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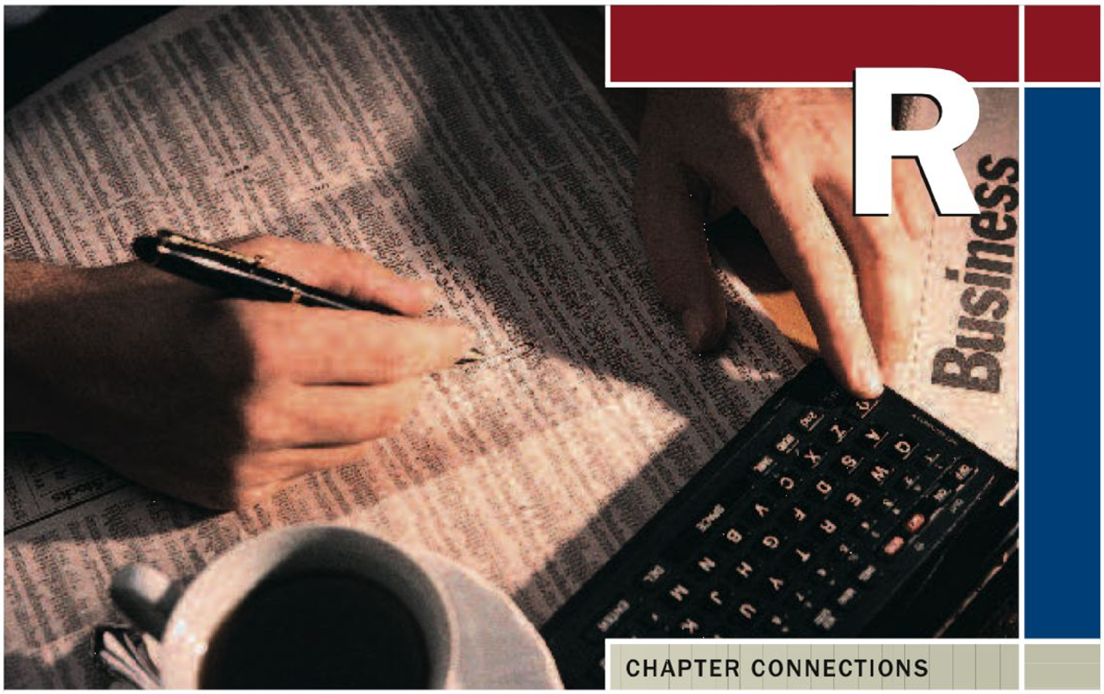

A Review of Basic Concepts and Skills

CHAPTER OUTLINE

CHAPTER CONNECTIONS

Jared places a small inheritance of \$2475 in a certificate of deposit that earns 6% interest compounded quarterly. The total in the CD after 10 years is given by the expression

$$2475\left(1 + \frac{0.06}{4}\right)^{4\cdot 10}.$$

This chapter reviews the skills required to correctly determine the CD's value, as well as other mathematical skills to be used throughout this course. This expression appears as Exercise 93 in Section R.1.

Check out these other real-world connections:

- Pediatric Dosages and Clark's Rule (Section R.1, Exercise 96)
- Maximizing Revenue of Video Game Sales (Section R.3, Exercise 143)
- Growth of a New Stock Hitting the Market (Section R.5, Exercise 83)
- Accident Investigation (Section R.6, Exercise 55)

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R.1 The Language, Notation, and Numbers of Mathematics

Learning Objectives

In Section R.1 you will review:

- ☐ **A.** Sets of numbers, graphing real numbers, and set notation
- ☐ **B.** Inequality symbols and order relations
- ☐ **C.** The absolute value of a real number
- ☐ **D.** The Order of Operations

The most fundamental requirement for learning algebra is mastering the words, symbols, and numbers used to express mathematical ideas. "Words are the symbols of knowledge, the keys to accurate learning" (Norman Lewis in *Word Power Made Easy,* Penguin Books).

A. Sets of Numbers, Graphing Real Numbers, and Set Notation

To effectively use mathematics as a problem-solving tool, we must first be familiar with the **sets of numbers** used to quantify (give a numeric value to) the things we investigate. Only then can we make comparisons and develop equations that lead to informed decisions.

Natural Numbers

The most basic numbers are those used to count physical objects: 1, 2, 3, 4, and so on. These are called **natural numbers** and are represented by the capital letter $\mathbb{N}$, often written in the special font shown. We use **set notation** to list or describe a set of numbers. Braces { } are used to group **members** or **elements** of the set, commas separate each member, and three dots (called an *ellipsis*) are used to indicate a pattern that continues indefinitely. The notation $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ is read, "$\mathbb{N}$ is the set of numbers 1, 2, 3, 4, 5, and so on." To show membership in a set, the symbol $\in$ is used. It is read "is an element of" or "belongs to." The statements $6 \in \mathbb{N}$ (6 is an element of $\mathbb{N}$) and $0 \notin \mathbb{N}$ (0 is not an element of $\mathbb{N}$) are true statements. A set having no elements is called the **empty** or **null set,** and is designated by empty braces { } or the symbol $\varnothing$.

EXAMPLE 1 ► **Writing Sets of Numbers Using Set Notation**

List the set of natural numbers that are

a. negative **b.** greater than 100

c. greater than or equal to 5 and less than 12

Solution ►

a. { }; all natural numbers are positive.

b. $\{101, 102, 103, 104, \ldots\}$

c. $\{5, 6, 7, 8, 9, 10, 11\}$

Now try Exercises 7 and 8 ►

Whole Numbers

Combining zero with the natural numbers produces a new set called the **whole numbers** $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$. We say that the natural numbers are a **proper subset** of the whole numbers, denoted $\mathbb{N} \subset \mathbb{W}$, since every natural number is also a whole number. The symbol $\subset$ means "is a proper subset of."

EXAMPLE 2 ► **Determining Membership in a Set**

Given $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4\}$, and $C = \{0, 1, 2, 3, 5, 8\}$, determine whether the following statements are true or false.

a. $B \subset A$ **b.** $B \subset C$ **c.** $C \subset \mathbb{W}$

d. $C \subset \mathbb{N}$ **e.** $104 \in \mathbb{W}$ **f.** $0 \in \mathbb{N}$

g. $2 \notin \mathbb{W}$

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Solution ►
a. True: Every element of B is in A. **b.** False: $4 \notin C$.
c. True: All elements are whole numbers. **d.** False: $0 \notin \mathbb{N}$.
e. True: 104 is a whole number. **f.** False: $0 \notin \mathbb{N}$.
g. False: 2 *is* a whole number.

Now try Exercises 9 through 14 ►

Integers

Numbers greater than zero are **positive numbers.** Every positive number has an *opposite* that is a **negative number** (a number less than zero). The set containing zero and the natural numbers with their opposites produces the set of **integers** $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We can illustrate the location of a number (in relation to other numbers) using a **number line** (see Figure R.1).

