. As part of a "free guest pass" promotion, the manager asks employees to try and determine the number of each size sold, given the following information: (1) the total revenue from soft drinks was \$719.80; (2) there were 9096 oz of soft drink sold; (3) there was a total of 394 soft drinks sold; and (4) the number of 24-oz and 12-oz drinks sold was 12 more than the number of 32-oz and 16-oz drinks sold. Write a system of equations that models this information, then solve the system using a matrix equation.

Solution ► If we let x, l, m, and s represent the number of 32-oz, 24-oz, 16-oz, and 12-oz soft drinks sold, the following system is produced:

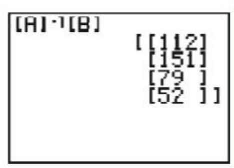
$$\begin{array}{r} \text{revenue:} \\ \text{ounces sold:} \\ \text{quantity sold:} \\ \text{relationship between amounts sold:} \end{array} \begin{cases} 2.25x + 1.90l + 1.50m + 1.20s = 719.8 \\ 32x + 24l + 16m + 12s = 9096 \\ x + l + m + s = 394 \\ l + s = x + m + 12 \end{cases}$$

D. You've just learned how to use determinants to find whether a matrix is invertible

When written as a matrix equation the system becomes:

$$\begin{bmatrix} 2.25 & 1.9 & 1.5 & 1.2 \\ 32 & 24 & 16 & 12 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ l \\ m \\ s \end{bmatrix} = \begin{bmatrix} 719.8 \\ 9096 \\ 394 \\ 12 \end{bmatrix}$$

To solve, carefully enter the matrix of coefficients as matrix A, and the matrix of constants as matrix B, then compute $A^{-1}B = X$ [verify $\det(A) \neq 0$]. This gives a solution of $(x, l, m, s) = (112, 151, 79, 52)$.



Now try Exercises 67 to 78 ►

9.3 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The $n \times n$ identity matrix I_n consists of 1's down the ______ and ______ for all other entries.

2. Given square matrices A and B of like size, B is the inverse of A if ___ = ___ = ___. Notationally we write $B =$ ____.

3. The product of a square matrix A and its inverse A^{-1} yields the ______ matrix.

4. If the determinant of a matrix is zero, the matrix is said to be ______ or ______, meaning no inverse exists.

5. Explain why inverses exist only for square matrices, then discuss why some square matrices do not have an inverse. Illustrate each point with an example.

6. What is the connection between the determinant of a 2×2 matrix and the formula for finding its inverse? Use the connection to create a 2×2 matrix that is invertible, and another that is not.

▶ DEVELOPING YOUR SKILLS

Use matrix multiplication, equality of matrices, and the arbitrary matrix given to show that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

7. $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$

8. $A = \begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 9 & -7 \\ -5 & 4 \end{bmatrix}$

9. $A = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.2 \end{bmatrix}$

10. $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{8} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{8} \end{bmatrix}$

For $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, **and** $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, **show** $AI = IA = A$ **for the matrices of like size. Use a calculator for Exercise 14.**

11. $\begin{bmatrix} -3 & 8 \\ -4 & 10 \end{bmatrix}$

12. $\begin{bmatrix} 0.5 & -0.2 \\ -0.7 & 0.3 \end{bmatrix}$

13. $\begin{bmatrix} -4 & 1 & 6 \\ 9 & 5 & 3 \\ 0 & -2 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 9 & 1 & 3 & -1 \\ 2 & 0 & -5 & 3 \\ 4 & 6 & 1 & 0 \\ 0 & -2 & 4 & 1 \end{bmatrix}$

Find the inverse of each 2×2 matrix using matrix multiplication, equality of matrices, and a system of equations.

15. $\begin{bmatrix} 5 & -4 \\ 2 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 1 & -5 \\ 0 & -4 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -3 \\ 4 & -10 \end{bmatrix}$

18. $\begin{bmatrix} -2 & 0.4 \\ 1 & 0.8 \end{bmatrix}$

Demonstrate that $B = A^{-1}$, by showing $AB = BA = I$. Do not use a calculator.

19. $A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}$, $B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$

20. $A = \begin{bmatrix} -2 & -6 \\ 4 & 11 \end{bmatrix}$, $B = \begin{bmatrix} 5.5 & 3 \\ -2 & -1 \end{bmatrix}$

21. $A = \begin{bmatrix} 4 & -5 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{4} & \frac{5}{8} \\ 0 & \frac{1}{2} \end{bmatrix}$

22. $A = \begin{bmatrix} -2 & 5 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} \frac{4}{7} & \frac{5}{7} \\ \frac{3}{7} & \frac{2}{7} \end{bmatrix}$

Use a calculator to find $A^{-1} = B$, then confirm the inverse by showing $AB = BA = I$.

23. $A = \begin{bmatrix} -2 & 3 & 1 \\ 5 & 2 & 4 \\ 2 & 0 & -1 \end{bmatrix}$

24. $A = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0 & 0.3 & 0.6 \\ 1 & 0.4 & -0.3 \end{bmatrix}$

25. $A = \begin{bmatrix} -7 & 5 & -3 \\ 1 & 9 & 0 \\ 2 & -2 & -5 \end{bmatrix}$

26. $A = \dfrac{1}{12}\begin{bmatrix} 12 & -6 & 3 & 0 \\ 0 & -4 & 8 & -12 \\ 12 & -12 & 0 & 0 \\ 0 & 12 & 0 & -12 \end{bmatrix}$

Write each system in the form of a matrix equation. Do not solve.

27. $\begin{cases} 2x - 3y = 9 \\ -5x + 7y = 8 \end{cases}$

28. $\begin{cases} 0.5x - 0.6y = 0.6 \\ -0.7x + 0.4y = -0.375 \end{cases}$

29. $\begin{cases} x + 2y - z = 1 \\ x + z = 3 \\ 2x - y + z = 3 \end{cases}$

30. $\begin{cases} 2x - 3y - 2z = 4 \\ \frac{1}{4}x - \frac{2}{5}y + \frac{3}{4}z = \frac{-1}{3} \\ -2x + 1.3y - 3z = 5 \end{cases}$

31. $\begin{cases} -2w + x - 4y + 5z = -3 \\ 2w - 5x + y - 3z = 4 \\ -3w + x + 6y + z = 1 \\ w + 4x - 5y + z = -9 \end{cases}$

32. $\begin{cases} 1.5w + 2.1x - 0.4y + z = 1 \\ 0.2w - 2.6x + y = 5.8 \\ 3.2x + z = 2.7 \\ 1.6w + 4x - 5y + 2.6z = -1.8 \end{cases}$

Write each system as a matrix equation and solve (if possible) using inverse matrices and your calculator. If the coefficient matrix is singular, write *no solution*.

33. $\begin{cases} 0.05x - 3.2y = -15.8 \\ 0.02x + 2.4y = 12.08 \end{cases}$

34. $\begin{cases} 0.3x + 1.1y = 3.5 \\ -0.5x - 2.9y = -10.1 \end{cases}$

35. $\begin{cases} \frac{-1}{6}u + \frac{1}{4}v = 1 \\ \frac{1}{2}u - \frac{2}{3}v = -2 \end{cases}$

36. $\begin{cases} \sqrt{2}a + \sqrt{3}b = \sqrt{5} \\ \sqrt{6}a + 3b = \sqrt{7} \end{cases}$

37. $\begin{cases} \frac{-1}{8}a + \frac{3}{5}b = \frac{5}{6} \\ \frac{5}{16}a - \frac{3}{2}b = \frac{-4}{5} \end{cases}$

38. $\begin{cases} 3\sqrt{2}a + 2\sqrt{3}b = 12 \\ 5\sqrt{2}a - 3\sqrt{3}b = 1 \end{cases}$

39. $\begin{cases} 0.2x - 1.6y + 2z = -1.9 \\ -0.4x - y + 0.6z = -1 \\ 0.8x + 3.2y - 0.4z = 0.2 \end{cases}$

40. $\begin{cases} 1.7x + 2.3y - 2z = 41.5 \\ 1.4x - 0.9y + 1.6z = -10 \\ -0.8x + 1.8y - 0.5z = 16.5 \end{cases}$

41. $\begin{cases} x - 2y + 2z = 6 \\ 2x - 1.5y + 1.8z = 2.8 \\ \frac{-2}{3}x + \frac{1}{2}y - \frac{3}{5}z = -\frac{11}{30} \end{cases}$

42. $\begin{cases} 4x - 5y - 6z = 5 \\ \frac{1}{8}x - \frac{3}{5}y + \frac{5}{4}z = \frac{-2}{3} \\ -0.5x + 2.4y - 5z = 5 \end{cases}$

43. $\begin{cases} -2w + 3x - 4y + 5z = -3 \\ 0.2w - 2.6x + y - 0.4z = 2.4 \\ -3w + 3.2x + 2.8y + z = 6.1 \\ 1.6w + 4x - 5y + 2.6z = -9.8 \end{cases}$

44. $\begin{cases} 2w - 5x + 3y - 4z = 7 \\ 1.6w + 4.2y - 1.8z = 5.4 \\ 3w + 6.7x - 9y + 4z = -8.5 \\ 0.7x - 0.9z = 0.9 \end{cases}$

Compute the determinant of each matrix and state whether an inverse matrix exists. Do not use a calculator.

45. $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}$

46. $\begin{bmatrix} 0.6 & 0.3 \\ 0.4 & 0.5 \end{bmatrix}$

47. $\begin{bmatrix} 1.2 & -0.8 \\ 0.3 & -0.2 \end{bmatrix}$

48. $\begin{bmatrix} -2 & 6 \\ -3 & 9 \end{bmatrix}$

Compute the determinant of each matrix without using a calculator. If the determinant is zero, write *singular matrix*.

49. $A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix}$

50. $B = \begin{bmatrix} -2 & 2 & 1 \\ 0 & -1 & 2 \\ 4 & -4 & 0 \end{bmatrix}$

51. $C = \begin{bmatrix} -2 & 3 & 4 \\ 0 & 6 & 2 \\ 1 & -1.5 & -2 \end{bmatrix}$

52. $D = \begin{bmatrix} 1 & 2 & -0.8 \\ 2.5 & 5 & -2 \\ 3 & 0 & -2.5 \end{bmatrix}$

Use a calculator to compute the determinant of each matrix. If the determinant is zero, write *singular matrix*. If the determinant is nonzero, find A^{-1} and store the result as matrix B (STO→ 2nd X^{-1} 2: [B] ENTER). Then verify each inverse by showing $AB = BA = I$.

53. $A = \begin{bmatrix} 1 & 0 & 3 & -4 \\ 2 & 5 & 0 & 1 \\ 8 & 15 & 6 & -5 \\ 0 & 8 & -4 & 1 \end{bmatrix}$

54. $M = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & -3 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & -1 & 1 & 4 \end{bmatrix}$

Compute the determinant of each matrix using the column rotation method.

55. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & -1 & 5 \\ 1 & 0 & -2 \end{bmatrix}$ **56.** $\begin{bmatrix} -3 & 2 & 4 \\ -1 & -2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$

57. $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 4 \\ 4 & 3 & 1 \end{bmatrix}$ **58.** $\begin{bmatrix} 5 & 6 & 2 \\ -2 & 1 & -2 \\ 3 & 4 & -1 \end{bmatrix}$

For each system shown, form the matrix equation $AX = B$; compute the determinant of the coefficient matrix and determine if you can proceed; and if possible, solve the system using the matrix equation.

59. $\begin{cases} x - 2y + 2z = 7 \\ 2x + 2y - z = 5 \\ 3x - y + z = 6 \end{cases}$ **60.** $\begin{cases} 2x - 3y - 2z = 7 \\ x - y + 2z = -5 \\ 3x + 2y - z = 11 \end{cases}$

61. $\begin{cases} x - 3y + 4z = -1 \\ 4x - y + 5z = 7 \\ 3x + 2y + z = -3 \end{cases}$ **62.** $\begin{cases} 5x - 2y + z = 1 \\ 3x - 4y + 9z = -2 \\ 4x - 3y + 5z = 6 \end{cases}$

▶ WORKING WITH FORMULAS

The inverse of a 2 × 2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to A^{-1} = \frac{1}{ad - bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The inverse of a 2 × 2 matrix can be found using the formula shown, as long as $ad - bc \neq 0$. Use the formula to find inverses for the matrices here (if possible), then verify by showing $A \cdot A^{-1} = A \cdot A^{-1} = I$.

63. $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$ **64.** $B = \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}$

65. $C = \begin{bmatrix} 0.3 & -0.4 \\ -0.6 & 0.8 \end{bmatrix}$ **66.** $\begin{bmatrix} 0.2 & 0.3 \\ -0.4 & -0.6 \end{bmatrix}$

▶ APPLICATIONS

Solve each application using a matrix equation.

Descriptive Translation

67. Convenience store sales: The local Moto-Mart sells four different sizes of Slushies—behemoth, 60 oz @ \$2.59; gargantuan, 48 oz @ \$2.29; mammoth, 36 oz @ \$1.99; and jumbo, 24 oz @ \$1.59. As part of a promotion, the owner offers free gas to any customer who can tell how many of each size were sold last week, given the following information: (1) The total revenue for the Slushies was \$402.29; (2) 7884 ounces were sold; (3) a total of 191 Slushies were sold; and (4) the number of behemoth Slushies sold was one more than the number of jumbo. How many of each size were sold?

68. Cartoon characters: In America, four of the most beloved cartoon characters are Foghorn Leghorn, Elmer Fudd, Bugs Bunny, and Tweety Bird. Suppose that Bugs Bunny is four times as tall as Tweety Bird. Elmer Fudd is as tall as the combined height of Bugs Bunny and Tweety Bird. Foghorn Leghorn is 20 cm taller than the combined height of Elmer Fudd and Tweety Bird. The combined height of all four characters is 500 cm. How tall is each one?

69. Rolling Stones music: One of the most prolific and popular rock-and-roll bands of all time is the Rolling Stones. Four of their many great hits include: *Jumpin' Jack Flash, Tumbling Dice, You Can't Always Get What You Want,* and *Wild Horses.* The total playing time of all four songs is 20.75 min.

The combined playing time of *Jumpin' Jack Flash* and *Tumbling Dice* equals that of *You Can't Always Get What You Want*. *Wild Horses* is 2 min longer than *Jumpin' Jack Flash*, and *You Can't Always Get What You Want* is twice as long as *Tumbling Dice*. Find the playing time of each song.

70. **Mozart's arias:** Mozart wrote some of vocal music's most memorable arias in his operas, including *Tamino's Aria, Papageno's Aria,* the *Champagne Aria,* and the *Catalogue Aria.* The total playing time of all four arias is 14.3 min. *Papageno's Aria* is 3 min shorter than the *Catalogue Aria.* The *Champagne Aria* is 2.7 min shorter than *Tamino's Aria.* The combined time of *Tamino's Aria* and *Papageno's Aria* is five times that of the *Champagne Aria.* Find the playing time of all four arias.