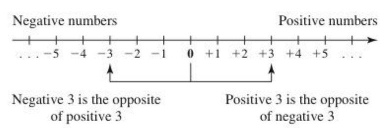

Figure R.1

The number that corresponds to a given point on the number line is called the **coordinate** of that point. When we want to note a specific location on the line, a bold dot "•" is used and we have then **graphed** the number. Since we need only one coordinate to denote a location on the number line, it can be referred to as a **one-dimensional graph.**

WORTHY OF NOTE
The integers are a subset of the rational numbers: $\mathbb{Z} \subset \mathbb{Q}$, since any integer can be written as a fraction using a denominator of one: $-2 = \frac{-2}{1}$ and $0 = \frac{0}{1}$.

Rational Numbers

Fractions and mixed numbers are part of a set called the **rational numbers** $\mathbb{Q}$. A rational number is one that can be written as a fraction with an integer numerator and an integer denominator other than zero. In set notation we write $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}; q \neq 0\}$. The vertical bar "|" is read "such that" and indicates that a description follows. In words, we say, "$\mathbb{Q}$ is the set of numbers of the form p over q, such that p and q are integers and q is not equal to zero."

EXAMPLE 3 ► Graphing Rational Numbers

Graph the fractions by converting to decimal form and estimating their location between two integers:

a. $-2\frac{1}{3}$ **b.** $\frac{7}{2}$

Solution ► **a.** $-2\frac{1}{3} = -2.3333333\ldots$ or $-2.\overline{3}$ **b.** $\frac{7}{2} = 3.5$

Now try Exercises 15 through 18 ►

Since the division $\frac{7}{2}$ **terminated,** the result is called a **terminating decimal.** The decimal form of $-2\frac{1}{3}$ is called **repeating** and **nonterminating.** Recall that a repeating decimal is written with a horizontal bar over the first block of digit(s) that repeat.

Irrational Numbers

Although any fraction can be written in decimal form, not all decimal numbers can be written as a fraction. One example is the number represented by the Greek letter π (pi),

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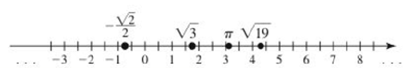
frequently seen in a study of circles. Although we often approximate π using 3.14, its true value has a **nonrepeating** and *nonterminating* decimal form. Other numbers of this type include 2.101001000100001 . . . (there is no block of digits that repeat), and $\sqrt{5} \approx 2.2360679\ldots$ (the decimal form never terminates). Numbers with a nonrepeating and nonterminating decimal form belong to the set of irrational numbers $\mathbb{H}$.

EXAMPLE 4 ► Approximating Irrational Numbers

Use a calculator as needed to approximate the value of each number given (round to 100ths), then graph them on the number line:

a. $\sqrt{3}$ **b.** π **c.** $\sqrt{19}$ **d.** $-\frac{\sqrt{2}}{2}$

Solution ► **a.** $\sqrt{3} \approx 1.73$ **b.** $\pi \approx 3.14$ **c.** $\sqrt{19} \approx 4.36$ **d.** $-\frac{\sqrt{2}}{2} = -0.71$



Now try Exercises 19 through 22 ►

WORTHY OF NOTE

Checking the approximation for $\sqrt{5}$ shown, we obtain $2.2360679^2 = 4.999999653$. While we can find better approximations by using more and more decimal places, we never obtain five *exactly* (although some calculators will say the result is 5 due to limitations in programming).

Real Numbers

The set of rational numbers combined with the set of irrational numbers produces the set of **real numbers** $\mathbb{R}$. Figure R.2 illustrates the relationship between the sets of numbers we've discussed so far. Notice how each subset appears "nested" in a larger set.

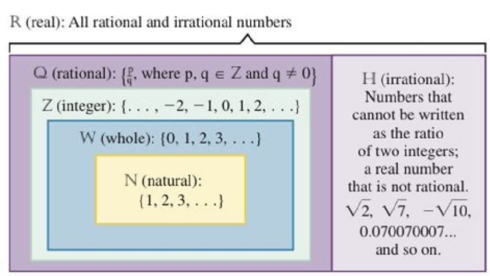


Figure R.2

EXAMPLE 5 ► Identifying Numbers

List the numbers in set $A = \{-2, 0, 5, \sqrt{7}, 12, \frac{2}{3}, 4.5, \sqrt{21}, \pi, -0.75\}$ that belong to

a. $\mathbb{Q}$ **b.** $\mathbb{H}$ **c.** $\mathbb{W}$ **d.** $\mathbb{Z}$

Solution ► **a.** $-2, 0, 5, 12, \frac{2}{3}, 4.5, -0.75 \in \mathbb{Q}$ **b.** $\sqrt{7}, \sqrt{21}, \pi \in \mathbb{H}$

c. $0, 5, 12 \in \mathbb{W}$ **d.** $-2, 0, 5, 12 \in \mathbb{Z}$

Now try Exercises 23 through 26 ►

EXAMPLE 6 ► Evaluating Statements about Sets of Numbers

Determine whether the statements are true or false.

a. $\mathbb{N} \subset \mathbb{Q}$ **b.** $\mathbb{H} \subset \mathbb{Q}$ **c.** $\mathbb{W} \subset \mathbb{Z}$ **d.** $\mathbb{Z} \subset \mathbb{R}$

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Solution ►

a. True: All natural numbers can be written as a fraction over 1.
b. False: No irrational number can be written in fraction form.
c. True: All whole numbers are integers.
d. True: Every integer is a real number.

Now try Exercises 27 through 38 ►

A. You've just reviewed sets of numbers, graphing real numbers, and set notation

B. Inequality Symbols and Order Relations

We compare numbers of different size using **inequality notation,** known as the **greater than** (>) and **less than** (<) symbols. Note that $-4 < 3$ is the same as saying -4 is to the left of 3 on the number line. In fact, on a number line, any given number is smaller than any number to the right of it (see Figure R.3).

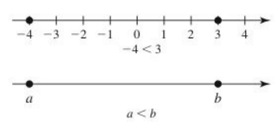

Figure R.3

Order Property of Real Numbers
Given any two real numbers a and b. **1.** $a < b$ if a is to the left of b on the number line. **2.** $a > b$ if a is to the right of b on the number line.

Inequality notation is used with numbers and variables to write mathematical statements. A **variable** is a symbol, commonly a letter of the alphabet, used to represent an unknown quantity. Over the years x, y, and n have become most common, although any letter (or symbol) can be used. Often we'll use variables that remind us of the quantities they represent, like L for length, and D for distance.

EXAMPLE 7 ► Writing Mathematical Models Using Inequalities

Use a variable and an inequality symbol to represent the statement: "To hit a home run out of Jacobi Park, the ball must travel over three hundred twenty-five feet."

Solution ► Let D represent distance: $D > 325$ ft.

Now try Exercises 39 through 42 ►

In Example 7, note the number 325 itself is not a possible value for D. If the ball traveled *exactly* 325 ft, it would hit the fence and stay in play. Numbers that mark the limit or boundary of an inequality are called **endpoints.** If the endpoint(s) are *not* included, the less than (<) or greater than (>) symbols are used. When the endpoints *are* included, the *less than or equal to symbol* (≤) or the *greater than or equal to symbol* (≥) is used. The decision to *include* or *exclude* an endpoint is often an important one, and many mathematical decisions (and real-life decisions) depend on a clear understanding of the distinction.