Manufacturing

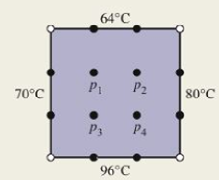
71. **Resource allocation:** Time Pieces Inc. manufactures four different types of grandfather clocks. Each clock requires these four stages: (1) assembly, (2) installing the clockworks, (3) inspection and testing, and (4) packaging for delivery. The time required for each stage is shown in the table, for each of the four clock types. At the end of a busy week, the owner determines that personnel on the assembly line worked for 262 hours, the installation crews for 160 hours, the testing department for 29 hours, and the packaging department for 68 hours. How many clocks of each type were made?

Dept.	Clock A	Clock B	Clock C	Clock D
Assemble	2.2	2.5	2.75	3
Install	1.2	1.4	1.8	2
Test	0.2	0.25	0.3	0.5
Pack	0.5	0.55	0.75	1.0

72. **Resource allocation:** Figurines Inc. makes and sells four sizes of metal figurines, mostly historical figures and celebrities. Each figurine goes through four stages of development: (1) casting, (2) trimming, (3) polishing, and (4) painting. The time required for each stage is shown in the table, for each of the four sizes. At the end of a busy week, the manager finds that the casting department put in 62 hr, and the trimming department worked for 93.5 hr, with the polishing and painting departments logging 138 hr and 358 hr respectively. How many figurines of each type were made?

Dept.	Small	Medium	Large	X-Large
Casting	0.5	0.6	0.75	1
Trimming	0.8	0.9	1.1	1.5
Polishing	1.2	1.4	1.7	2
Painting	2.5	3.5	4.5	6

73. **Thermal conductivity:** In lab experiments designed to measure the heat conductivity of a square metal plate of uniform density, the edges are held at four different (constant) temperatures. The *mean-value principle* from physics tells us that the temperature at a given point p_i on the plate, is equal to the average temperature of nearby points. Use this information to form a system of four equations in four variables, and determine the temperature at interior points p_1, p_2, p_3, and p_4 on the plate shown. (*Hint:* Use the temperature of the four points closest to each.)



74. **Thermal conductivity:** Repeat Exercise 73 if (a) the temperatures at the top and bottom of the plate were *increased* by 10°, with the temperatures at the left and right edges *decreased* by 10° (what do you notice?); (b) the temperature at the top and the temperature to the left were *decreased* by 10°, with the temperatures at the bottom and right held at their original temperature.

Curve Fitting

75. **Cubic fit:** Find a cubic function of the form $y = ax^3 + bx^2 + cx + d$ such that $(-4, -6)$, $(-1, 0)$, $(1, -16)$, and $(3, 8)$ are on the graph of the function.

76. **Cubic fit:** Find a cubic function of the form $y = ax^3 + bx^2 + cx + d$ such that $(-2, 5)$, $(0, 1)$, $(2, -3)$, and $(3, 25)$ are on the graph of the function.

Nutrition

77. **Animal diets:** A zoo dietician needs to create a specialized diet that regulates an animal's intake of fat, carbohydrates, and protein during a meal. The table given shows three different foods and the amount of these nutrients (in grams) that each ounce of food provides. How many ounces of each should the dietician recommend to supply 20 g of fat, 30 g of carbohydrates, and 44 g of protein?

Nutrient	Food I	Food II	Food III
Fat	2	4	3
Carb.	4	2	5
Protein	5	6	7

78. **Training diet:** A physical trainer is designing a workout diet for one of her clients, and wants to supply him with 24 g of fat, 244 g of

carbohydrates, and 40 g of protein for the noontime meal. The table given shows three different foods and the amount of these nutrients (in grams) that each ounce of food provides. How many ounces of each should the trainer recommend?

Nutrient	Food I	Food II	Food III
Fat	2	5	0
Carb.	10	15	18
Protein	2	10	0.75

▶ EXTENDING THE CONCEPT

79. Some matrix applications require that you solve a matrix equation of the form $AX + B = C$, where A, B, and C are matrices with the appropriate number of rows and columns and A^{-1} exists. Investigate the solution process for such equations using $A = \begin{bmatrix} 2 & 3 \\ -5 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, $C = \begin{bmatrix} 12 \\ -4 \end{bmatrix}$, and $X = \begin{bmatrix} x \\ y \end{bmatrix}$, then solve $AX + B = C$ for X *symbolically* (using A^{-1}, I, and so on).

80. It is possible for the matrix of coefficients to be singular, yet for solutions to exist. If the system is dependent instead of inconsistent, there may be infinitely many solutions and the solution set must be written using a parameter or the set notation seen previously. Try solving the exercise given here using matrix equations. If this is not possible, discuss why, then solve using elimination. If the system is dependent, find at least *two* sets of three fractions that fit the criteria.The sum of the two smaller fractions equals the larger, the larger less the smaller equals the "middle" fraction, and four times the smaller fraction equals the sum of the other two.

81. Another alternative for finding determinants uses the triangularized form of a matrix and is offered without proof: *If nonsingular matrix A is written in triangularized form without exchanging any rows and without using the operations kR_i to replace any row (k a constant), then det(A) is equal to the product of resulting diagonal entries*. Compute the determinant of each matrix using this method. Be careful not to interchange rows and do not replace any row by a multiple of that row in the process.

a. $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 2 & 5 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 2 & 5 & -1 \\ -2 & -3 & 4 \\ 4 & 6 & 5 \end{bmatrix}$

c. $\begin{bmatrix} -2 & 4 & 1 \\ 5 & 7 & -2 \\ 3 & -8 & -1 \end{bmatrix}$ **d.** $\begin{bmatrix} 3 & -1 & 4 \\ 0 & -2 & 6 \\ -2 & 1 & -3 \end{bmatrix}$

82. Find 2×2 nonzero matrices A and B whose product gives the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

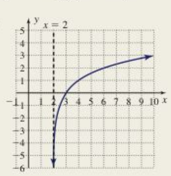
▶ MAINTAINING YOUR SKILLS

83. **(5.5)** Find the amplitude and period of $y = -125\cos(3t)$.

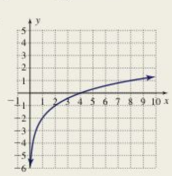
84. **(2.5/4.3)** Match each equation to its related graph. Justify your answers.

$y = \log_2(x - 2)$ $y = \log_2 x - 2$

a.



b.



85. **(1.3)** Solve the absolute value inequality: $-3|2x + 5| - 7 \leq -19$.

86. **(6.6)** Find all solutions of $-7\tan^2 x = -21$ in $[0, 2\pi)$.

9.4 Applications of Matrices and Determinants: Cramer's Rule, Partial Fractions, and More

Learning Objectives

In Section 9.4 you will learn how to:

- ❑ **A.** Solve a system using determinants and Cramer's rule
- ❑ **B.** Decompose a rational expression into partial fractions
- ❑ **C.** Use determinants in applications involving geometry in the coordinate plane

In addition to solving systems, matrices can be used to accomplish such diverse things as finding the volume of a three-dimensional solid or establishing certain geometrical relationships in the coordinate plane. Numerous uses are also found in higher mathematics, such as checking whether solutions to a differential equation are linearly independent.

A. Solving Systems Using Determinants and Cramer's Rule

In addition to identifying singular matrices, determinants can actually be used to *develop a formula* for the solution of a system. Consider the following solution to a *general* 2×2 system, which parallels the solution to a specific 2×2 system. With a view toward a solution involving determinants, the coefficients of x are written as a_{11} and a_{21} in the general system, and the coefficients of y are a_{12} and a_{22}.

Specific System

$$\begin{cases} 2x + 5y = 9 \\ 3x + 4y = 10 \end{cases}$$

eliminate the x-term
$-3R1 + 2R2$

sums to zero

$$\begin{cases} -3 \cdot 2x - 3 \cdot 5y = -3 \cdot 9 \\ 2 \cdot 3x + 2 \cdot 4y = 2 \cdot 10 \end{cases}$$

$$2 \cdot 4y - 3 \cdot 5y = 2 \cdot 10 - 3 \cdot 9$$

General System

$$\begin{cases} a_{11}x + a_{12}y = c_1 \\ a_{21}x + a_{22}y = c_2 \end{cases}$$

eliminate the x-term
$-a_{21}R1 + a_{11}R2$

sums to zero

$$\begin{cases} -a_{21}a_{11}x - a_{21}a_{12}y = -a_{21}c_1 \\ a_{11}a_{21}x + a_{11}a_{22}y = a_{11}c_2 \end{cases}$$

$$a_{11}a_{22}y - a_{21}a_{12}y = a_{11}c_2 - a_{21}c_1$$

Notice the x-terms sum to zero in both systems. We are deliberately leaving the solution on the left unsimplified to show the pattern developing on the right. Next we solve for y.

Factor Out y

$$(2 \cdot 4 - 3 \cdot 5)y = 2 \cdot 10 - 3 \cdot 9$$

$$y = \frac{2 \cdot 10 - 3 \cdot 9}{2 \cdot 4 - 3 \cdot 5} \quad \text{divide}$$

Factor Out y

$$(a_{11}a_{22} - a_{21}a_{12})y = a_{11}c_2 - a_{21}c_1$$

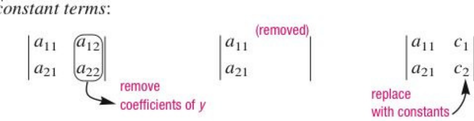
$$\text{divide} \quad y = \frac{a_{11}c_2 - a_{21}c_1}{a_{11}a_{22} - a_{21}a_{12}}$$

On the left we find $y = \frac{-7}{-7} = 1$ and back-substitution shows $x = 2$. But more importantly, on the right we obtain a formula for the y-value of *any* 2×2 system: $y = \frac{a_{11}c_2 - a_{21}c_1}{a_{11}a_{22} - a_{21}a_{12}}$. If we had chosen to solve for x, the solution would be $x = \frac{a_{22}c_1 - a_{12}c_2}{a_{11}a_{22} - a_{21}a_{12}}$. Note these formulas are defined only if $a_{11}a_{22} - a_{21}a_{12} \neq 0$. You may have already noticed, but this denominator is the *determinant of the matrix of coefficients* $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ from the previous section! Since the numerator is also a difference of two products, we investigate the possibility that it too can be expressed as a determinant. Working backward, we're able to reconstruct the numerator for x in determinant form as $\begin{bmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{bmatrix}$, where it is apparent this matrix was formed by *replacing the coefficients of the x-variables with the constant terms.*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} \phantom{a_{11}} & a_{12} \\ \phantom{a_{21}} & a_{22} \end{vmatrix} \qquad \begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}$$

remove coefficients of x — (removed) — replace with constants

It is also apparent the numerator for y can be also written in determinant form as $\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}$, or the determinant formed by *replacing the coefficients of the y-variables with the constant terms*:



If we use the notation D_y for this determinant, D_x for the determinant where x coefficients were replaced by the constants, and D as the determinant for the matrix of coefficients—the solutions can be written as shown next, with the result known as **Cramer's rule.**

Cramer's Rule for 2 × 2 Systems

Given a 2 × 2 system of linear equations

$$\begin{cases} a_{11}x + a_{12}y = c_1 \\ a_{21}x + a_{22}y = c_2 \end{cases}$$

the solution is the ordered pair (x, y), where

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \quad \text{and} \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

provided $D \neq 0$.

EXAMPLE 1 ► Solving a System Using Cramer's Rule

Use Cramer's rule to solve the system $\begin{cases} 2x - 5y = 9 \\ -3x + 4y = -10 \end{cases}$.

Solution ► Begin by finding the value of D, D_x, and D_y.

$$D = \begin{vmatrix} 2 & -5 \\ -3 & 4 \end{vmatrix} \qquad D_x = \begin{vmatrix} 9 & -5 \\ -10 & 4 \end{vmatrix} \qquad D_y = \begin{vmatrix} 2 & 9 \\ -3 & -10 \end{vmatrix}$$

$$(2)(4) - (-3)(-5) = -7 \qquad (9)(4) - (-10)(-5) = -14 \qquad (2)(-10) - (-3)(9) = 7$$

This gives $x = \frac{D_x}{D} = \frac{-14}{-7} = 2$ and $y = \frac{D_y}{D} = \frac{7}{-7} = -1$. The solution is $(2, -1)$. Check by substituting these values into the original equations.

Now try Exercises 7 through 14 ►

Regardless of the method used to solve a system, always be aware that a consistent, inconsistent, or dependent system is possible. The system $\begin{cases} y - 2x = -3 \\ 4x + 6 = 2y \end{cases}$ yields $\begin{cases} -2x + y = -3 \\ 4x - 2y = -6 \end{cases}$ in standard form, with $D = \begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} = (-2)(-2) - (4)(1) = 0$. Since $\det(D) = 0$, Cramer's rule cannot be applied, and the system is either inconsistent or dependent. To find out which, we write the equations in function form (solve for y). The result is $\begin{cases} y = 2x - 3 \\ y = 2x + 3 \end{cases}$, showing the system consists of two parallel lines and has no solutions.

Cramer's Rule for 3 × 3 Systems

Cramer's rule can be extended to a 3 × 3 system of linear equations, using the same pattern as for 2 × 2 systems. Given the general 3 × 3 system

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

the solutions are $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, and $z = \frac{D_z}{D}$, where D_x, D_y, and D_z are again formed by replacing the coefficients of the indicated variable with the constants, and D is the determinant of the matrix of coefficients $(D \neq 0)$.

Cramer's Rule Applied to 3 × 3 Systems

Given a 3 × 3 system of linear equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

The solution is an ordered triple (x, y, z), where

$$x = \frac{\begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad y = \frac{\begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}} \quad z = \frac{\begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}},$$

provided $D \neq 0$.

A. You've just learned how to solve a system using determinants and Cramer's rule

EXAMPLE 2 ► **Solving a 3 × 3 System Using Cramer's Rule**

Solve using Cramer's rule: $\begin{cases} x - 2y + 3z = -1 \\ -2x + y - 5z = 1 \\ 3x + 3y + 4z = 2 \end{cases}$

Solution ► Begin by computing the determinant of the matrix of coefficients, to ensure that Cramer's rule can be applied. Using the third row, we have

$$D = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 1 & -5 \\ 3 & 3 & 4 \end{vmatrix} = +3\begin{vmatrix} -2 & 3 \\ 1 & -5 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ -2 & -5 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}$$

$$= 3(7) - 3(1) + 4(-3) = 6$$

Since $D \neq 0$ we continue, electing to compute the remaining determinants using a calculator.

$$D_x = \begin{vmatrix} -1 & -2 & 3 \\ 1 & 1 & -5 \\ 2 & 3 & 4 \end{vmatrix} = 12 \quad D_y = \begin{vmatrix} 1 & -1 & 3 \\ -2 & 1 & -5 \\ 3 & 2 & 4 \end{vmatrix} = 0 \quad D_z = \begin{vmatrix} 1 & -2 & -1 \\ -2 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix} = -6$$

The solution is $x = \frac{D_x}{D} = \frac{12}{6} = 2$, $y = \frac{D_y}{D} = \frac{0}{6} = 0$, and $z = \frac{D_z}{D} = \frac{-6}{6} = -1$, or $(2, 0, -1)$ in triple form. Check this solution in the original equations.

Now try Exercises 15 through 22 ►

B. Rational Expressions and Partial Fractions

Recall that a rational expression is one of the form $\frac{P(x)}{Q(x)}$, where P and Q are polynomials and $Q(x) \neq 0$. The addition of rational expressions is widely taught in courses prior to college algebra, and involves combining two rational expressions into a single term using a common denominator. In some applications of higher mathematics, we seek to reverse this process and *decompose* a rational expression into a sum of its **partial fractions.** To begin, we make the following observations:

1. Consider the sum $\frac{7}{x+2} + \frac{5}{x-3}$, noting both terms are proper fractions (the degree of the numerator is less than the degree of the denominator) and have distinct linear denominators.

$$\frac{7}{x+2} + \frac{5}{x-3} = \frac{7(x-3)}{(x+2)(x-3)} + \frac{5(x+2)}{(x-3)(x+2)} \quad \text{common denominators}$$
$$= \frac{7(x-3) + 5(x+2)}{(x+2)(x-3)} \quad \text{combine numerators}$$
$$= \frac{12x - 11}{(x+2)(x-3)} \quad \text{result}$$

Assuming we didn't have the original sum to look at, reversing the process would require us to begin with a **decomposition template** such as

$$\frac{12x - 11}{(x+2)(x-3)} = \frac{A}{x+2} + \frac{B}{x-3}$$

and solve for the *constants* A and B. We know the numerators must be constant, else the fraction(s) would be improper while the original expression is not.

2. Consider the sum $\frac{3}{x-1} + \frac{5}{x^2 - 2x + 1}$, again noting both terms are proper fractions.

$$\frac{3}{x-1} + \frac{5}{x^2 - 2x + 1} = \frac{3}{x-1} + \frac{5}{(x-1)(x-1)} \quad \text{factor denominators}$$
$$= \frac{3(x-1)}{(x-1)(x-1)} + \frac{5}{(x-1)(x-1)} \quad \text{common denominators}$$
$$= \frac{(3x-3) + 5}{(x-1)(x-1)} \quad \text{combine numerators}$$
$$= \frac{3x + 2}{(x-1)^2} \quad \text{result}$$

Note that while the new denominator is the repeated factor $(x-1)^2$, *both* $(x-1)$ and $(x-1)^2$ were denominators in the original sum. Assuming we didn't know the original sum, reversing the process would require us to begin with the template

$$\frac{3x + 2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

and solve for the constants A and B. As with observation 1, we know the numerator of the first term must be constant. While the second term would still be a proper fraction if the numerator were linear (degree 1), the denominator is a *repeated* linear factor and using a single constant in the numerator of *all such fractions* will ensure we obtain unique values for A and B. In the end, for any repeated linear factor $(ax + b)^n$ in the original denominator, terms of the form $\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_{n-1}}{(ax+b)^{n-1}}+\frac{A_n}{(ax+b)^n}$ must appear in the decomposition template, although some of these numerators may turn out to be zero.

EXAMPLE 3 ▶ Writing the Decomposition Template for Unique and Repeated Linear Factors

Write the decomposition template for

a. $\frac{x-8}{2x^2+5x+3}$ **b.** $\frac{x+1}{x^2-6x+9}$

Solution ▶ **a.** Factoring the denominator gives $\frac{x-8}{(2x+3)(x+1)}$. With two distinct linear factors in the denominator, the decomposition template is

$$\frac{x-8}{(2x+3)(x+1)}=\frac{A}{2x+3}+\frac{B}{x+1}$$ decomposition template

b. After factoring we have $\frac{x+1}{(x-3)^2}$, and the denominator is a repeated linear factor. Using our previous observations the template would be

$$\frac{x+1}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}$$ decomposition template

Now try Exercises 23 through 28 ▶

When both distinct and repeated linear factors are present in the denominator, the decomposition template maintains the elements illustrated in both observations 1 and 2.

EXAMPLE 4 ▶ Writing the Decomposition Template for Unique and Repeated Linear Factors

Write the decomposition template for $\frac{x^2-4x-15}{x^3-2x^2+x}$.

Solution ▶ Factoring the denominator gives $\frac{x^2-4x-15}{x(x^2-2x+1)}$ or $\frac{x^2-4x-15}{x(x-1)^2}$ after factoring completely. With a distinct linear factor of x, and the repeated linear factor $(x-1)^2$, the decomposition template becomes

$$\frac{x^2-4x-15}{x(x-1)^2}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$ decomposition template

Now try Exercises 29 and 30 ▶

To continue our observations,

3. Consider the sum $\frac{4}{x}+\frac{2x+3}{x^2+1}$, noting the denominator of the first term is linear, while the denominator of the second is an irreducible quadratic.

$$\frac{4}{x}+\frac{2x+3}{x^2+1}=\frac{4(x^2+1)}{x(x^2+1)}+\frac{(2x+3)x}{(x^2+1)x}$$ find common denominator

$$=\frac{(4x^2+4)+(2x^2+3x)}{x(x^2+1)}$$ combine numerators

$$=\frac{6x^2+3x+4}{x(x^2+1)}$$ result

Here, reversing the process would require us to begin with the template

$$\frac{6x^2 + 3x + 4}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1},$$

allowing that the numerator of the second term might be linear since the denominator is quadratic but *not due to a repeated linear factor*.

4. Finally, consider the sum $\frac{1}{x^2 + 3} + \frac{x - 2}{(x^2 + 3)^2}$, where the denominator of the first term is an irreducible quadratic, with the second being *the same factor* with multiplicity two.

$$\frac{1}{x^2 + 3} + \frac{x - 2}{(x^2 + 3)^2} = \frac{1(x^2 + 3)}{(x^2 + 3)(x^2 + 3)} + \frac{x - 2}{(x^2 + 3)(x^2 + 3)} \quad \text{common denominators}$$

$$= \frac{(x^2 + 3) + (x - 2)}{(x^2 + 3)(x^2 + 3)} \quad \text{combine numerators}$$

$$= \frac{x^2 + x + 1}{(x^2 + 3)^2} \quad \text{result after simplifying}$$

Reversing the process would require us to begin with the template

$$\frac{x^2 + x + 1}{(x^2 + 3)^2} = \frac{Ax + B}{x^2 + 3} + \frac{Cx + D}{(x^2 + 3)^2}$$

allowing that the numerator of either term might be nonconstant for the reasons in observation 3. Similar to our reasoning in observation 2, all powers of a repeated quadratic factor must be present in the template.

WORTHY OF NOTE

Note that the second term in the decomposition template would still be a proper fraction if the numerator were quadratic or cubic, but since the denominator is a *repeated* quadratic factor, using only a linear form ensures we obtain unique values for all coefficients.

EXAMPLE 5 ► Writing the Decomposition Template for Linear and Quadratic Factors

Write the decomposition template for

a. $\frac{x^2 + 10x + 1}{(x + 1)(x^2 + 3x + 1)}$ **b.** $\frac{x^2}{(x^2 + 2)^3}$

Solution ► **a.** One factor of the denominator is a distinct linear factor, and the other is an irreducible quadratic. The decomposition template is

$$\frac{x^2 + 10x + 1}{(x + 1)(x^2 + 3x + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 3x + 1} \quad \text{decomposition template}$$

b. The denominator consists of a repeated, irreducible quadratic factor. Using our previous observations the template would be

$$\frac{x^2}{(x^2 + 2)^3} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2} + \frac{Ex + F}{(x^2 + 2)^3} \quad \text{decomposition template}$$

Now try Exercises 31 and 32 ►

When both distinct and repeated factors are present in the denominator, the decomposition template maintains the essential elements determined by observations 1 through 4. Using these observations, we can formulate a general approach to the decomposition template.

Decomposition Template for Rational Expressions

For the rational expression $\frac{P(x)}{Q(x)}$ in lowest terms and constants $A, B, C, D, \ldots$

1. Factor Q completely into linear factors and irreducible quadratic factors.
2. For the linear factors, each distinct linear factor and each power of a repeated linear factor must appear in the decomposition template with a constant numerator.
3. For the irreducible quadratic factors, each distinct quadratic factor and each power of a repeated quadratic factor must appear in the decomposition template with a linear numerator.
4. If the degree of P is greater than or equal to the degree of Q, find the quotient and remainder using polynomial division. Only the remainder portion need be decomposed into partial fractions.

Once the template is obtained, we multiply both sides of the equation by the factored form of the original denominator and simplify. The resulting equation is an identity — a true statement for all real numbers x, and in many cases the constants A, B, C, and so on can be identified using a choice of **convenient values** for x, as in Example 6.

EXAMPLE 6 ► Decomposing a Rational Expression with Linear Factors

Decompose the expression $\frac{4x + 11}{x^2 + 7x + 10}$ into partial fractions.

Solution ► Factoring the denominator gives $\frac{4x + 11}{(x + 5)(x + 2)}$, with two distinct linear factors in the denominator. The required template is

$$\frac{4x + 11}{(x + 5)(x + 2)} = \frac{A}{x + 5} + \frac{B}{x + 2} \qquad \text{decomposition template}$$

Multiplying both sides by $(x + 5)(x + 2)$ clears all denominators and yields

$$4x + 11 = A(x + 2) + B(x + 5) \qquad \text{clear denominators}$$

Since the equation must be true for all x, using $x = -5$ will *conveniently* eliminate the term with B, and enable us to solve for A directly:

$$\begin{aligned} 4(-5) + 11 &= A(-5 + 2) + B(-5 + 5) && \text{substitute } -5 \text{ for } x \\ -20 + 11 &= -3A + B(0) && \text{simplify} \\ -9 &= -3A && \text{term with } B \text{ is eliminated} \\ 3 &= A && \text{solve for } A \end{aligned}$$

To find B, we repeat this procedure, using an x-value that *conveniently* eliminates the term with A, namely, $x = -2$.

$$\begin{aligned} 4x + 11 &= A(x + 2) + B(x + 5) && \text{original equation} \\ 4(-2) + 11 &= A(-2 + 2) + B(-2 + 5) && \text{substitute } -2 \text{ for } x \\ -8 + 11 &= A(0) + 3B && \text{simplify} \\ 3 &= 3B && \text{term with } A \text{ is eliminated} \\ 1 &= B && \text{solve for } B \end{aligned}$$

With $A = 3$ and $B = 1$, the complete decomposition is

$$\frac{4x + 11}{(x + 5)(x + 2)} = \frac{3}{x + 5} + \frac{1}{x + 2}$$

which can be checked by adding the fractions on the right.

Now try Exercises 33 through 38 ►

EXAMPLE 7 ▶ Decomposing a Rational Expression with Repeated Linear Factors

Decompose the expression $\frac{9}{(x + 5)(x^2 + 7x + 10)}$ into partial fractions.

Solution ▶ Factoring the denominator gives $\frac{9}{(x + 5)(x + 2)(x + 5)} = \frac{9}{(x + 2)(x + 5)^2}$, (one distinct linear factor, one repeated linear factor). The decomposition template is $\frac{9}{(x + 2)(x + 5)^2} = \frac{A}{x + 2} + \frac{B}{x + 5} + \frac{C}{(x + 5)^2}$. Multiplying both sides by $(x + 2)(x + 5)^2$ clears all denominators and yields

$$9 = A(x + 5)^2 + B(x + 2)(x + 5) + C(x + 2).$$

Using $x = -5$ will eliminate the terms with A and B, giving

$$\begin{aligned} 9 &= A(-5 + 5)^2 + B(-5 + 2)(-5 + 5) + C(-5 + 2) && \text{substitute } -5 \text{ for } x \\ 9 &= A(0) + B(-3)(0) - 3C && \text{simplify} \\ 9 &= -3C && \text{terms with } A \text{ and } B \text{ are eliminated} \\ -3 &= C && \text{solve for } C \end{aligned}$$

Using $x = -2$ will eliminate the terms with B and C, and we have

$$\begin{aligned} 9 &= A(x + 5)^2 + B(x + 2)(x + 5) + C(x + 2) && \text{original equation} \\ 9 &= A(-2 + 5)^2 + B(-2 + 2)(-2 + 5) + C(-2 + 2) && \text{substitute } -2 \text{ for } x \\ 9 &= A(3)^2 + B(0)(3) + C(0) && \text{simplify} \\ 9 &= 9A && \text{terms with } B \text{ and } C \text{ are eliminated} \\ 1 &= A && \text{solve for } A \end{aligned}$$

To find B, we substitute $A = 1$ and $C = -3$ into the original equation, *with any value of x that does not eliminate B*. For efficiency, we'll often use $x = 0$ or $x = 1$ for this purpose (if possible).

$$\begin{aligned} 9 &= A(x + 5)^2 + B(x + 2)(x + 5) + C(x + 2) && \text{original equation} \\ 9 &= 1(0 + 5)^2 + B(0 + 2)(0 + 5) - 3(0 + 2) && \text{substitute 1 for } A, -3 \text{ for } C, 0 \text{ for } x \\ 9 &= 25 + 10B - 6 && \text{simplify} \\ -1 &= B && \text{solve for } B \end{aligned}$$

With $A = 1$, $B = -1$, and $C = -3$ the complete decomposition is

$$\begin{aligned} \frac{9}{(x + 2)(x + 5)^2} &= \frac{1}{x + 2} + \frac{-1}{x + 5} + \frac{-3}{(x + 5)^2} \\ &= \frac{1}{x + 2} - \frac{1}{x + 5} - \frac{3}{(x + 5)^2} \end{aligned}$$

Now try Exercises 39 and 40 ▶

As an alternative to using convenient values, a system of equations can be set up by multiplying out the right-hand side (after clearing fractions) and equating coefficients of the terms with like degrees.

EXAMPLE 8 ▶ Decomposing a Rational Expression with Linear and Quadratic Factors

Decompose the given expression into partial fractions: $\frac{3x^2 - x - 11}{x^3 - 3x^2 + 4x - 12}$.