B. You've just reviewed inequality symbols and order relations

C. The Absolute Value of a Real Number

Any nonzero real number "n" is either a positive number or a negative number. But in some applications, our primary interest is simply the *size* of n, rather than its sign. This is called the **absolute value** of n, denoted $|n|$, and can be thought of as its *distance from*

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zero on the number line, regardless of the direction (see Figure R.4). Since distance is always positive or zero, $|n| \geq 0$.

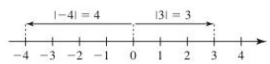

Figure R.4

EXAMPLE 8 ▶ Absolute Value Reading and Reasoning

In the table shown, the absolute value of a number is given in column 1. Complete the remaining columns.

Solution ▶

Column 1 (In Symbols)	Column 2 (Spoken)	Column 3 (Result)	Column 4 (Reason)
$\|7.5\|$	"the absolute value of seven and five-tenths"	7.5	the distance between 7.5 and 0 is 7.5 units
$\|-2\|$	"the absolute value of negative two"	2	the distance between −2 and 0 is 2 units
$-\|-6\|$	"the opposite of the absolute value of negative six"	−6	the distance between −6 and 0 is 6 units, the opposite of 6 is −6

Now try Exercises 43 through 50 ▶

Example 8 shows the absolute value of a positive number is the number itself, while the absolute value of a negative number is the *opposite of that number* (recall that $-n$ is positive if n itself is negative). For this reason the formal definition of absolute value is stated as follows.

Absolute Value

For any real number n,

$$|n| = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$$

The concept of absolute value can actually be used to find the distance between *any* two numbers on a number line. For instance, we know the distance between 2 and 8 is 6 (by counting). Using absolute values, we write $|8 - 2| = |6| = 6$, or $|2 - 8| = |-6| = 6$. Generally, if a and b are two numbers on the real number line, the distance between them is $|a - b|$ or $|b - a|$.

EXAMPLE 9 ▶ Using Absolute Value to Find the Distance between Points

Find the distance between −5 and 3 on the number line.

Solution ▶ $|-5 - 3| = |-8| = 8$ or $|3 - (-5)| = |8| = 8$.

Now try Exercises 51 through 58 ▶

☑ **C.** You've just reviewed the absolute value of a real number

D. The Order of Operations

The operations of addition, subtraction, multiplication, and division are defined for the set of real numbers, and the concept of absolute value plays an important role. Prior to our study of the order of operations, we will review fundamental concepts related to division and zero, exponential notation, and square roots/cube roots.

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Division and Zero

EXAMPLE 10 ► **Understanding Division with Zero by Writing the Related Product**

Rewrite each quotient *using the related product.*

a. $0 \div 8 = p$ **b.** $\frac{16}{0} = q$ **c.** $\frac{0}{12} = n$

Solution ► **a.** $0 \div 8 = p$, if $p \cdot 8 = 0$. This shows $p = 0$.

b. $\frac{16}{0} = q$, if $q \cdot 0 = 16$. There is no such number q.

c. $\frac{0}{12} = n$, if $n \cdot 12 = 0$. This shows $n = 0$.

Now try Exercises 59 through 62 ►

WORTHY OF NOTE

When a pizza is delivered to your home, it often has "8 parts to the whole," and in fraction form we have $\frac{8}{8}$. When all 8 pieces are eaten, 0 pieces remain and the fraction form becomes $\frac{0}{8} = 0$. However, the expression $\frac{8}{0}$ is meaningless, since it would indicate a pizza that has "0 parts to the whole (??)." The special case of $\frac{0}{0}$ is said to be indeterminate, as $\frac{0}{0} = n$ is true for all real numbers n (since the check gives $n \cdot 0 = 0$✓).

In Example 10(a), a dividend of 0 and a divisor of 8 means we are going to divide zero into eight groups. The related multiplication shows there will be zero in each group. As in Example 10(b), an expression with a divisor of 0 *cannot be computed or checked*. Although it seems trivial, division by zero has many implications in a study of mathematics, so make an effort to know the facts: The quotient of zero and any nonzero number is zero, but division *by* zero is undefined.

Division and Zero

The quotient of zero and any real number n is zero ($n \neq 0$):

$$0 \div n = 0 \qquad \frac{0}{n} = 0.$$

The expressions $n \div 0$ and $\frac{n}{0}$ are undefined.

Squares, Cubes, and Exponential Form

When a number is repeatedly multiplied by itself as in (10)(10)(10)(10), we write it using **exponential notation** as 10^4. The number used for repeated multiplication (in this case 10) is called the **base,** and the superscript number is called an **exponent.** The exponent tells how many times the base occurs as a factor, and we say 10^4 is written in **exponential form.** Numbers that result from squaring an integer are called **perfect squares,** while numbers that result from cubing an integer are called **perfect cubes.** These are often collected into a table, such as Table R.1, and memorized to help complete many common calculations mentally. Only the square and cube of selected positive integers are shown.

Table R.1

Perfect Squares				Perfect Cubes	
N	N^2	N	N^2	N	N^3
1	1	7	49	1	1
2	4	8	64	2	8
3	9	9	81	3	27
4	16	10	100	4	64
5	25	11	121	5	125
6	36	12	144	6	216

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EXAMPLE 11 ► **Evaluating Numbers in Exponential Form**

Write each exponential in expanded form, then determine its value.

a. 4^3 **b.** $(-6)^2$ **c.** -6^2 **d.** $(\frac{2}{3})^3$

Solution ► **a.** $4^3 = 4 \cdot 4 \cdot 4 = 64$ **b.** $(-6)^2 = (-6) \cdot (-6) = 36$

c. $-6^2 = -(6 \cdot 6) = -36$ **d.** $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$

Now try Exercises 63 and 64 ►

Examples 11(b) and 11(c) illustrate an important distinction. The expression $(-6)^2$ is read, "the square of negative six" and the negative sign is included in both factors. The expression -6^2 is read, "the opposite of six squared," and the square of six is calculated first, then made negative.

Square Roots and Cube Roots

For the square root operation, either the $\sqrt{\ }$ or $\sqrt[2]{\ }$ notation can be used. The $\sqrt{\ }$ symbol is called a **radical,** the number under the radical is called the **radicand,** and the small case number used is called the **index.** The index tells how many factors are needed to obtain the radicand. For example, $\sqrt{25} = 5$, since $5 \cdot 5 = 5^2 = 25$ (when the $\sqrt{\ }$ symbol is used, the index is understood to be 2). In general, $\sqrt{a} = b$ only if $b^2 = a$. All numbers greater than zero have one positive and one negative square root. The *positive* or **principal square root** of 49 is 7 ($\sqrt{49} = 7$) since $7^2 = 49$. The *negative* square root of 49 is -7 ($-\sqrt{49} = -7$). The cube root of a number has the form $\sqrt[3]{a} = b$, where $b^3 = a$. This means $\sqrt[3]{27} = 3$ since $3^3 = 27$, and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$. The cube root of a real number has one unique real value. In general, we have the following:

Square Roots	Cube Roots
$\sqrt{a} = b$ if $b^2 = a$	$\sqrt[3]{a} = b$ if $b^3 = a$
$(a \geq 0)$	$(a \in \mathbb{R})$
This indicates that	This indicates that
$\sqrt{a} \cdot \sqrt{a} = a$	$\sqrt[3]{a} \cdot \sqrt[3]{a} \cdot \sqrt[3]{a} = a$
or $(\sqrt{a})^2 = a$	or $(\sqrt[3]{a})^3 = a$

WORTHY OF NOTE

It is helpful to note that both 0 and 1 are their own square root, cube root, and *n*th root. That is, $\sqrt{0} = 0$, $\sqrt[3]{0} = 0, \ldots,$ $\sqrt[n]{0} = 0$; and $\sqrt{1} = 1$, $\sqrt[3]{1} = 1, \ldots, \sqrt[n]{1} = 1$.