Solution ► A careful inspection indicates the denominator will factor by grouping, giving $x^3 - 3x^2 + 4x - 12 = x^2(x - 3) + 4(x - 3) = (x - 3)(x^2 + 4)$. With one linear factor and one irreducible quadratic factor, the required template is

$$\frac{3x^2 - x - 11}{(x-3)(x^2+4)} = \frac{A}{x-3} + \frac{Bx + C}{x^2 + 4} \quad \text{decomposition template}$$

$$3x^2 - x - 11 = A(x^2 + 4) + (Bx + C)(x - 3) \quad \text{multiply by } (x-3)(x^2+4) \text{ (clear denominators)}$$

$$= Ax^2 + 4A + Bx^2 - 3Bx + Cx - 3C \quad \text{distribute/F-O-I-L}$$

$$= (A + B)x^2 + (C - 3B)x + 4A - 3C \quad \text{group and factor}$$

For the left side to equal the right, we must equate coefficients of terms with like degree: $A + B = 3$, $C - 3B = -1$, and $4A - 3C = -11$. This gives the 3×3 system $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -3 & 1 & -1 \\ 4 & 0 & -3 & -11 \end{array}\right]$ in matrix form (verify this). Using the matrix of coefficients we find that $D = 13$, and we complete the solution using Cramer's rule.

$$D_A = \begin{vmatrix} 3 & 1 & 0 \\ -1 & -3 & 1 \\ -11 & 0 & -3 \end{vmatrix} = 13 \quad D_B = \begin{vmatrix} 1 & 3 & 0 \\ 0 & -1 & 1 \\ 4 & -11 & -3 \end{vmatrix} = 26 \quad D_C = \begin{vmatrix} 1 & 1 & 3 \\ 0 & -3 & -1 \\ 4 & 0 & -11 \end{vmatrix} = 65$$

The result is $A = \frac{13}{13} = 1$, $B = \frac{26}{13} = 2$, and $C = \frac{65}{13} = 5$, giving the decomposition

$$\frac{3x^2 - x - 11}{(x-3)(x^2+4)} = \frac{1}{x-3} + \frac{2x+5}{x^2+4}.$$

Now try Exercises 41 through 46 ►

In some cases, the "convenient values" method cannot be applied and a system of equations is our *only* option. Also, if the decomposition template produces a large or cumbersome system, a graphing calculator can be used to assist the solution process.

EXAMPLE 9 ► **Decomposing a Rational Expression Using Technology**

Use matrix equations and a graphing calculator to decompose the given expression into partial fractions: $\frac{12x^3 + 62x^2 + 102x + 56}{x(x+2)^3}$.

Solution ► The decomposition template will be

$$\frac{12x^3 + 62x^2 + 102x + 56}{x(x+2)^3} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}.$$

Clearing denominators and simplifying yields

$$12x^3 + 62x^2 + 102x + 56 = A(x + 2)^3 + Bx(x + 2)^2 + Cx(x + 2) + Dx \quad \text{clear denominators}$$

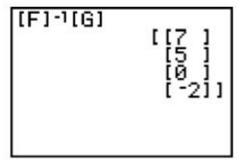
After expanding the powers on the right and factoring we obtain

$$12x^3 + 62x^2 + 102x + 56 = (A + B)x^3 + (6A + 4B + C)x^2 + (12A + 4B + 2C + D)x + 8A.$$

By equating the coefficients of like terms, the following system and matrix equation are obtained:

$$\begin{cases} A + B = 12 \\ 6A + 4B + C = 62 \\ 12A + 4B + 2C + D = 102 \\ 8A = 56 \end{cases} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 6 & 4 & 1 & 0 \\ 12 & 4 & 2 & 1 \\ 8 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 12 \\ 62 \\ 102 \\ 56 \end{bmatrix}.$$

After carefully entering the matrices F (coefficients) and G (constants), we obtain the solution $A = 7$, $B = 5$, $C = 0$, and $D = -2$ as shown in the figure. The decomposed form is



$$\frac{12x^3 + 62x^2 + 102x + 56}{x(x+2)^3} = \frac{7}{x} + \frac{5}{x+2} + \frac{-2}{(x+2)^3}.$$

Now try Exercises 47 and 48 ►

As a final reminder, if the degree of the numerator is *greater than* the degree of the denominator, divide using long division and apply the preceding methods to the remainder polynomial. For instance, you can check that $\frac{3x^3 + 6x^2 + 5x - 7}{x^2 + 2x + 1} = 3x + \frac{2x - 7}{(x+1)^2}$, and decomposing the remainder polynomial gives a final result of $3x + \frac{2}{x+1} - \frac{9}{(x+1)^2}$.

B. You've just learned how to decompose a rational expression into partial fractions

C. Determinants, Geometry, and the Coordinate Plane

As mentioned in the introduction, the use of determinants extends far beyond solving systems of equations. Here, we'll demonstrate how determinants can be used to find the area of a triangle whose vertices are given as three points in the coordinate plane.

The Area of a Triangle in the *xy*-Plane

Given a triangle with vertices at (x_1, y_1), (x_2, y_2), and (x_3, y_3),

$$\text{Area} = \left|\frac{\det(T)}{2}\right| \quad \text{where } T = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

EXAMPLE 10 ► Finding the Area of a Triangle Using Determinants

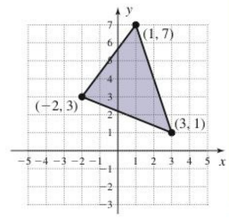
Find the area of a triangle with vertices at (3, 1), (−2, 3), and (1, 7).

Solution ► Begin by forming matrix T and computing $\det(T)$:

$$\det(T) = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ -2 & 3 & 1 \\ 1 & 7 & 1 \end{vmatrix}$$

$$= 3(3 - 7) - 1(-2 - 1) + 1(-14 - 3)$$

$$= -12 + 3 + (-17) = -26$$



Compute the area: $A = \left|\frac{\det(T)}{2}\right| = \left|\frac{-26}{2}\right|$

$= 13$

The area of this triangle is 13 units^2.

Now try Exercises 51 through 56 ▶

As an extension of this formula, what if the three points were collinear? After a moment, it may occur to you that the formula would give an area of 0 units^2, since no triangle could be formed. This gives rise to a **test for collinear points.**

Test for Collinear Points

Three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if

$$\det(A) = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

C. You've just learned how to use determinants in applications involving geometry in the coordinate plane

See Exercises 57 through 62.

9.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in the blank with the appropriate word or phrase. Carefully reread the section if needed.

1. The determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is evaluated as: ____________.

2. ________ rule uses a ratio of determinants to solve for the unknowns in a system.

3. Given the matrix of coefficients D, the matrix D_x is formed by replacing the coefficients of x with the ________ terms.

4. The three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear if $|T| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ has a value of ____.

5. Discuss/Explain the process of writing $\frac{8x - 3}{x^2 - x}$ as a sum of partial fractions.

6. Discuss/Explain why Cramer's rule cannot be applied if $D = 0$. Use an example to illustrate.

▶ DEVELOPING YOUR SKILLS

Write the determinants D, D_x, and D_y for the systems given. Do not solve.

7. $\begin{cases} 2x + 5y = 7 \\ -3x + 4y = 1 \end{cases}$

8. $\begin{cases} -x + 5y = 12 \\ 3x - 2y = -8 \end{cases}$

Solve each system of equations using Cramer's rule, if possible. Do not use a calculator.

9. $\begin{cases} 4x + y = -11 \\ 3x - 5y = -60 \end{cases}$

10. $\begin{cases} x = -2y - 11 \\ y = 2x - 13 \end{cases}$

11. $\begin{cases} \frac{x}{8} + \frac{y}{4} = 1 \\ \frac{y}{5} = \frac{x}{2} + 6 \end{cases}$

12. $\begin{cases} \frac{2}{3}x - \frac{3}{8}y = \frac{7}{5} \\ \frac{5}{6}x + \frac{3}{4}y = \frac{11}{10} \end{cases}$

13. $\begin{cases} 0.6x - 0.3y = 8 \\ 0.8x - 0.4y = -3 \end{cases}$

14. $\begin{cases} -2.5x + 6y = -1.5 \\ 0.5x - 1.2y = 3.6 \end{cases}$

Write the determinants D, D_x, D_y, and D_z for the systems given, then determine if a solution using Cramer's rule is possible by computing the value of D without the use of a calculator (do not solve the system). Try to determine how the system from part (a) is related to the system in part (b).

15. a. $\begin{cases} 4x - y + 2z = -5 \\ -3x + 2y - z = 8 \\ x - 5y + 3z = -3 \end{cases}$ b. $\begin{cases} 4x - y + 2z = -5 \\ -3x + 2y - z = 8 \\ x + y + z = -3 \end{cases}$

16. a. $\begin{cases} 2x + 3z = -2 \\ -x + 5y + z = 12 \\ 3x - 2y + z = -8 \end{cases}$ b. $\begin{cases} 2x + 3z = -2 \\ -x + 5y + z = 12 \\ x + 5y + 4z = -8 \end{cases}$

Use Cramer's rule to solve each system of equations.

17. $\begin{cases} x + 2y + 5z = 10 \\ 3x + 4y - z = 10 \\ x - y - z = -2 \end{cases}$

18. $\begin{cases} x + 3y + 5z = 6 \\ 2x - 4y + 6z = 14 \\ 9x - 6y + 3z = 3 \end{cases}$

19. $\begin{cases} y + 2z = 1 \\ 4x - 5y + 8z = -8 \\ 8x - 9z = 9 \end{cases}$

20. $\begin{cases} x + 2y + 5z = 10 \\ 3x - z = 8 \\ -y - z = -3 \end{cases}$

21. $\begin{cases} w + 2x - 3y = -8 \\ x - 3y + 5z = -22 \\ 4w - 5x = 5 \\ -y + 3z = -11 \end{cases}$

22. $\begin{cases} w - 2x + 3y - z = 11 \\ 3w - 2y + 6z = -13 \\ 2x + 4y - 5z = 16 \\ 3x - 4z = 5 \end{cases}$

▶ DECOMPOSITION OF RATIONAL EXPRESSIONS

Exercises 23 through 32 are designed solely to reinforce the various possibilities for decomposing a rational expression. All are proper fractions whose denominators are completely factored. Set up the partial fraction decomposition using appropriate numerators, but *do not solve*.

23. $\frac{3x + 2}{(x + 3)(x - 2)}$

24. $\frac{-4x + 1}{(x - 2)(x - 5)}$

25. $\frac{3x^2 - 2x + 5}{(x - 1)(x + 2)(x - 3)}$

26. $\frac{-2x^2 + 3x - 4}{(x + 3)(x + 1)(x - 2)}$

27. $\frac{x^2 + 5}{x(x - 3)(x + 1)}$

28. $\frac{x^2 - 7}{(x + 4)(x - 2)x}$

29. $\frac{x^2 + x - 1}{x^2(x + 2)}$

30. $\frac{x^2 - 3x + 5}{(x - 3)(x + 2)^2}$

31. $\frac{x^3 + 3x - 2}{(x + 1)(x^2 + 2)^2}$

32. $\frac{2x^3 + 3x^2 - 4x + 1}{x(x^2 + 3)^2}$

Decompose each rational expression into partial fractions.

33. $\frac{4 - x}{x^2 + x}$

34. $\frac{3x + 13}{x^2 + 5x + 6}$

35. $\frac{2x - 27}{2x^2 + x - 15}$

36. $\frac{-11x + 6}{5x^2 - 4x - 12}$

37. $\frac{8x^2 - 3x - 7}{x^3 - x}$

38. $\frac{x^2 + 24x - 12}{x^3 - 4x}$

39. $\frac{3x^2 + 7x - 1}{x^3 + 2x^2 + x}$

40. $\frac{-2x^2 - 7x + 28}{x^3 - 4x^2 + 4x}$

41. $\frac{3x^2 + 10x + 4}{8 - x^3}$

42. $\frac{3x^2 + 4x - 1}{x^3 - 1}$

43. $\frac{6x^2 + x + 13}{x^3 + 2x^2 + 3x + 6}$

44. $\frac{2x^2 - 14x - 7}{x^3 - 2x^2 + 5x - 10}$

45. $\frac{x^4 - x^2 - 2x + 1}{x^5 + 2x^3 + x}$

46. $\frac{-3x^4 + 13x^2 + x - 12}{x^5 + 4x^3 + 4x}$

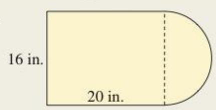
47. $\frac{x^3 - 17x^2 + 76x - 98}{(x^2 - 6x + 9)(x^2 - 2x - 3)}$

48. $\frac{16x^3 - 66x^2 + 98x - 54}{(2x^2 - 3x)(4x^2 - 12x + 9)}$

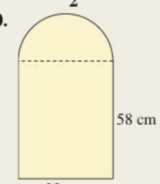
▶ WORKING WITH FORMULAS

Area of a Norman window: $A = \begin{vmatrix} L & r^2 \\ -\frac{\pi}{2} & W \end{vmatrix}$**. The determinant shown can be used to find the area of a Norman window (rectangle + half-circle) with length *L*, width *W*, and radius** $r = \frac{W}{2}$**. Find the area of the following windows.**

49.



50.



▶ APPLICATIONS

Geometric Applications

Find the area of the triangle with the vertices given. Assume units are in centimeters.

51. (2, 1), (3, 7), and (5, 3)

52. (−2, 3), (−3, −4), and (−6, 1)

Find the area of the parallelogram with vertices given. Assume units are in feet.

53. (−4, 2), (−6, −1), (3, −1), and (5, 2)

54. (−5, −6), (5, 0), (5, 4), and (−5, −2)

The volume of a triangular pyramid is given by the formula $V = \frac{1}{3}Bh$**, where *B* represents the area of the triangular base and *h* is the height of the pyramid. Find the volume of a triangular pyramid whose height is given and whose base has the coordinates shown. Assume units are in meters.**

55. $h = 6$ m; vertices (3, 5), (−4, 2), and (−1, 6)

56. $h = 7.5$ m; vertices (−2, 3), (−3, −4), and (−6, 1)

Determine if the following sets of points are collinear.

57. (1, 5), (−2, −1), and (4, 11)

58. (1, 1), (3, −5), and (−2, 9)

59. (−2.5, 5.2), (1.2, −5.6), and (2.2, −8.5)

60. (−0.5, 1.25), (−2.8, 3.75), and (3, 6.25)

For each linear equation given, substitute the first two points to verify they are solutions. Then use the test for collinear points to determine if the third point is also a solution.

61. $2x - 3y = 7$; (2, −1), (−1.3, −3.2), (−3.1, −4.4)

62. $5x + 2y = 4$; (2, −3), (3.5, −6.75), (−2.7, 8.75)

Write a linear system that models each application. Then solve using Cramer's rule.

63. Return on investments: If \$15,000 is invested at a certain interest rate and \$25,000 is invested at another interest rate, the total return was \$2900. If the investments were reversed the return would be \$2700. What was the interest rate paid on each investment?

64. Cost of fruit: Many years ago, two pounds of apples, 2 lb of kiwi, and 10 lb of pears cost \$3.26. Three pounds of apples, 2 lb of kiwi, and 7 lb of pears cost \$2.98. Two pounds of apples, 3 lb of kiwi, and 6 lb of pears cost \$2.89. Find the cost of a pound of each fruit.

▶ EXTENDING THE CONCEPT

65. Solve the given system four different ways: (1) elimination, (2) row reduction, (3) Cramer's rule, and (4) using a matrix equation. Which method seems to be the least error-prone? Which method seems most efficient (takes the least time)? Discuss the advantages and drawbacks of each method.

$$\begin{cases} x + 3y + 5z = 6 \\ 2x - 4y + 6z = 14 \\ 9x - 6y + 3z = 3 \end{cases}$$

66. Find the area of the pentagon whose vertices are: (−5, −5), (5, −5), (8, 6), (−8, 6), and (0, 12.5).

67. The polynomial form for the equation of a circle is $x^2 + y^2 + Dx + Ey + F = 0$. Find the equation of the circle that contains the points (−1, 7), (2, 8), and (5, −1).