EXAMPLE 12 ► **Evaluating Square Roots and Cube Roots**

Determine the value of each expression.

a. $\sqrt{49}$ **b.** $\sqrt[3]{125}$ **c.** $\sqrt{\frac{9}{16}}$ **d.** $-\sqrt{16}$ **e.** $\sqrt{-25}$

Solution ► **a.** 7 since $7 \cdot 7 = 49$ **b.** 5 since $5 \cdot 5 \cdot 5 = 125$

c. $\frac{3}{4}$ since $\frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$ **d.** -4 since $\sqrt{16} = 4$

e. not a real number since $5 \cdot 5 = (-5)(-5) = 25$

Now try Exercises 65 through 70 ►

For square roots, if the radicand is a perfect square or has perfect squares in both the numerator and denominator, the result is a rational number as in Examples 12(a) and 12(c). If the radicand is not a perfect square, the result is an irrational number. Similar statements can be made regarding cube roots [see Example 12(b)].

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WORTHY OF NOTE

Sometimes the acronym **PEMDAS** is used as a more concise way to recall the order of operations: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition, and **S**ubtraction. The idea has merit, so long as you remember that multiplication and division *have an equal rank,* as do addition and subtraction, and these must be computed in the order they occur (from left to right).

The Order of Operations

When basic operations are combined into a larger mathematical expression, we use a specified **priority** or **order of operations** to evaluate them.

The Order of Operations

1. Simplify within grouping symbols (parentheses, brackets, braces, etc.). If there are "nested" symbols of grouping, begin with the innermost group. If a fraction bar is used, simplify the numerator and denominator separately.
2. Evaluate all exponents and roots.
3. Compute all multiplications or divisions *in the order they occur from left to right.*
4. Compute all additions or subtractions *in the order they occur from left to right.*

EXAMPLE 13 ► **Evaluating Expressions Using the Order of Operations**

Simplify using the order of operations:

a. $5 + 2 \cdot 3$ **b.** $8 + 36 \div 4(12 - 3^2)$

c. $7500\left(1 + \frac{0.075}{12}\right)^{12 \cdot 15}$ **d.** $\frac{-4.5(8) - 3}{\sqrt[3]{125} + 2^3}$

Solution ►

a. $5 + 2 \cdot 3 = 5 + 6$ multiplication before addition

$= 11$ result

b. $8 + 36 \div 4(12 - 3^2)$

$= 8 + 36 \div 4(12 - 9)$ simplify within parentheses

$= 8 + 36 \div 4(3)$ $12 - 9 = 3$

$= 8 + 9(3)$ division before multiplication

$= 8 + 27$ multiply

$= 35$ result

c. $7500\left(1 + \frac{0.075}{12}\right)^{12 \cdot 15}$ original expression

$= 7500(1.00625)^{12 \cdot 15}$ simplify within the parenthesis (division before addition)

$= 7500(1.00625)^{180}$ simplify the exponent

$\approx 7500(3.069451727)$ exponents before multiplication

$\approx 23{,}020.89$ result (rounded to hundredths)

d. $\frac{-4.5(8) - 3}{\sqrt[3]{125} + 2^3}$ original expression

$= \frac{-36 - 3}{5 + 8}$ simplify terms in the numerator and denominator

$= \frac{-39}{13}$ combine terms

$= -3$ result

WORTHY OF NOTE

Many common tendencies are hard to overcome. For instance, evaluate the expressions $3 + 4 \cdot 5$ and $24 \div 6 \cdot 2$. For the first, the correct result is 23 (multiplication before addition), though some will get 35 by adding first. For the second, the correct result is 8 (multiplication or division *in order*), though some will get 2 by multiplying first.

☑ **D.** You've just reviewed the order of operations

Now try Exercises 71 through 94 ►

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R.1 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. The symbol $\subset$ means: is a ____ ____ of and the symbol $\in$ means: is an ____ of.
2. A number corresponding to a point on the number line is called the ______ of that point.
3. Every positive number has two square roots, one ______ and one ______. The two square roots of 49 are __ and __; $\sqrt{49}$ represents the ______ square root of 49.
4. The decimal form of $\sqrt{7}$ contains an infinite number of non ______ and non ______ digits. This means that $\sqrt{7}$ is a(n) ______ number.
5. Discuss/Explain why the value of $12 \cdot \frac{1}{3} + \frac{2}{3}$ is $4\frac{2}{3}$ and not 12.
6. Discuss/Explain (a) why $(-5)^2 = 25$, while $-5^2 = -25$; and (b) why $-5^3 = (-5)^3 = -125$.

▶ DEVELOPING YOUR SKILLS

7. List the natural numbers that are
   a. less than 6.
   b. less than 1.
8. List the natural numbers that are
   a. between 0 and 1.
   b. greater than 50.

Identify each of the following statements as either true or false. If false, give an example that shows why.

9. $\mathbb{N} \subset \mathbb{W}$
10. $\mathbb{W} \not\subset \mathbb{N}$
11. $\{33, 35, 37, 39\} \subset \mathbb{W}$
12. $\{2.2, 2.3, 2.4, 2.5\} \subset \mathbb{W}$
13. $6 \in \{0, 1, 2, 3, \ldots\}$
14. $1297 \notin \{0, 1, 2, 3, \ldots\}$

Convert to decimal form and graph by estimating the number's location between two integers.

15. $\frac{4}{3}$
16. $-\frac{7}{8}$
17. $2\frac{5}{9}$
18. $-1\frac{5}{6}$

Use a calculator to find the principal square root of each number (round to hundredths as needed). Then graph each number by estimating its location between two integers.

19. 7
20. $\frac{75}{4}$
21. 3
22. $\frac{25\pi}{2}$

For the sets in Exercises 23 through 26:

a. List all numbers that are elements of (i) $\mathbb{N}$, (ii) $\mathbb{W}$, (iii) $\mathbb{Z}$, (iv) $\mathbb{Q}$, (v) $\mathbb{H}$, and (vi) $\mathbb{R}$.

b. Reorder the elements of each set from smallest to largest.

c. Graph the elements of each set on a number line.

23. $\{-1, 8, 0.75, \frac{9}{2}, 5.\overline{6}, 7, \frac{3}{5}, 6\}$
24. $\{-7, 2.\overline{1}, 5.73, -3\frac{5}{6}, 0, -1.12, \frac{7}{8}\}$
25. $\{-5, \sqrt{49}, 2, -3, 6, -1, \sqrt{3}, 0, 4, \pi\}$
26. $\{-8, 5, -2\frac{3}{5}, 1.75, -\sqrt{2}, -0.6, \pi, \frac{7}{2}, \sqrt{64}\}$

State true or false. If false, state why.