MAINTAINING YOUR SKILLS

68. **(3.4)** Graph the polynomial using information about end behavior, y-intercept, x-intercept(s), and midinterval points: $f(x) = x^3 - 2x^2 - 7x + 6$.

69. **(7.1)** Solve the triangle with the following measures:
side $a = 8.7$ in.
side $b = 11.2$ in.
$\angle A = 49.0°$

70. **(4.2/4.5)** Solve the equation $3^{2x-1} = 9^{2-x}$ two ways. First using logarithms, then by equating the bases and using properties of equality.

71. **(5.6)** Graph $y = 3\tan(2\pi t)$ over the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Note the period, asymptotes, zeroes, and value of A.

SUMMARY AND CONCEPT REVIEW

SECTION 9.1 Solving Linear Systems Using Matrices and Row Operations

KEY CONCEPTS

- A *matrix* is a rectangular arrangement of numbers. An $m \times n$ matrix has m rows and n columns.
- An *augmented matrix* is derived from a system of linear equations by augmenting the *coefficient matrix* (formed by the variable coefficients) with the *matrix of constants*.
- One matrix method for solving systems of equations is by triangularizing the augmented matrix.
- An inconsistent system with no solutions will yield a contradictory statement such as $0 = 1$. A dependent system with infinitely many solutions will yield an identity statement such as $0 = 0$.

EXERCISES

1. Write an example of the following matrices:

a. 2×3 **b.** 3×2 **c.** 3×4, in triangular form

Solve by triangularizing the matrix. Use a calculator for Exercise 4.

2. $\begin{cases} x - 2y = 6 \\ 4x - 3y = 4 \end{cases}$

3. $\begin{cases} x - 2y + 2z = 7 \\ 2x + 2y - z = 5 \\ 3x - y + z = 6 \end{cases}$

4. $\begin{cases} 2w + x + 2y - 3z = -19 \\ w - 2x - y + 4z = 15 \\ x + 2y - z = 1 \\ 3w - 2x - 5z = -60 \end{cases}$

5. $\begin{cases} 2x - y + 2z = -1 \\ x + 2y + 2z = -3 \\ 3x - 4y + 2z = 1 \end{cases}$

SECTION 9.2 The Algebra of Matrices

KEY CONCEPTS

- The entries of a matrix are denoted a_{ij}, where i gives the row and j gives the column of its location.
- Two matrices A and B of equal size (or *order*) are equal if corresponding entries are equal.
- The sum or difference of two matrices of equal order is found by combining corresponding entries: $A + B = [a_{ij} + b_{ij}]$
- The *identity matrix for addition* is an $m \times n$ matrix whose entries are all zeroes.
- To perform *scalar multiplication,* take the product of the constant with each entry in the matrix, forming a new matrix of like size. For matrix A: $kA = [ka_{ij}]$.

- *Matrix multiplication* is performed as row entry × column entry. For an $m \times n$ matrix $A = [a_{ij}]$ and an $s \times t$ matrix $B = [b_{ij}]$, AB is possible if $n = s$. The result will be an $m \times t$ matrix $P = [p_{ij}]$, where p_{ij} is the product of the ith row of A with the jth column of B.
- When technology is used to perform operations on matrices, carefully enter each matrix into the calculator. Then double check that each entry is correct and appraise the results to see if they are reasonable.

EXERCISES

Compute the operations indicated below (if possible), using the following matrices.

$$A = \begin{bmatrix} \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{8} & \frac{-7}{8} \end{bmatrix} \qquad B = \begin{bmatrix} -7 & 6 \\ 1 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} -1 & 3 & 4 \\ 5 & -2 & 0 \\ 6 & -3 & 2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & -3 & 0 \\ 0.5 & 1 & -1 \\ 4 & 0.1 & 5 \end{bmatrix}$$

6. $A + B$ **7.** $B - A$ **8.** $C - B$ **9.** $8A$ **10.** BA

11. $C + D$ **12.** $D - C$ **13.** BC **14.** $-4D$ **15.** CD

0
2 4 4
16

SECTION 9.3 Solving Linear Systems Using Matrix Equations

KEY CONCEPTS

- I, the *identity matrix for multiplication,* has 1's on the main diagonal and 0's for all other entries. For any $n \times n$ matrix A, the identity matrix is also an $n \times n$ matrix, is called I_n, and $AI_n = I_nA = A$.
- For an $n \times n$ (square) matrix A, the *inverse matrix* for multiplication is a matrix B such that $AB = BA = I_n$. For matrix A the inverse is denoted A^{-1}. Only square matrices have an inverse.
- Any $n \times n$ system of equations can be written as a matrix equation and solved (if a unique solution exists) using an inverse matrix. The system

$$\begin{cases} 2x + 3y = 7 \\ x - 4y = -2 \end{cases} \text{ is written as } \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}.$$

- Every square matrix has a real number associated with it, called its *determinant*. For 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A) = a_{11}a_{22} - a_{21}a_{12}$.
- If the determinant of a matrix is zero, the matrix is said to be *singular* or *noninvertible*. If the coefficient matrix of a matrix equation is noninvertible, the system is either inconsistent or dependent.

EXERCISES

Complete Exercises 16 through 18 using the following matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 0.2 & 0.2 \\ -0.6 & 0.4 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 10 & -6 \\ -15 & 9 \end{bmatrix}$$

16. Exactly one of the matrices given is singular. Compute each determinant to identify it.

17. Show that $AB = BA = B$. What can you conclude about the matrix A?

18. Show that $BC = CB = I$. What can you conclude about the matrix C?

Complete Exercises 19 through 21 using the matrices given:

$$E = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 1 & -5 \\ -1 & -1 & -2 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ -2 & 1 & -1 \end{bmatrix} \qquad G = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

19. Exactly one of the matrices is singular. Use a calculator to determine which.

20. Compute the products FG and GF. What can you conclude about matrix G?

21. Verify that $EG \neq GE$ and $EF \neq FE$. What can you conclude?

Solve manually using a matrix equation.

22. $\begin{cases} 2x - 5y = 14 \\ -3y + 4x = -14 \end{cases}$

Solve using a matrix equation and your calculator.

23. $\begin{cases} 0.5x - 2.2y + 3z = -8 \\ -0.6x - y + 2z = -7.2 \\ x + 1.5y - 0.2z = 2.6 \end{cases}$

SECTION 9.4 Applications of Matrices and Determinants: Cramer's Rule, Partial Fractions, and More

KEY CONCEPTS

- Cramer's rule uses a ratio of determinants to solve systems of equations (if solutions exist).
- The determinant of the 2×2 matrix $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is $a_{11}a_{22} - a_{21}a_{12}$.
- To compute the value of 3×3 and larger determinants, a calculator is generally used.
- Determinants can be used to find the area of a triangle in the plane if the vertices of the triangle are known, and as a test to see if three points are collinear.
- A system of equations can be used to write a rational expression as a sum of its partial fractions.

EXERCISES

Solve using Cramer's rule.

24. $\begin{cases} 5x + 6y = 8 \\ 10x - 2y = -9 \end{cases}$

25. $\begin{cases} 2x + y = -2 \\ -x + y + 5z = 12 \\ 3x - 2y + z = -8 \end{cases}$

26. $\begin{cases} 2x + y - z = -1 \\ x - 2y + z = 5 \\ 3x - y + 2z = 8 \end{cases}$

27. Find the area of a triangle whose vertices have the coordinates (6, 1), (−1, −6), and (−6, 2).

28. Find the partial fraction decomposition for $\dfrac{7x^2 - 5x + 17}{x^3 - 2x^2 + 3x - 6}$.

MIXED REVIEW

Solve using row operations to triangularize the matrix.

1. $\begin{cases} \frac{1}{2}x + \frac{2}{3}y = 3 \\ \frac{-2}{5}x - \frac{1}{4}y = 1 \end{cases}$

2. $\begin{cases} 2x - 5y = 5 \\ y = 0.4x + 1.2 \end{cases}$

3. $\begin{cases} 3x - 4y + 5z = 5 \\ -x + 2y - 3z = -3 \\ 3x - 2y + z = 1 \end{cases}$

4. $\begin{cases} -2x + y - 4z = -11 \\ x + 3y - z = -4 \\ 3x - 2y + z = 7 \end{cases}$

Compute as indicated for

$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}$

$C = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 0 & -1 \end{bmatrix}$ $\quad D = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

5. a. $-2AC$ **b.** CD

6. a. BA **b.** $CB - 4A$

7. Find: **a.** a_{22} **b.** b_{21} **c.** c_{12} **d.** d_{32}

8. Find values of x, y, and z so that $A = B$, given:
$A = \begin{bmatrix} 2x & y - z \\ y + z & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -1 \\ 5 & 4 \end{bmatrix}$

9. Solve using a matrix equation:
$\begin{cases} -x - 2z = 5 \\ 2y + z = -4 \\ -x + 2y = 3 \end{cases}$

10. Use a matrix equation and a calculator to solve:
$\begin{cases} w + \frac{1}{2}x + \frac{2}{3}y - z = -3 \\ \frac{3}{4}x - y + \frac{5}{8}z = \frac{41}{8} \\ \frac{2}{5}w - x - \frac{3}{10}z = -\frac{27}{10} \\ w + 2x - 3y + 4z = 16 \end{cases}$

11. Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$.

12. Compute the determinant of the following matrix without the use of a calculator, and state whether an inverse exists. $A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 7 & 4 \\ 0 & -1 & -1 \end{bmatrix}$

13. Given $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$, write the system of equations represented by the matrix equation $AX = B$.

14. Solve using Cramer's rule: $\begin{cases} -2x + y = -8 \\ x - 2y = 7 \end{cases}$

15. Solve using Cramer's rule:
$$\begin{cases} -x + 5y - 2z = 1 \\ 2x + 3y - z = 3 \\ 3x - y + 3z = -2 \end{cases}$$

16. Decompose the expression into partial fractions: $\dfrac{x^2 + 1}{x^3 - 3x + 2}$

17. Find the area of a triangle with vertices at $(-1, 2)$, $(3, 1)$, and $(0, 4)$.

18. Determine if the following points are collinear:
a. $(1, -1)$, $(2, 1)$, and $(-2, -7)$
b. $(2, 0)$, $(-3, 2)$, $(0, 1)$

19. The ancient Gaelic game of Hurling is one of the most demanding and skillful field sports being played today. The field (or pitch) is a rectangle whose perimeter is 438 m. Its length is 55 m longer than its width. Use a system of linear equations and the augmented matrix to find the dimensions of the pitch.

20. A local fitness center is offering incentives in an effort to boost membership. If you buy a year's membership (Y), you receive a \$50 rebate and six tickets to a St. Louis Cardinals home game. For a half-year membership (H), you receive a \$30 rebate and four tickets to a Cardinals home game. For a monthly trial membership (M), you receive a \$10 rebate and two tickets to a Cardinals home game. During the last month, male clients purchased 40 one-year, 52 half-year, and 70 monthly memberships, while female clients purchased 50 one-year, 44 half-year, and 60 monthly memberships. Write the number of sales of each type to males and females as a 2×3 matrix, and the amount of the rebates and number of Cards tickets awarded per type of membership as a 3×2 matrix. Use these matrices to determine (a) the total amount of rebate money paid to males and (b) the number of Cardinals tickets awarded to females.

PRACTICE TEST

Solve each system by triangularizing the augmented matrix and using back-substitution.

1. $\begin{cases} 3x + 8y = -5 \\ x + 10y = 2 \end{cases}$

2. $\begin{cases} 3x - y + 5z = 1 \\ 3x + y + 4z = 4 \\ x + y + z = \frac{7}{3} \end{cases}$

3. $\begin{cases} 4x - 5y - 6z = 5 \\ 2x - 3y + 3z = 0 \\ x + 2y - 3z = 5 \end{cases}$

4. Given matrices A and B, compute:

   a. $A - B$ **b.** $\frac{2}{5}B$ **c.** AB **d.** A^{-1} **e.** $|A|$

   $A = \begin{bmatrix} -3 & -2 \\ 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 3 \\ -3 & -5 \end{bmatrix}$

5. Given matrices C and D, use a calculator to find:

   a. $C - D$ **b.** $-0.6D$ **c.** DC **d.** D^{-1} **e.** $|D|$

   $C = \begin{bmatrix} 0.5 & 0 & 0.2 \\ 0.4 & -0.5 & 0 \\ 0.1 & -0.4 & -0.1 \end{bmatrix}$

   $D = \begin{bmatrix} 0.5 & 0.1 & 0.2 \\ -0.1 & 0.1 & 0 \\ 0.3 & 0.4 & 0.8 \end{bmatrix}$

6. Use matrices to find three different solutions of the dependent system:

   $\begin{cases} 2x - y + z = 4 \\ 3x - 2y + 4z = 9 \\ x - 2y + 8z = 11 \end{cases}$

7. Solve using Cramer's rule: $\begin{cases} 2x - 3y = 2 \\ x - 6y = -2 \end{cases}$

8. Solve using a calculator and Cramer's rule:
$$\begin{cases} 2x + 3y + z = 3 \\ x - 2y - z = 4 \\ x - y - 2z = -1 \end{cases}$$

9. Solve using a matrix equation and your calculator:
$$\begin{cases} 2x - 5y = 11 \\ 4x + 7y = 4 \end{cases}$$

10. Solve using a matrix equation and your calculator:
$$\begin{cases} x - 2y + 2z = 7 \\ 2x + 2y - z = 5 \\ 3x - y + z = 6 \end{cases}$$

11. Find values of x, y, and z so that $A = B$, given
$A = \begin{bmatrix} 2x + y & 3 \\ x + z & 3x + 2z \end{bmatrix}$ and
$B = \begin{bmatrix} z - 1 & 3 \\ 2y + 5 & y + 8 \end{bmatrix}$

12. Given matrix X is a solution to $AX = B$ for the matrix A given. Find matrix B.
$X = \begin{bmatrix} -1 \\ \frac{-3}{2} \\ 2 \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -6 & 3 \\ 3 & 4 & -1 \end{bmatrix}$

13. Use matrices to determine which three of the following four points are collinear: $(-1, 4)$, $(1, 3)$, $(2, 1)$, $(4, -1)$

14. A farmer plants a triangular field with wheat. The first vertex of the triangular field is 1 mi east and 1 mi north of his house. The second vertex is 3 mi east and 1 mi south of his house. The third vertex is 1 mi west and 2 mi south of his house. What is the area of the field?

15. For $A = \begin{bmatrix} r & 2 \\ 3 & s \end{bmatrix}$ and $A^2 = \begin{bmatrix} 10 & -2 \\ -3 & 7 \end{bmatrix}$ given, find r and s.

Create a system of equations to model each exercise, then solve using any matrix method.

16. Dr. Brown and Dr. Stamper graduate from medical school with $155,000 worth of student loans. Due to her state's tuition reimbursement plan, Dr. Brown owes one fourth of what Dr. Stamper owes. How much does each doctor owe?