27. $\mathbb{R} \subset \mathbb{H}$
28. $\mathbb{N} \subset \mathbb{R}$
29. $\mathbb{Q} \subset \mathbb{Z}$
30. $\mathbb{Z} \subset \mathbb{Q}$
31. $\sqrt{25} \in \mathbb{H}$
32. $\sqrt{19} \in \mathbb{H}$

Match each set with its correct symbol and description/illustration.

33. __ Irrational numbers	a. $\mathbb{R}$	I. $\{1, 2, 3, 4, \ldots\}$
34. __ Integers	b. $\mathbb{Q}$	II. $\{\frac{a}{b}, \mid a, b \in \mathbb{Z}; b \neq 0\}$
35. __ Real numbers	c. $\mathbb{H}$	III. $\{0, 1, 2, 3, 4, \ldots\}$
36. __ Rational numbers	d. $\mathbb{W}$	IV. $\{\pi, \sqrt{7}, -\sqrt{13}$, etc.$\}$
37. __ Whole numbers	e. $\mathbb{N}$	V. $\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
38. __ Natural numbers	f. $\mathbb{Z}$	VI. $\mathbb{N}, \mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{H}$

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Use a descriptive variable and an inequality symbol ($<, >, \le, \ge$) to write a model for each statement.

39. To spend the night at a friend's house, Kylie must be at least 6 years old.

40. Monty can spend at most \$2500 on the purchase of a used automobile.

41. If Jerod gets no more than two words incorrect on his spelling test he can play in the soccer game this weekend.

42. Andy must weigh less than 112 lb to be allowed to wrestle in his weight class at the meet.

Evaluate/simplify each expression.

43. $|-2.75|$ **44.** $|-7.24|$

45. $-|-4|$ **46.** $-|-6|$

47. $\left|\frac{1}{2}\right|$ **48.** $\left|\frac{2}{5}\right|$

49. $\left|-\frac{3}{4}\right|$ **50.** $\left|-\frac{3}{7}\right|$

Use the concept of absolute value to complete Exercises 51 to 58.

51. Write the statement two ways, then simplify. "The distance between -7.5 and 2.5 is . . ."

52. Write the statement two ways, then simplify. "The distance between $13\frac{2}{5}$ and $-2\frac{3}{5}$ is . . ."

53. What two numbers on the number line are five units from negative three?

54. What two numbers on the number line are three units from two?

55. If n is positive, then $-n$ is ______.

56. If n is negative, then $-n$ is ______.

57. If $n < 0$, then $|n| =$ ______.

58. If $n > 0$, then $|n| =$ ______.

Determine which expressions are equal to zero and which are undefined. Justify your responses by writing the related multiplication.

59. $12 \div 0$ **60.** $0 \div 12$

61. $\frac{7}{0}$ **62.** $\frac{0}{7}$

Without computing the actual answer, state whether the result will be positive or negative. Be careful to note what power is used and whether the negative sign is included in parentheses.

63. **a.** $(-7)^2$ **b.** -7^2 **c.** $(-7)^5$ **d.** -7^5

64. **a.** $(-7)^3$ **b.** -7^3 **c.** $(-7)^4$ **d.** -7^4

Evaluate without the aid of a calculator.

65. $-\sqrt{\frac{121}{36}}$ **66.** $-\sqrt{\frac{25}{49}}$

67. $\sqrt[3]{-8}$ **68.** $\sqrt[3]{-64}$

69. What perfect square is closest to 78?

70. What perfect cube is closest to -71?

Perform the operation indicated without the aid of a calculator.

71. $-24 - (-31)$ **72.** $-45 - (-54)$

73. $7.045 - 9.23$ **74.** $0.0762 - 0.9034$

75. $4\frac{5}{6} + \left(-\frac{1}{2}\right)$ **76.** $1\frac{1}{8} + \left(-\frac{3}{4}\right)$

77. $\left(-\frac{2}{3}\right)\left(3\frac{5}{8}\right)$ **78.** $(-8)\left(2\frac{1}{4}\right)$

79. $(12)(-3)(0)$ **80.** $(-1)(0)(-5)$

81. $-60 \div 12$ **82.** $75 \div (-15)$

83. $\frac{4}{5} \div (-8)$ **84.** $-15 \div \frac{1}{2}$

85. $-\frac{2}{3} \div \frac{16}{21}$ **86.** $-\frac{3}{4} \div \frac{7}{8}$

Evaluate without a calculator, using the order of operations.

87. $12 - 10 \div 2 \times 5 + (-3)^2$

88. $(5 - 2)^2 - 16 \div 4 \cdot 2 - 1$

89. $\sqrt{\frac{9}{16}} - \frac{3}{5}\cdot\left(\frac{5}{3}\right)^2$ **90.** $\left(\frac{3}{2}\right)^2 \div \left(\frac{9}{4}\right) - \sqrt{\frac{25}{64}}$

91. $\frac{4(-7) - 6^2}{6 - \sqrt{49}}$ **92.** $\frac{5(-6) - 3^2}{9 - \sqrt{64}}$

Evaluate using a calculator (round to hundredths).

93. $2475\left(1 + \frac{0.06}{4}\right)^{4\cdot 10}$

94. $5100\left(1 + \frac{0.078}{52}\right)^{52\cdot 20}$

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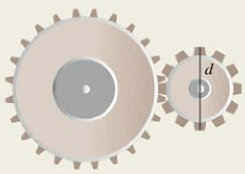
▶ WORKING WITH FORMULAS

95. Pitch diameter: $D = \frac{d \cdot n}{n + 2}$

Mesh gears are used to transfer rotary motion and power from one shaft to another. The *pitch diameter D* of a drive gear is given by the formula shown, where *d* is the outer diameter of the gear and *n* is the number of teeth on the gear. Find the pitch diameter of a gear with 12 teeth and an outer diameter of 5 cm.



96. Pediatric dosages and Clark's rule: $D_C = \frac{D_A \cdot W}{150}$

The amount of medication prescribed for young children depends on their weight, height, age, body surface area and other factors. **Clark's rule** is a formula that helps estimate the correct child's dose D_C based on the adult dose D_A and the weight W of the child (an average adult weight of 150 lb is assumed). Compute a child's dose if the adult dose is 50 mg and the child weighs 30 lb.

▶ APPLICATIONS

Use positive and negative numbers **to model the situation, then compute.**

97. Temperature changes: At 6:00 P.M., the temperature was 50°F. A cold front moves through that causes the temperature to *drop* 3°F each hour until midnight. What is the temperature at midnight?

98. Air conditioning: Most air conditioning systems are designed to create a 2° *drop* in the air temperature each hour. How long would it take to reduce the air temperature from 86° to 71°?

99. Record temperatures: The state of California holds the record for the greatest temperature swing between a record high and a record low. The record high was 134°F and the record low was −45°F. How many degrees *difference* are there between the record high and the record low?

100. Cold fronts: In Juneau, Alaska, the temperature was 17°F early one morning. A cold front later moved in and the temperature *dropped* 32°F by lunch time. What was the temperature at lunch time?

▶ EXTENDING THE CONCEPT

101. Here are some historical approximations for π. Which one is closest to the true value?

Archimedes: $3\frac{1}{7}$ Tsu Ch'ung-chih: $\frac{355}{113}$
Aryabhata: $\frac{62,832}{20,000}$ Brahmagupta: $\sqrt{10}$

102. If $A > 0$ and $B < 0$, is the product $A \cdot (-B)$ positive or negative?

103. If $A < 0$ and $B < 0$, is the quotient $-(A \div B)$ positive or negative?

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R.2 Algebraic Expressions and the Properties of Real Numbers

Learning Objectives

In Section R.2 you will review how to:

- **A.** Identify terms, coefficients, and expressions
- **B.** Create mathematical models
- **C.** Evaluate algebraic expressions
- **D.** Identify and use properties of real numbers
- **E.** Simplify algebraic expressions

To effectively use mathematics as a problem-solving tool, you must develop the ability to translate written or verbal information into a mathematical model. After obtaining a model, many applications require that you work effectively with algebraic terms and expressions. The basic ideas involved are reviewed here.

A. Terms, Coefficients, and Algebraic Expressions

An **algebraic term** is a *collection of factors* that may include numbers, variables, or expressions within parentheses. Here are some examples:

(1) 3 (2) $-6P$ (3) $5xy$ (4) $-8n^2$ (5) n (6) $2(x + 3)$

If a term consists of a single nonvariable number, it is called a **constant** term. In (1), 3 is a constant term. Any term that contains a variable is called a **variable term.** We call the constant factor of a term the **numerical coefficient** or simply the **coefficient.** The coefficients for (1), (2), (3), and (4) are 3, −6, 5, and −8, respectively. In (5), the coefficient of n is 1, since $1 \cdot n = 1n = n$. The term in (6) has two factors as written, 2 and $(x + 3)$. The coefficient is 2.

An **algebraic expression** can be a single term or a sum or difference of terms. To avoid confusion when identifying the coefficient of each term, the expression can be rewritten using algebraic addition if desired: $A - B = A + (-B)$. To identify the coefficient of a rational term, it sometimes helps to **decompose** the term, rewriting it using a unit fraction as in $\frac{n-2}{5} = \frac{1}{5}(n - 2)$ and $\frac{x}{2} = \frac{1}{2}x$.

EXAMPLE 1 ▶ Identifying Terms and Coefficients

State the number of terms in each expression as given, then identify the coefficient of each term.

a. $2x - 5y$ **b.** $\frac{x+3}{7} - 2x$ **c.** $-(x - 12)$ **d.** $-2x^2 - x + 5$

Solution ▶

Rewritten:	**a.** $2x + (-5y)$	**b.** $\frac{1}{7}(x + 3) + (-2x)$	**c.** $-1(x - 12)$	**d.** $-2x^2 + (-1x) + 5$
Number of terms:	two	two	one	three
Coefficient(s):	2 and −5	$\frac{1}{7}$ and −2	−1	−2, −1, and 5

A. You've just reviewed how to identify terms, coefficients, and expressions

Now try Exercises 7 through 14 ▶

B. Translating Written or Verbal Information into a Mathematical Model

The key to solving many applied problems is finding an algebraic expression that accurately models the situation. First, we assign a variable to represent an unknown quantity, then build related expressions using words from the English language that suggest a mathematical operation.

As mentioned earlier, variables that remind us of what they represent are often used in the modeling process. Capital letters are also used due to their widespread appearance in other fields.

EXAMPLE 2 ▶ Translating English Phrases into Algebraic Expressions

Assign a variable to the unknown number, then translate each phrase into an algebraic expression.

a. twice a number, increased by five

b. six less than three times the width

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c. ten less than triple the payment
d. two hundred fifty feet more than double the length

Solution ► **a.** Let n represent the number. Then $2n$ represents twice the number, and $2n + 5$ represents twice a number, increased by five.
b. Let W represent the width. Then $3W$ represents three times the width, and $3W - 6$ represents six less than three times the width.
c. Let p represent the payment. Then $3p$ represents a triple payment, and $3p - 10$ represents 10 less than triple the payment.
d. Let L represent the length in feet. Then $2L$ represents double the length, and $2L + 250$ represents 250 feet more than double the length.

Now try Exercises 15 through 32 ►

Identifying and translating such phrases *when they occur in context* is an important problem-solving skill. Note how this is done in Example 3.

EXAMPLE 3 ► **Creating a Mathematical Model**

The cost for a rental car is \$35 plus 15 cents per mile. Express the cost of renting a car in terms of the number of miles driven.

Solution ► Let m represent the number of miles driven. Then $0.15m$ represents the cost for each mile and $C = 35 + 0.15m$ represents the total cost for renting the car.

Now try Exercises 33 through 40 ►

☑ **B.** You've just reviewed how to create mathematical models

C. Evaluating Algebraic Expressions

We often need to **evaluate** expressions to investigate patterns and note relationships.

Evaluating a Mathematical Expression

1. Replace each variable with open parentheses ( ).
2. Substitute the given values for each variable.
3. Simplify using the order of operations.

In this evaluation, it's best to use a **vertical format,** with the original expression written first, the substitutions shown next, followed by the simplified forms and the final result. The numbers substituted or "plugged into" the expression are often called the **input values,** with the resulting values called **outputs.**

EXAMPLE 4 ► **Evaluating an Algebraic Expression**

Evaluate the expression $x^3 - 2x^2 + 5$ for $x = -3$.

Solution ► For $x = -3$:

$$\begin{aligned} x^3 - 2x^2 + 5 &= (-3)^3 - 2(-3)^2 + 5 && \text{substitute } -3 \text{ for } x \\ &= -27 - 2(9) + 5 && \text{simplify: } (-3)^3 = -27, (-3)^2 = 9 \\ &= -27 - 18 + 5 && \text{simplify: } 2(9) = 18 \\ &= -40 && \text{result} \end{aligned}$$

Now try Exercises 41 through 60 ►

If the same expression is evaluated repeatedly, results are often collected and analyzed in a table of values, as shown in Example 5. As a practical matter, the substitutions

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and simplifications are often done mentally or on scratch paper, with the table showing only the input and output values.

EXAMPLE 5 ► **Evaluating an Algebraic Expression**

Evaluate $x^2 - 2x - 3$ to complete the table shown. Which input value(s) of x cause the expression to have an output of 0?

Solution ►

Input x	Output $x^2 - 2x - 3$
-2	5
-1	0
0	-3
1	-4
2	-3
3	0
4	5

The expression has an output of 0 when $x = -1$ and $x = 3$.