17. Justin is rehabbing two old houses simultaneously. He calculates that last week he spent 23 hr working on these houses. If he spent 8 more hours on one of the houses, how many hours did he spend on each house?

18. In his first month as assistant principal of Washington High School, Mr. Johnson gave out 20 detentions. They were either for 1 day, 2 days, or 5 days. He recorded a total of 38 days of detention served. He also noted that there were twice as many 2-day detentions as 5-day detentions. How many of each type of detention did Mr. Johnson give out?

19. The city of Cherrywood has approved a $1,800,000 plan to renovate its historic commercial district. The money will be coming from three separate sources. The first is a federal program that charges a low 2% interest annually. The second is a municipal bond offering that will cost 5% annually. The third is a standard loan from a neighborhood bank, but it will cost 8.5% annually. In the first year, the city will not make any repayment on these loans and will accrue $94,500 more debt. The federal program and bank loan together are responsible for $29,500 of this interest. How much money was originally provided by each source?

20. Decompose the expression into partial fractions:
$$\frac{4x^2 - 4x + 3}{x^3 - 27}$$

CALCULATOR EXPLORATION AND DISCOVERY

Cramer's Rule

In Section 9.4, we saw that one interesting application of matrices is Cramer's rule. You may have noticed that when technology is used with Cramer's rule, the chances of making an error are fairly high, as we need to input the entries for numerous matrices. However, as we mentioned in the chapter introduction, one of the advantages of matrices is that they *are easily programmable,* and we can actually write a very simple program that will make Cramer's rule a more efficient method.

To begin, press the [PRGM] key, and then the right arrow [▶] twice to enter a name for our program. At the prompt, we'll enter **CRAMER2.** As we write the program, note that the needed commands **(ClrHome, Disp, Prompt, Stop)** are all located in the submenus of the [PRGM] key, and the [=] sign is found under the **TEST** menu, accessed using the [2nd] [MATH] keys (the arrows "→" are used to indicate the [STO▶] key.

This program asks for the coefficients of two linear equations in two variables, written in standard form. Then, using Cramer's rule and the formula for the determinant of a 2×2 matrix, it returns the solutions of the system.

```
:PROGRAM:CRAMER2
:ClrHome
:Disp "2×2 SYSTEM"
:Disp "AX+BY = C"
:Disp "DX+EY = F"
:Prompt A,B,C,D,E,F
:(CE–BF)/(AE–BD)→X
:(AF–DC)/(AE–BD)→Y
:Disp "X =",X,"Y = ",Y
:Stop
```

Exercise 1: Create 2×2 systems of your own that are (a) consistent, (b) inconsistent, and (c) dependent. Then verify results using the program.

Exercise 2: Use the formula from page 598 of Section 9.4 to write a similar program for 3×3 systems. Call the program CRAMER3, and repeat parts (a), (b), and (c) from Exercise 1.

STRENGTHENING CORE SKILLS

Augmented Matrices and Matrix Inverses

The formula for finding the inverse of a 2×2 matrix has its roots in the more general method of computing the inverse of an $n \times n$ matrix. This involves augmenting a square matrix M with its corresponding identity I_n on the right (forming an $n \times 2n$ matrix), and using row operations to *transform M into the identity*. In some sense, as the original matrix is transformed, the "identity part" keeps track of the operations we used to convert M and we can use the results to "get back home," so to speak. We'll illustrate with the 2×2 matrix from Section 9.3, Example 2B, where we found that $\begin{bmatrix} 1 & -2.5 \\ -1 & 3 \end{bmatrix}$ was the inverse matrix for $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$. We begin by augmenting $\begin{bmatrix} 6 & 5 \\ 2 & 2 \end{bmatrix}$ with the 2×2 identity.

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] -2\text{R1} + 6\text{R2} \rightarrow \text{R2} \left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 0 & 2 & -2 & 6 \end{array}\right] \frac{\text{R2}}{2} \rightarrow \text{R2} \left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 6 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right] -5\text{R2} + \text{R1} \rightarrow \text{R1} \left[\begin{array}{cc|cc} 6 & 0 & 6 & -15 \\ 0 & 1 & -1 & 3 \end{array}\right] \frac{\text{R1}}{6} \rightarrow \text{R1} \left[\begin{array}{cc|cc} 1 & 0 & 1 & -2.5 \\ 0 & 1 & -1 & 3 \end{array}\right]$$

As you can see, the identity is automatically transformed into the inverse matrix when this method is applied. Performing similar row operations on the general matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ results in the formula given earlier.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad -c\text{R1} + a\text{R2} \rightarrow \text{R2} \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right]$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad-bc & c & a \end{array}\right] \quad \frac{\text{R2}}{ad - bc} \rightarrow \text{R2} \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{array}\right] \quad -b\text{R2} + \text{R1} \rightarrow \text{R1} \quad \left[\begin{array}{cc|cc} a & 0 & \frac{bc}{ad - bc} + 1 & \frac{-ba}{ad - bc} \\ 0 & 1 & \frac{-c}{ad - bc} & \frac{a}{ad - bc} \end{array}\right]$$

Finding a common denominator for $\frac{bc}{ad - bc} + 1$ and combining like terms gives $\frac{bc}{ad - bc} + \frac{ad - bc}{ad - bc} = \frac{ad}{ad - bc}$, and making this replacement we then have

$$\left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ba}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \xrightarrow{\frac{R1}{a} \to R1} \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$. This shows

$A^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$ and factoring out $\frac{1}{ad-bc}$ produces the familiar formula.

As you might imagine, attempting this on a general 3 × 3 matrix is problematic at best, and instead we simply apply the augmented matrix method to find A^{-1} for the 3 × 3 matrix shown in blue.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & -2 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-3R1 + 2R3 \to R3]{R1 + 2R2 \to R2} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 7 & -4 & 1 & 2 & 0 \\ 0 & -5 & 4 & -3 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[5R2 + 7R3 \to R3]{R2 - 7R1 \to R1} \left[\begin{array}{ccc|ccc} -14 & 0 & -4 & -6 & 2 & 0 \\ 0 & 7 & -4 & 1 & 2 & 0 \\ 0 & 0 & 8 & -16 & 10 & 14 \end{array}\right]$$

$$\xrightarrow{\frac{R3}{8} \to R3} \left[\begin{array}{ccc|ccc} -14 & 0 & -4 & -6 & 2 & 0 \\ 0 & 7 & -4 & 1 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1.25 & 1.75 \end{array}\right]$$

$$\xrightarrow[4R3 + R1 \to R1]{4R3 + R2 \to R2} \left[\begin{array}{ccc|ccc} -14 & 0 & 0 & -14 & 7 & 7 \\ 0 & 7 & 0 & -7 & 7 & 7 \\ 0 & 0 & 1 & -2 & 1.25 & 1.75 \end{array}\right]$$

Using $\frac{R1}{-14} \to R1$ and $\frac{R2}{7} \to R2$ produces the inverse matrix $A^{-1} = \begin{bmatrix} 1 & -0.5 & -0.5 \\ -1 & 1 & 1 \\ -2 & 1.25 & 1.75 \end{bmatrix}$.

To verify, we show $AA^{-1} = I$: $\begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & -2 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -0.5 & -0.5 \\ -1 & 1 & 1 \\ -2 & 1.25 & 1.75 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓ ($A^{-1}A$ is also true).

Exercise 1: Use the preceding inverse and a matrix equation to solve the system

$$\begin{cases} 2x + y = -2 \\ -x + 3y - 2z = -15. \\ 3x - y + 2z = 9 \end{cases}$$

CUMULATIVE REVIEW CHAPTERS 1–9

1. Solve each equation by factoring.

a. $9x^2 - 12x = -4$

b. $x^2 - 7x = 0$

c. $3x^3 - 15x^2 + 6x = 30$

d. $x^3 = 4x + 3x^2$

2. Solve for x.

$$\frac{-3}{x-3} + \frac{3x}{x^2 - x - 6} = \frac{1}{x+2}$$

3. A torus is a donut-shaped solid figure. Its surface area is given by the formula $A = \pi^2(R^2 - r^2)$, where R is the outer radius of the donut, and r is the inner radius. Solve the formula for R in terms of r and A.

4. State the value of all six trig functions given $(21, -28)$ is a point on the terminal side of θ.

5. Sketch the graph of $y = 3\cos\left(\frac{\pi}{6}x - \frac{\pi}{3}\right)$ using transformations of $y = \cos x$.

6. A jai alai player in an open-air court becomes frustrated and flings the pelota out and beyond the court. If the initial velocity of the ball is 136 mph and it is released at height of 5 ft, (a) how high is the ball after 3 sec? (b) What is the maximum height of the ball? (c) How long until the ball hits the ground (hopefully in an unpopulated area)? Recall the projectile equation is $h(t) = -16t^2 + v_0t + k$.

7. For a complex number $a + bi$, (a) verify the sum of a complex number and its conjugate is a real number. (b) Verify the product of a complex number and its conjuate is a real number.

8. Solve using the quadratic formula: $5x^2 + 8x + 2 = 0$.

9. Solve by completing the square: $3x^2 - 72x + 427 = 0$.

10. Given $\cos 53° \approx 0.6$ and $\cos 72° \approx 0.3$, approximate the value of $\cos 19°$ and $\cos 125°$ without using a calculator.

11. Given $\left(\frac{\sqrt{3}}{4}, y\right)$ is a point on the unit circle, find the value of $\sin\theta$, $\cos\theta$, and $\tan\theta$ if $y > 0$.

12. State the domain of each function:

a. $f(x) = \sqrt{2x - 3}$

b. $g(x) = \log_b(x + 3)$

c. $h(x) = \dfrac{x + 3}{x^2 - 5}$

13. Write the following formulas from memory:

a. slope formula

b. midpoint formula

c. quadratic formula

d. distance formula

e. interest formula (compounded continuously)

14. Find the equation of the line perpendicular to the line $4x + 5y = -20$, that contains the point (0, 1).

15. For the force vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ given, find a third force vector $\mathbf{F}_3$ that will bring these vectors into equilibrium: $\mathbf{F}_1 \langle -5, 12\rangle$ $\mathbf{F}_2 \langle 8, 6\rangle$

16. A commercial fishery stocks a lake with 250 fish. Based on previous experience, the population of fish is expected to grow according to the model $P(t) = \dfrac{12{,}000}{1 + 25e^{-0.2t}}$, where t is the time in months. From on this model, (a) how many months are required for the population to grow to 7500 fish? (b) If the fishery expects to harvest three-fourths of the fish population in 2 yr, approximately how many fish will be taken?

17. Evaluate each expression by drawing a right triangle and labeling the sides.

a. $\sec\left[\sin^{-1}\left(\dfrac{x}{\sqrt{121 + x^2}}\right)\right]$

b. $\sin\left[\csc^{-1}\left(\dfrac{\sqrt{9 + x^2}}{x}\right)\right]$

18. An luxury ship is traveling at 15 mph on a heading of 10°. There is a strong, 12 mph ocean current flowing from the southeast, at a heading of 330°. What is the true course and speed of the ship?

19. Use the *Guidelines for Graphing* to sketch the graph of function f given, then use it to solve $f(x) < 0$: $f(x) = x^3 - 4x^2 + x + 6$

20. Use the *Guidelines for Graphing* to sketch the graph of function g given, then use it to name the intervals where $g(x)\downarrow$ and $g(x)\uparrow$: $g(x) = \dfrac{x^2 - 4}{x^2 - 1}$

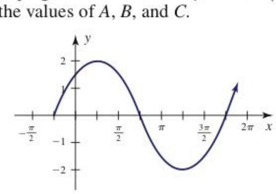
21. Find $(1 - \sqrt{3}i)^8$ using De Moivre's theorem.

22. Solve $\ln(x + 2) + \ln(x - 3) = \ln(4x)$.

23. If I saved \$200 each month in an annuity program that paid 8% annual interest compounded monthly, how long would it take to save \$10,000?

24. Mount Tortolas lies on the Argentine-Chilean border. When viewed from a distance of 5 mi, the angle of elevation to the top of the peak is 38°. How tall is Mount Tortolas? State the answer in feet.

25. The graph given is of the form $y = A\sin(Bx + C)$. Find the values of A, B, and C.



Modeling With Technology IV Matrix Applications

Learning Objectives

In this section, you will learn how to:

- ❑ **A.** Find market equilibrium graphically
- ❑ **B.** Use matrix equations to solve static systems
- ❑ **C.** Use matrices for encryption/decryption

Most of the skills needed for this study have been presented in previous sections. Here we'll use various types of regression, combined with systems of equations, to solve practical applications from business and industry.

A. Supply/Demand Curves and Market Equilibrium

In a "free-market" economy, also referred to as a "supply-and-demand" economy, there are naturally occurring forces that invariably come into play if no outside forces act on the producers (suppliers) and consumers (demanders). Generally speaking, the higher the price of an item, the lower the demand. A good advertising campaign can increase the demand, but the increasing demand brings an increase in price, which moderates the demand—and so it goes until a balance is reached. There are limitations to this model, and this interplay can be affected by the number of available consumers, production limits, "shelf-life" issues, and so on, but at any given moment in the life cycle of a product, consumer demand responds to price in this way. On the other hand, producers also respond to price in a very predictable way. The higher the price of an item, the more producers are willing to supply, but as the price of the item decreases, the producers' willingness to supply the item also decreases. These free-market forces ebb and flow until **market equilibrium** occurs, at the specific price where the supply and demand are equal. In the mathematical model for these market forces, it seems reasonable that *price* is the independent variable, with quantity demanded and quantity produced as the dependent variables.

EXAMPLE 1 ► Finding Market Equilibrium Graphically

An electronics company manufactures mp3 players. The monthly demand for their player can be modeled by the function $D(p) = -95p + 11{,}900$, where $D(p)$ represents the number of players *purchased* (demanded) at price p. The supply function for these players is $S(p) = 130p - 1600$, where $S(p)$ represents the number of players manufactured (supplied) at price p. Use a graphing calculator to help find the price at which market equilibrium will occur.

Solution ► Enter the demand equation as Y_1 and the supply equation as Y_2, as shown in Figure MWT IV.1. To set an appropriate viewing window, note that negative values do not fit the context, so we restrict our attention to QI. Since the y-intercepts are 11,900 and 600, respectively, appropriate settings for the y-values would be Ymin $= 0$ and Ymax $= 12{,}000$. The x-intercepts are approximately 12 and 125, so appropriate settings for the x-values would be Xmin $= 0$ and Xmax $= 130$ to allow for a frame around the viewing window. Pressing the GRAPH key at this point should produce the graph shown in Figure MWT IV.2. To find the equilibrium point (where supply and demand meet), we'll use the 2nd TRACE **(CALC)**, **5:intersect** option. As before, we press ENTER three times to identify the first curve,

Figure MWT IV.1

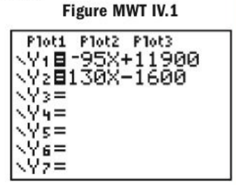
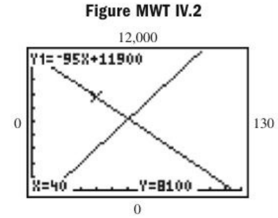


Figure MWT IV.2



the second curve, and to bypass the "Guess" option. The result shows (60, 6200) is the point of intersection, meaning equilibrium will occur at a price of \$60/player, and 6200 units will be demanded and supplied at this price.