Now try Exercises 61 through 66 ►

WORTHY OF NOTE

In Example 4, note the importance of the first step in the evaluation process: *replace each variable with open parentheses.* Skipping this step could easily lead to confusion as we try to evaluate the squared term, since $-3^2 = -9$, while $(-3)^2 = 9$. Also see **Exercises 55 and 56.**

C. You've just reviewed how to evaluate algebraic expressions

For exercises that combine the skills from Examples 3 through 5, **see Exercises 91 to 98.**

D. Properties of Real Numbers

While the phrase, "an unknown number times five," is accurately modeled by the expression $n5$ for some number n, in algebra we prefer to have numerical coefficients precede variable factors. When we reorder the factors as $5n$, we are using the **commutative property of multiplication.** A reordering of terms involves the **commutative property of addition.**

The Commutative Properties

Given that a and b represent real numbers:

ADDITION: $a + b = b + a$	MULTIPLICATION: $a \cdot b = b \cdot a$
Terms can be combined in any order without changing the sum.	Factors can be multiplied in any order without changing the product.

Each property can be extended to include any number of terms or factors. While the commutative property implies a *reordering* or *movement* of terms (to commute implies back-and-forth movement), the **associative property** implies a *regrouping* or reassociation of terms. For example, the sum $\left(\frac{3}{4} + \frac{3}{5}\right) + \frac{2}{5}$ is easier to compute if we regroup the addends as $\frac{3}{4} + \left(\frac{3}{5} + \frac{2}{5}\right)$. This illustrates the **associative property of addition.** Multiplication is also associative.

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The Associative Properties

Given that a, b, and c represent real numbers:

ADDITION: $(a + b) + c = a + (b + c)$ Terms can be regrouped.

MULTIPLICATION: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ Factors can be regrouped.

EXAMPLE 6 ► Simplifying Expressions Using Properties of Real Numbers

Use the commutative and associative properties to simplify each calculation.

a. $\frac{3}{8} - 19 + \frac{5}{8}$ **b.** $[-2.5 \cdot (-1.2)] \cdot 10$

Solution ► **a.**

$\frac{3}{8} - 19 + \frac{5}{8} = -19 + \frac{3}{8} + \frac{5}{8}$ commutative property

$= -19 + \left(\frac{3}{8} + \frac{5}{8}\right)$ associative property

$= -19 + 1$ simplify

$= -18$ result

b.

$[-2.5 \cdot (-1.2)] \cdot 10 = -2.5 \cdot [(-1.2) \cdot 10]$ associative property

$= -2.5 \cdot (-12)$ simplify

$= 30$ result

Now try Exercises 67 and 68 ►

WORTHY OF NOTE

Is subtraction commutative? Consider a situation involving money. If you had \$100, you could easily buy an item costing \$20: \$100 − \$20 leaves you with \$80. But if you had \$20, could you buy an item costing \$100? Obviously \$100 − \$20 is not the same as \$20 − \$100. Subtraction is *not* commutative. Likewise, 100 ÷ 20 is not the same as 20 ÷ 100, and division is *not* commutative.

For any real number x, $x + 0 = x$ and 0 is called the **additive identity** since the original number was returned or "identified." Similarly, 1 is called the **multiplicative identity** since $1 \cdot x = x$. The identity properties are used extensively in the process of solving equations.

The Additive and Multiplicative Identities

Given that x is a real number,

$x + 0 = x$ Zero is the identity for addition.

$1 \cdot x = x$ One is the identity for multiplication.

For any real number x, there is a real number $-x$ such that $x + (-x) = 0$. The number $-x$ is called the **additive inverse** of x, since their sum results in the additive identity. Similarly, the **multiplicative inverse** of any nonzero number x is $\frac{1}{x}$, since $x \cdot \frac{1}{x} = 1$ (the multiplicative identity). This property can also be stated as $\frac{p}{q} \cdot \frac{q}{p} = 1$ $(p, q \neq 0)$ for any rational number $\frac{p}{q}$. Note that $\frac{p}{q}$ and $\frac{q}{p}$ are **reciprocals.**

The Additive and Multiplicative Inverses

Given that p, q, and x represent real numbers $(p, q \neq 0)$:

$x + (-x) = 0$ x and $-x$ are additive inverses.

$\frac{p}{q} \cdot \frac{q}{p} = 1$ $\frac{p}{q}$ and $\frac{q}{p}$ are multiplicative inverses.

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EXAMPLE 7 ► **Determining Additive and Multiplicative Inverses**

Replace the box to create a true statement:

a. $\square \cdot \frac{-3}{5}x = 1 \cdot x$ **b.** $x + 4.7 + \square = x$

Solution ► **a.** $\square = \frac{5}{-3}$, since $\frac{5}{-3} \cdot \frac{-3}{5} = 1$

b. $\square = -4.7$, since $4.7 + (-4.7) = 0$

Now try Exercises 69 and 70 ►

The **distributive property of multiplication over addition** is widely used in a study of algebra, because it enables us to rewrite a product as an equivalent sum and vice versa.

The Distributive Property of Multiplication over Addition

Given that a, b, and c represent real numbers:

$a(b + c) = ab + ac$	$ab + ac = a(b + c)$
A factor outside a sum can be distributed to each addend in the sum.	A factor common to each addend in a sum can be "undistributed" and written outside a group.

EXAMPLE 8 ► **Simplifying Expressions Using the Distributive Property**

Apply the distributive property as appropriate. Simplify if possible.

a. $7(p + 5.2)$ **b.** $-(2.5 - x)$ **c.** $7x^3 - x^3$ **d.** $\frac{5}{2}n + \frac{1}{2}n$

Solution ►

a. $7(p + 5.2) = 7p + 7(5.2)$
$= 7p + 36.4$

b. $-(2.5 - x) = -1(2.5 - x)$
$= -1(2.5) - (-1)(x)$
$= -2.5 + x$

c. $7x^3 - x^3 = 7x^3 - 1x^3$
$= (7 - 1)x^3$
$= 6x^3$

d. $\frac{5}{2}n + \frac{1}{2}n = \left(\frac{5}{2} + \frac{1}{2}\right)n$
$= \left(\frac{6}{2}\right)n$
$= 3n$

Now try Exercises 71 through 78 ►

WORTHY OF NOTE

From Example 8(b) we learn that a negative sign outside a group changes the sign of all terms within the group: $-(2.5 - x) = -2.5 + x$.

D. You've just reviewed how to identify and use properties of real numbers

E. Simplifying Algebraic Expressions

Two terms are **like terms** only if they have the *same variable factors* (the coefficient is not used to identify like terms). For instance, $3x^2$ and $-\frac{1}{7}x^2$ are like terms, while $5x^3$ and $5x^2$ are not. We simplify expressions by **combining like terms** using the distributive property, along with the commutative and associative properties. Many times the distributive property is used to eliminate grouping symbols *and* combine like terms within the same expression.

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EXAMPLE 9 ▶ Simplifying an Algebraic Expression

Simplify the expression completely: $7(2p^2 + 1) - (p^2 + 3)$.

Solution ▶

$7(2p^2 + 1) - 1(p^2 + 3)$	original expression; note coefficient of -1
$= 14p^2 + 7 - 1p^2 - 3$	distributive property
$= (14p^2 - 1p^2) + (7 - 3)$	commutative and associative properties (collect like terms)
$= (14 - 1)p^2 + 4$	distributive property
$= 13p^2 + 4$	result

Now try Exercises 79 through 88 ▶

The steps for simplifying an algebraic expression are summarized here:

To Simplify an Expression

1. Eliminate parentheses by applying the distributive property.
2. Use the commutative and associative properties to group like terms.
3. Use the distributive property to combine like terms.