Now try Exercises 1 through 4 ►

As a follow-up to Example 1, a person could reasonably ask, "Where did the equations come from?" In this instance, they were artificially constructed to yield a "nice" solution. In actual practice, the equations and coefficients are not so "well behaved" and are based on the collection and interpretation of real data. While market analysts have sophisticated programs and numerous models to help develop these equations, in this study we'll use our experience with regression to develop the supply and demand curves.

EXAMPLE 2 ► Using Technology to Find Market Equilibrium

The company from Exercise 1 hired a consulting firm to do market research on their "next generation" mp3 player. Over a 10-week period, the firm collected the data shown for the mp3 player market (data includes mp3 players sold and expected to sell).

Price (dollars)	Demand	Supply (Inventory)
107.10	6,900	12,200
85.50	7,900	9,900
64.80	13,200	8,000
52.20	13,500	7,900
108.00	6,700	14,000
91.80	7,600	12,000
77.40	9,200	9,400
46.80	13,800	6,100
74.70	10,600	8,800
68.40	12,800	8,600

a. Use a graphing calculator to simultaneously display the demand and supply scatter-plots.

b. Calculate a line of best fit for each and graph them with the scatter-plots (identify each curve).

c. Find the equilibrium point.

Solution ► **a.** Begin by clearing all lists. This can be done manually, or by pressing 2nd + **(MEM)** and selecting option **4:ClrAllLists** (the command appears on the home screen). Pressing ENTER will execute the command, and the word DONE will appear. Carefully input price in L1, demand in L2, and supply in L3 (see Figure MWT IV.3). With the window settings given in Figure MWT IV.4, pressing GRAPH will display the price/demand and price/supply scatter-plots shown. If this is not the case, use 2nd Y= **(STAT PLOT)** to be sure that "On" is highlighted in Plot1 and Plot2, and that Plot1 uses L1 and L2, while Plot2 uses L1 and L3 (Figure MWT IV.5). Note we've chosen a different mark to indicate the data points in Plot2.

b. Calculate the linear regression equation for L1 and L2 (demand), and paste it in Y_1: **LinReg (ax + b) L1, L2, Y_1** ENTER. Next, calculate the linear regression for L1 and L3 (supply) and paste it in

Figure MWT IV.3

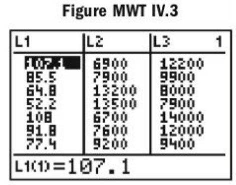


Figure MWT IV.4

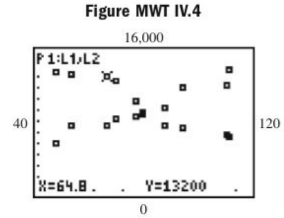


Figure MWT IV.5

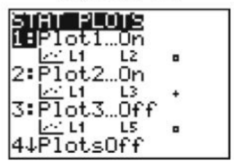
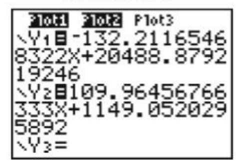
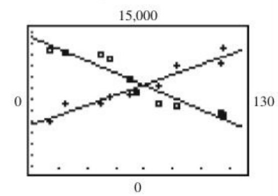


Figure MWT IV.6



Y_2: **LinReg(ax + b) L1, L3, Y_2** ENTER (recall that Y_1 and Y_2 are accessed using the VARS key). The resulting equations and graphs are shown in Figures MWT IV.6 and MWT IV.7.

Figure MWT IV.7



c. Once again we use 2nd TRACE **(CALC) 5:intersect** to find the equilibrium point, which is approximately (80, 9931). Supply and demand for this mp3 player model are approximately equal at a price of about \$80, with 9931 mp3 players bought and sold.

A. You've just learned how to find market equilibrium graphically

Now try Exercises 5 through 8 ►

B. Solving Static Systems with Varying Constraints

When the considerations of a business or industry involve more than two variables, solutions using matrix methods have a distinct advantage over other methods. Companies often have to perform calculations using basic systems weekly, daily, or even hourly, to keep up with trends, market changes, changes in cost of raw materials, and so on. In many situations, the basic requirements remain the same, but the frequently changing inputs require a recalculation each time they change.

EXAMPLE 3 ► Determining Supply Inventories Using Matrices

BNN Soft Drinks receives new orders daily for its most popular drink, Saratoga Cola. It can deliver the carbonated beverage in a twelve-pack of 12-ounce (oz) cans, a six-pack of 20-oz bottles, or in a 2-L bottle. The ingredients required to produce a twelve-pack include 1 gallon (gal) of carbonated water, 1.25 pounds (lb) of sugar, 2 cups (c) of flavoring, and 0.5 grams (g) of caffeine. For the six-pack, 0.8 gal of carbonated water, 1 lb of sugar, 1.6 c of flavoring, and 0.4 g of caffeine are needed. The 2-L bottle contains 0.47 gal of carbonated water, 0.59 lb of sugar, 0.94 c of flavoring, and 0.24 g of caffeine. How much of each ingredient must be on hand for Monday's order of 300 twelve-packs, 200 six-packs, and 500 2-L bottles? What quantities must be on hand for Tuesday's order: 410 twelve-packs, 320 six-packs, and 275 2-L bottles?

Solution ► Begin by setting up a general system of equations, letting x represent the number of twelve-packs, y the number of six-packs, and z the number of 2-L bottles:

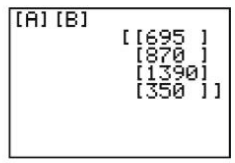
$$\begin{cases} 1x + 0.8y + 0.47z = \text{gallons of carbonated water} \\ 1.25x + 1y + 0.59z = \text{pounds of sugar} \\ 2x + 1.6y + 0.94z = \text{cups of flavoring} \\ 0.5x + 0.4y + 0.24z = \text{grams of caffeine} \end{cases}$$

As a matrix equation we have

$$\begin{bmatrix} 1 & 0.8 & 0.47 \\ 1.25 & 1 & 0.59 \\ 2 & 1.6 & 0.94 \\ 0.5 & 0.4 & 0.24 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} w \\ s \\ f \\ c \end{bmatrix}$$

Enter the 4×3 matrix as matrix A, and the size of the order as matrix as B. Using a calculator, we find

$$AB = \begin{bmatrix} 1 & 0.8 & 0.47 \\ 1.25 & 1 & 0.59 \\ 2 & 1.6 & 0.94 \\ 0.5 & 0.4 & 0.24 \end{bmatrix}\begin{bmatrix} 300 \\ 200 \\ 500 \end{bmatrix} = \begin{bmatrix} 695 \\ 870 \\ 1390 \\ 350 \end{bmatrix},$$



and BNN Soft Drinks will need 695 gal of carbonated water, 870 lb of sugar, 1390 c of flavoring, and 350 g of caffeine for Monday's order. After entering $C = \begin{bmatrix} 410 \\ 320 \\ 275 \end{bmatrix}$ for Tuesday's orders, computing the product AC shows 795.25 gal of carbonated water, 994.75 lb of sugar, 1590.5 c of flavoring, and 399 g of caffeine are needed for Tuesday.

Now try Exercises 9 through 14 ►

Example 3 showed how the creation of a static matrix can help track and control inventory requirements. In Example 4, we use a static matrix to solve a system that will identify the amount of data traffic used by a company during various hours of the day.

EXAMPLE 4 ► Identifying the Source of Data Traffic Using Matrices

Mariño Imports is a medium-size company that is considering upgrading from a 1.544 megabytes per sec (Mbps) T1 Internet line to a fractional T3 line with a bandwidth of 7.72 Mbps. They currently use their bandwidth for phone traffic, office data, and Internet commerce. The IT (Internet Technology) director devises a plan to monitor how much data traffic each resource uses on an hourly basis. Because of the physical arrangement of the hardware, she cannot monitor each resource individually. The table shows the information she collected for the first 3 hr. Determine how many gigabytes (GB) each resource used individually during these 3 hrs.

	Phone, Data, and Commerce	Phone and Data	Data and Commerce
9–10:00 A.M.	5.4 GB	4.0 GB	4.2 GB
10–11:00 A.M.	5.3 GB	3.8 GB	4.2 GB
11–12:00 P.M.	5.1 GB	3.5 GB	3.6 GB

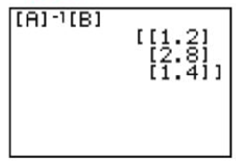
Solution ► Using p to represent the phone traffic, d for office data, and c for Internet commerce, we create the system shown, which models data use for the 9:00 o'clock hour. Since we actually need to solve two more systems whose only difference is the constant terms (for the 10 and 11 o'clock hours), using a matrix equation to solve the system (Section 6.3)

$$\begin{cases} p + d + c = 5.4 \\ p + d = 4.0 \\ d + c = 4.2 \end{cases}$$

will be most convenient. Begin by writing the related matrix equation for this system:

$$AX = B: \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} p \\ d \\ c \end{bmatrix} = \begin{bmatrix} 5.4 \\ 4.0 \\ 4.2 \end{bmatrix}$$

Using $X = A^{-1}B$ to solve the system (see figure), we find there was 1.2 GB of phone traffic, 2.8 GB of office data, and 1.4 GB of Internet commerce during this hour. Note that the IT director may make this calculation 10 or more times a day (once for every hour of business). While we *could* solve for the 10 and 11 o'clock hours using



$C = \begin{bmatrix} 5.3 \\ 3.8 \\ 4.2 \end{bmatrix}$ and $D = \begin{bmatrix} 5.1 \\ 3.5 \\ 3.6 \end{bmatrix}$, then calculate $A^{-1}C$ and $A^{-1}D$, these calculations *can all be performed simultaneously* by combining the matrices B, C, and D into one 3×3 matrix and multiplying by A^{-1}. Due to the properties of matrix multiplication, each column of the product will represent the data information for a given hour, as shown here:

$$[A^{-1}]\begin{bmatrix} 5.4 & 5.3 & 5.1 \\ 4.0 & 3.8 & 3.5 \\ 4.2 & 4.2 & 3.6 \end{bmatrix} = \begin{bmatrix} 1.2 & 1.1 & 1.5 \\ 2.8 & 2.7 & 2.0 \\ 1.4 & 1.5 & 1.6 \end{bmatrix}$$

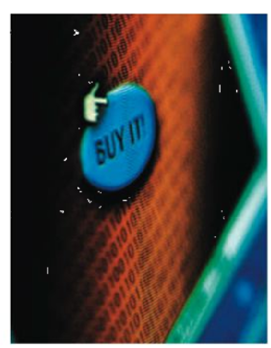
B. You've just learned how to use matrix equations to solve static systems

In the second hour, there was 1.1 GB of phone traffic, 2.7 GB of office data, and 1.5 GB of Internet commerce. In the third hour, 1.5 GB of phone traffic, 2.0 GB of office data, and 1.6 GB of Internet commerce bandwidth was used.

Now try Exercises 15 through 18 ►

C. Using Matrices to Encode Messages

In *The Gold-Bug,* by Edgar Allan Poe, the narrator deciphers the secret message on a treasure map by accounting for the frequencies of specific letters and words. The coding used was a simple substitution cipher, which today can be broken with relative ease. In modern times where information wields great power, a simple cipher like the one used here will not suffice. When you pay your tuition, register for classes, make on-line purchases, and so on, you are publicly transmitting very private data, which needs to be protected. While there are many complex encryption methods available (including the now famous symmetric-key and public-key techniques), we will use a matrix-based technique. The nature of matrix multiplication makes it very difficult to determine the exact matrices that yield a given product, and we'll use this fact to our advantage. Beginning with a fixed, invertible matrix A, we will develop a matrix B such that the product AB is possible, and our secret message is encrypted in AB. At the receiving end, they will need to know A^{-1} to decipher the message, since $A^{-1}(AB) = (A^{-1}A)B = B$, which is the original message. Note that in case an intruder were to find matrix A (perhaps purchasing the information from a disgruntled employee), we must be able to change it easily.



This means we should develop a method for generating a matrix A, with integer entries, where A is invertible and A^{-1} also has integer entries.

EXAMPLE 5 ► Finding an Invertible Matrix A Where Both A and A^{-1} Have Integer Entries

Find an invertible 3×3 matrix A as just described, and its inverse A^{-1}.

Solution ► Begin with any 3×3 matrix that has only 1s or -1s on its main diagonal, and 0s below the diagonal. The upper triangle can consist of any integer values you choose, as in

$$\begin{bmatrix} -1 & 5 & -1 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, use **any** of the elementary row operations to make the matrix more complex. For instance, we'll using a calculator to create a new matrix by (1) using $R1 + R2 \rightarrow R2$, and $R1 + R3 \rightarrow R3$ to create matrix [C], then (2) using $R2 + R3 \rightarrow R2$ to create matrix [D], then (3) using $R3 + R2 \rightarrow R2$ to create matrix [E], and finally (4) $-2\,R1 + R3 \rightarrow R3$ to obtain our final matrix [A]. To begin, enter the initial matrix as matrix B. For (1) $R1 + R2 \rightarrow R2$, go to the (MATRIX) **MATH** submenu, select option **D:row+(** and press ENTER to bring this option to the home screen (Figure MWT IV.8). This feature requires us to name the matrix we're using, and to indicate what rows to add, so we enter **D:row+([B], 1, 2).** The screen shown in Figure MWT IV.9 indicates we've placed the result in matrix [C]. For $R1 + R3 \rightarrow R3$, recall **D:row+([B], 1, 2)** using 2nd ENTER and change it to **D:row+([C], 1, 3)** STO→ [D]. Repeat this process for (3) $R3 + R2 \rightarrow R2$ to create matrix [E]: **D:row+([D], 3, 2)** STO→ [E] (Figure MWT IV.10). Finally, we compute (4) $-2R1 + R3 \rightarrow R3$ using the (new) option **F:*row+(−2, [E], 1, 3)** STO→ [A] and the process is complete. The entries of matrix A are all integers, and A^{-1} exists and also has integer entries (see Figures MWT IV.11 and MWT IV.12). This will always be the case for matrices created in this way.

Figure MWT IV.8

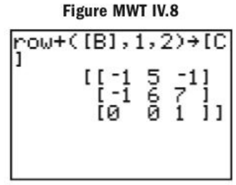


Figure MWT IV.9

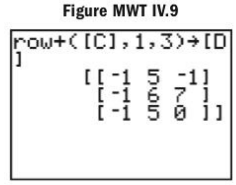


Figure MWT IV.10

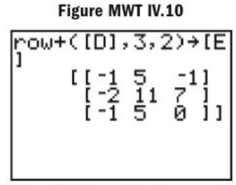


Figure MWT IV.11

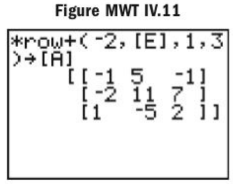
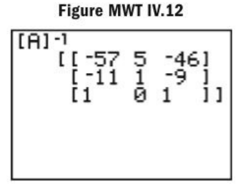


Figure MWT IV.12



Now try Exercises 19 through 24 ►

EXAMPLE 6 ▶ Using Matrices to Encode Messages

Set up a substitution cipher to encode the message MATH IS SWEET, and then use the matrix A from Example 5 to encrypt it.

Solution ▶ For the cipher, we will associate a unique number to every letter in the alphabet. This can be done randomly or using a systematic approach. Here we choose to associate 0 with a blank space, and assign 1 to A, −1 to B, 2 to C, −2 to D, and so on.

Blank	A	B	C	D	E	F	G	H	I	J	K	L	M
0	1	−1	2	−2	3	−3	4	−4	5	−5	6	−6	7

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
−7	8	−8	9	−9	10	−10	11	−11	12	−12	13	−13

Now encode the secret message as shown:

M	A	T	H		I	S		S	W	E	E	T
7	1	−10	−4	0	5	10	0	10	12	3	3	−10

We next enter the coded message into a new matrix B, by entering it letter by letter *into the columns* of B. Note that since the encrypting matrix A is 3×3, B must have 3 rows for multiplication to be possible. The result is

$$B = \begin{bmatrix} M & H & S & W & T \\ A & * & * & E & * \\ T & I & S & E & * \end{bmatrix} = \begin{bmatrix} 7 & -4 & 10 & 12 & -10 \\ 1 & 0 & 0 & 3 & 0 \\ -10 & 5 & 10 & 3 & 0 \end{bmatrix}$$

If the message is too short to fill matrix B, we use blank spaces to complete the final column. Computing the product AB encrypts the message, and only someone with access to A^{-1} will be able to read it:

$$AB = \begin{bmatrix} -1 & 5 & -1 \\ -2 & 11 & 7 \\ 1 & -5 & 2 \end{bmatrix} \begin{bmatrix} 7 & -4 & 10 & 12 & -10 \\ 1 & 0 & 0 & 3 & 0 \\ -10 & 5 & 10 & 3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -1 & -20 & 0 & 10 \\ -73 & 43 & 50 & 30 & 20 \\ -18 & 6 & 30 & 3 & -10 \end{bmatrix}$$

The coded message is 8, −73, −18, −1, 43, 6, −20, 50, 30, 0, 30, 3, 10, 20, −10.

Now try Exercises 25 through 30 ▶

EXAMPLE 7 ▶ Deciphering Coded Messages Using an Inverse Matrix

Decipher the coded message from Example 6, using A^{-1} from Example 5.

Solution ▶ The received message is 8, −73, −18, −1, 43, 6, −20, 50, 30, 0, 30, 3, 10, 20, −10, and is the result of the product AB. To find matrix B, we apply A^{-1} since $A^{-1}(AB) = (A^{-1}A)B = B$. Writing the received message in matrix form we have

$$AB = \begin{bmatrix} 8 & -1 & -20 & 0 & 10 \\ -73 & 43 & 50 & 30 & 20 \\ -18 & 6 & 30 & 3 & -10 \end{bmatrix}$$

Next multiply AB by A^{-1} on the left, to determine matrix B:

$$A^{-1}(AB) = \begin{bmatrix} -57 & 5 & -46 \\ -11 & 1 & -9 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & -1 & -20 & 0 & 10 \\ -73 & 43 & 50 & 30 & 20 \\ -18 & 6 & 30 & 3 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -4 & 10 & 12 & -10 \\ 1 & 0 & 0 & 3 & 0 \\ -10 & 5 & 10 & 3 & 0 \end{bmatrix} = B$$

Writing matrix B in sentence form gives 7, 1, −10, −4, 0, 5, 10, 0, 10, 12, 3, 3, −10, 0, 0, and using the substitution cipher to replace numbers with letters, reveals the message MATH IS SWEET.

C. You've just learned how to use matrices for encryption/decryption

Now try Exercises 31 through 36 ►

MODELING WITH TECHNOLOGY EXERCISES

1. A water sports company manufactures high-end wakeboards. The monthly demand for their boards is modeled by $D(p) = -2.25p + 1125$, where $D(p)$ represents the number of boards bought at price p. The supply function is $S(p) = 0.75p - 75$, where $S(p)$ represents the number of boards supplied at price p. Use a graphing calculator to find the market equilibrium for this product.

2. A local metal shop manufactures rabbit cages. The monthly demand for their cage is related to its price. The function $D(p) = -\frac{1}{2}p + \frac{45}{2}$ models this demand, where $D(p)$ represents the number of cages bought at price p. The supply function is $S(p) = p - 15$, where $S(p)$ represents the number of cages supplied at price p. Use a graphing calculator to find the market equilibrium for this product.

3. The Bureau of Labor and Statistics (BLS) keeps track of important statistics for many different markets. In studying the supply and demand for refined gasoline in Atlanta, Georgia, for the month of April, the BLS stated that monthly demand was modeled by the function D shown, where $D(p)$ represents the number of gallons bought at price p. The monthly supply was modeled by the function S shown, where $S(p)$ represents the number of gallons supplied at price p. Use a graphing calculator to find the market equilibrium for this product:
$D(p) = (-5.000 \times 10^7)p + (2.435 \times 10^8)$,
$S(p) = (5.000 \times 10^7)p - (6.350 \times 10^7)$

4. The Bureau of Labor and Statistics has been closely following the energy-efficient fluorescent lightbulb market over the past year. The yearly demand for these "green" lightbulbs is related to its price. The demand function is $D(p) = (-4.500 \times 10^6)p + (5.191 \times 10^7)$, where $D(p)$ represents the number of 30-W bulbs bought at price p. The supply function is $S(p) = (8.5 \times 10^6)p - (6.483 \times 10^7)$, where $S(p)$ represents the number of 30-watt bulbs supplied at price p. Use a graphing calculator to find the market equilibrium for this product.

5. The company from Exercise 1 has hired an outside consulting firm to do some market research on their wakeboard. This consulting firm collected the following supply and demand data for this and comparable wakeboards over a 10-week period. Find the equilibrium point. Round your answer to the nearest integer and dollar.

Average Price (in U.S. dollars)	Quantity Demanded	Available Inventory
424.85	175	232
445.25	166	247
389.55	291	215
349.98	391	201
402.22	218	226
413.87	200	222
481.73	139	251
419.45	177	235
397.05	220	219
361.90	317	212

6. The metal shop from Exercise 2 has collected some data on sales and production of its rabbit cages over the past 8 weeks. The following table shows the supply and demand data for this cage. Find the equilibrium point (round to the nearest cent and whole cage).

Average Price (in U.S. dollars)	Quantity Sold (Demand)	Production (Supply)
22.99	12	7
21.49	14	6
23.99	11	7
26.99	9	11
25.99	8	10
27.99	8	13
24.49	10	9
26.49	9	11

7. The Bureau of Labor and Statistics from Exercise 3 collected the following supply-and-demand data for refined gasoline for the month of May. Data were collected every Tuesday and Friday. Find the equilibrium point, rounding your answer to the nearest hundred thousand gallons and whole cent.

Average Price (in U.S. dollars)	Quantity Demanded (1×10^7 gal)	Available Inventory (1×10^7 gal)
3.17	8.82	9.10
3.12	8.87	9.05
3.04	9.08	8.97
2.84	9.22	8.91
3.11	8.92	9.02
3.15	8.76	9.08
3.10	9.01	8.99
3.11	8.94	9.01
2.93	9.13	8.93

8. The Bureau of Labor and Statistics from Exercise 4 collected the following supply and demand data for the energy-efficient fluorescent lightbulbs sold each month for the past year. Find the equilibrium point, rounding your answer to the nearest ten thousand lightbulbs and whole cent. What is the yearly demand at the equilibrium point?

Average Price (in U.S. dollars)	Quantity Demanded (in millions)	Available Inventory (in millions)
9.40	0.84	1.23
8.51	1.17	0.95
8.78	1.05	1.11
10.82	0.68	1.29
6.77	1.47	0.77
9.33	0.91	1.21
8.34	1.25	0.88
10.37	0.76	1.27
8.62	1.09	1.02
8.44	1.21	0.92
8.58	1.18	0.97
8.96	1.01	1.17

9. Slammin' Drums manufactures several different types of drums. Its most popular drums are the 22″ bass drum, the 12″ tom, and the 14″ snare drum. The 22″ bass drum requires 7 ft^2 of skin, 8.5 ft^2 of wood veneer, 8 tension rods, and 11.5 ft of hoop. The 12″ tom requires 2 ft^2 of skin, 3 ft^2 of wood veneer, 6 tension rods, and 6.5 ft of hoop. The 14″ snare requires 2.5 ft^2 of skin, 1.5 ft^2 of wood veneer, 10 tension rods, and 7 ft of hoop. In February, Slammin' Drums received orders for 15 bass drums, 21 toms, and 27 snares. Use your calculator and a matrix equation to determine how much of each raw material they need to have on hand, to fill these orders.

10. In March, Slammin' Drum's orders consisted of 19 bass drums, 19 toms, and 25 snares. Use your calculator and a matrix equation to determine how much of each raw material they need to have on hand, to fill their orders. (See Exercise 9.)

11. The following table represents Slammin's orders for the months of April through July. Use your calculator and a matrix equation to determine how much of each raw material they need to have on hand to fill these orders. (See Exercise 9.) (*Hint:* Using a clever 4×3 and 3×1 matrix can reduce this problem to a single step.)

	April	May	June	July
Bass drum	23	21	17	14
Tom	20	18	15	17
Snare drum	29	35	27	25

12. The following table represents Slammin's orders for the months of August through November. Use your calculator and a matrix equation to determine how much of each raw material they need to have on hand to fill their orders. (See Exercise 9.) (*Hint:* Using a clever 4×3 and 3×1 matrix can reduce this problem to a single step.)

	August	September	October	November
Bass drum	17	22	16	12
Tom	15	14	13	11
Snare drum	32	28	27	21

13. Midwest Petroleum (MP) produces three types of combustibles using common refined gasoline and vegetable products. The first is E10 (also known as gasohol), the second is E85, and the third is biodiesel. One gallon of E10 requires 0.90 gal of gasoline, 2 lb of corn, 1 oz of yeast, and 0.5 gal of water. One gallon of E85 requires 0.15 gal of gasoline, 17 lb of corn, 8.5 oz of yeast, and 4.25 gal of water. One gallon of biodiesel requires 20 lb of corn and 3 gal of water. One week's production at MP consisted of 100,000 gal of E10, 15,000 gal of E85, and 7000 gal of biodiesel. Use your calculator and a matrix equation to determine how much of each raw material they used to fill their orders.

14. The following table represents Midwest Petroleum's production for the next 3 weeks. Use your calculator and a matrix equation to determine the total amount of raw material they used to fill their orders. See Exercise 13.

	Week 2	Week 3	Week 4
E10	110,000	95,000	105,000
E85	17,000	18,000	20,000
Biodiesel	6,000	8,000	10,000

15. Roll-X Watches makes some of the finest wristwatches in the world. Their most popular model is the Clam. It comes in three versions: Silver, Gold, and Platinum. Management thinks there might be a thief in the production line, so they decide to closely monitor the precious metal consumption. A Silver Clam contains 1.2 oz of silver and 0.2 oz of gold. A Gold Clam contains 0.5 oz of silver, 0.8 oz of gold, and 0.1 oz of platinum. A Platinum Clam contains 0.2 oz of silver, 0.5 oz of gold, and 0.7 oz of platinum. During the first week of monitoring, the production team used 10.9 oz of silver, 9.2 oz of gold, and 2.3 oz of platinum. Use your calculator and a matrix equation to determine the number of each type of watch that should have been produced.

16. The following table contains the precious metal consumption of the Roll-X Watch production line during the next five weeks (see Exercise 15). Use your graphing calculator to determine the number of each type of watch that should have been produced each week. For which week does the data seem to indicate a possible theft of precious metal?

Ounces	Week 1	Week 2	Week 3	Week 4	Week 5
Silver	13.1	9	12.9	11.9	11.2
Gold	11	7.7	8.6	8.4	9.5
Platinum	2.5	1.5	0.9	2.8	1.7

17. There are three classes of grain, of which three bundles from the first class, two from the second, and one from the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second, and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class? *(This is the historic problem from the Chiu chang suan shu.)*

18. During a given week, the measures of grain that make up the bundles in Exercise 17 can vary slightly. Three local Chinese bakeries always buy the same amount of bundles, as outlined in Exercise 17. That is to say, bakery 1 buys three bundles of the first class, two of the second, and one of the third. Bakery 2 buys two of the first, three of the second, and one of the third. And finally, bakery 3 buys one of the first, two of the second, and three of the third. The following table outlines how many measures of grain each bakery received each day. How many measures of grain were contained in one bundle of each class, on each day?

	Mon	Tues	Wed	Thurs	Fri
Bakery 1 (measures)	39	38	38	37.75	39.75
Bakery 2 (measures)	34	33	33.5	32.5	35
Bakery 3 (measures)	26	26	27	26.25	27.25

For Exercises 19–24, use the criteria indicated to find 3×3 matrices A and A^{-1}, where the entries of both are all integers.

19. The lower triangle is all zeroes.

20. The upper triangle is all zeroes.

21. $a_{2,1} = 5$

22. $a_{3,2} = -2$

23. $a_{3,1} = 1$ and $a_{2,3} = 2$

24. $a_{2,1} = -3$ and $a_{1,3} = 1$

25. Use the matrix A you created in Exercise 19 and the substitution cipher from Example 6 to encrypt your full name.

26. Use the matrix A you created in Exercise 20 and the substitution cipher from Example 6 to encrypt your school's name.

27. Design your own substitution cipher. Then use it and the matrix A you created in Exercise 21 to encrypt the title of your favorite movie.

28. Design your own substitution cipher. Then use it and the matrix A you created in Exercise 22 to encrypt the title of your favorite snack food.

29. Design your own substitution cipher. Then use it and the matrix A you created in Exercise 23 to encrypt the White House switchboard phone number, 202-456-1414.

30. Design your own substitution cipher. Then use it and the matrix A you created in Exercise 24 to encrypt the Casa Rosada switchboard phone number 54-11-4344-3600. The Casa Rosada, or Pink House, consists of the offices of the president of Argentina.

31. Use the matrix A^{-1} from Exercise 19, and the appropriate substitution cipher to decrypt the message from Exercise 25.

32. Use the matrix A^{-1} from Exercise 20, and the appropriate substitution cipher to decrypt the message from Exercise 26.

33. Use the matrix A^{-1} from Exercise 21, and the appropriate substitution cipher to decrypt the message from Exercise 27.

34. Use the matrix A^{-1} from Exercise 22, and the appropriate substitution cipher to decrypt the message from Exercise 28.

35. Use the matrix A^{-1} from Exercise 23, and the appropriate substitution cipher to decrypt the message from Exercise 29.

36. Use the matrix A^{-1} from Exercise 24, and the appropriate substitution cipher to decrypt the message from Exercise 30.