E. You've just reviewed how to simplify algebraic expressions

As you practice with these ideas, many of the steps will become more automatic. At some point, the distributive property, the commutative and associative properties, as well as the use of algebraic addition will all be performed mentally.

R.2 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. A term consisting of a single number is called a(n) ________ term.
2. A term containing a variable is called a(n) ________ term.
3. The constant factor in a variable term is called the ________.
4. When $3 \cdot 14 \cdot \frac{2}{3}$ is written as $3 \cdot \frac{2}{3} \cdot 14$, the ________ property has been used.
5. Discuss/Explain why the additive inverse of -5 is 5, while the multiplicative inverse of -5 is $-\frac{1}{5}$.
6. Discuss/Explain how we can rewrite the sum $3x + 6y$ as a product, and the product $2(x + 7)$ as a sum.

▶ DEVELOPING YOUR SKILLS

Identify the number of terms in each expression and the coefficient of each term.

7. $3x - 5y$
8. $-2a - 3b$
9. $2x + \dfrac{x + 3}{4}$
10. $\dfrac{n - 5}{3} + 7n$
11. $-2x^2 + x - 5$
12. $3n^2 + n - 7$
13. $-(x + 5)$
14. $-(n - 3)$

Translate each phrase into an algebraic expression.

15. seven fewer than a number
16. x decreased by six
17. the sum of a number and four
18. a number increased by nine
19. the difference between a number and five is squared

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20. the sum of a number and two is cubed

21. thirteen less than twice a number

22. five less than double a number

23. a number squared plus the number doubled

24. a number cubed less the number tripled

25. five fewer than two-thirds of a number

26. fourteen more than one-half of a number

27. three times the sum of a number and five, decreased by seven

28. five times the difference of a number and two, increased by six

Create a mathematical model using descriptive variables.

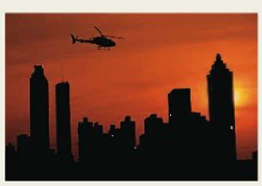
29. The length of the rectangle is three meters less than twice the width.

30. The height of the triangle is six centimeters less than three times the base.

31. The speed of the car was fifteen miles per hour more than the speed of the bus.

32. It took Romulus three minutes more time than Remus to finish the race.

33. Hovering altitude: The helicopter was hovering 150 ft above the top of the building. Express the altitude of the helicopter in terms of the building's height.



34. Stacks on a cruise liner: The smoke stacks of the luxury liner cleared the bridge by 25 ft as it passed beneath it. Express the height of the stacks in terms of the bridge's height.

35. Dimensions of a city park: The length of a rectangular city park is 20 m more than twice its width. Express the length of the park in terms of the width.

36. Dimensions of a parking lot: In order to meet the city code while using the available space, a contractor planned to construct a parking lot with a length that was 50 ft less than three times its width. Express the length of the lot in terms of the width.

37. Cost of milk: In 2008, a gallon of milk cost two and one-half times what it did in 1990. Express the cost of a gallon of milk in 2008 in terms of the 1990 cost.

38. Cost of gas: In 2008, a gallon of gasoline cost one and one-half times what it did in 1990. Express the cost of a gallon of gas in 2008 in terms of the 1990 cost.

39. Pest control: In her pest control business, Judy charges $50 per call plus $12.50 per gallon of insecticide for the control of spiders and other insects. Express the total charge in terms of the number of gallons of insecticide used.

40. Computer repairs: As his reputation and referral business grew, Keith began to charge $75 per service call plus an hourly rate of $50 for the repair and maintenance of home computers. Express the cost of a service call in terms of the number of hours spent on the call.

Evaluate each algebraic expression given $x = 2$ and $y = -3$.

41. $4x - 2y$

42. $5x - 3y$

43. $-2x^2 + 3y^2$

44. $-5x^2 + 4y^2$

45. $2y^2 + 5y - 3$

46. $3x^2 + 2x - 5$

47. $-2(3y + 1)$

48. $-3(2y + 5)$

49. $3x^2y$

50. $6xy^2$

51. $(-3x)^2 - 4xy - y^2$

52. $(-2x)^2 - 5xy - y^2$

53. $\frac{1}{2}x - \frac{1}{3}y$

54. $\frac{2}{3}x - \frac{1}{2}y$

55. $(3x - 2y)^2$

56. $(2x - 3y)^2$

57. $\dfrac{-12y + 5}{-3x + 1}$

58. $\dfrac{12x + (-3)}{-3y + 1}$

59. $\sqrt{-12y} \cdot 4$

60. $7 \cdot \sqrt{-27y}$

Evaluate each expression for integers from -3 to 3 inclusive. What input(s) give an output of zero?

61. $x^2 - 3x - 4$

62. $x^2 - 2x - 3$

63. $-3(1 - x) - 6$

64. $5(3 - x) - 10$

65. $x^3 - 6x + 4$

66. $x^3 + 5x + 18$

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Rewrite each expression using the given property and simplify if possible.

67. Commutative property of addition

a. $-5 + 7$ **b.** $-2 + n$

c. $-4.2 + a + 13.6$ **d.** $7 + x - 7$

68. Associative property of multiplication

a. $2 \cdot (3 \cdot 6)$ **b.** $3 \cdot (4 \cdot b)$

c. $-1.5 \cdot (6 \cdot a)$ **d.** $-6 \cdot (-\frac{5}{6} \cdot x)$

Replace the box so that a true statement results.

69. a. $x + (-3.2) + \square = x$

b. $n - \frac{5}{6} + \square = n$

70. a. $\square \cdot \frac{2}{3}x = 1x$

b. $\square \cdot \frac{n}{-3} = 1n$

Simplify by removing all grouping symbols (as needed) and combining like terms.

71. $-5(x - 2.6)$

72. $-12(v - 3.2)$

73. $\frac{2}{3}(-\frac{1}{5}p + 9)$

74. $\frac{5}{6}(-\frac{2}{15}q + 24)$

75. $3a + (-5a)$

76. $13m + (-5m)$

77. $\frac{2}{3}x + \frac{3}{4}x$

78. $\frac{5}{12}y - \frac{3}{8}y$

79. $3(a^2 + 3a) - (5a^2 + 7a)$

80. $2(b^2 + 5b) - (6b^2 + 9b)$

81. $x^2 - (3x - 5x^2)$

82. $n^2 - (5n - 4n^2)$

83. $(3a + 2b - 5c) - (a - b - 7c)$

84. $(x - 4y + 8z) - (8x - 5y - 2z)$

85. $\frac{3}{5}(5n - 4) + \frac{5}{8}(n + 16)$

86. $\frac{2}{3}(2x - 9) + \frac{3}{4}(x + 12)$

87. $(3a^2 - 5a + 7) + 2(2a^2 - 4a - 6)$

88. $2(3m^2 + 2m - 7) - (m^2 - 5m + 4)$

▶ WORKING WITH FORMULAS

89. Electrical resistance: $R = \frac{kL}{d^2}$

The electrical resistance in a wire depends on the length and diameter of the wire. This resistance can be modeled by the formula shown, where R is the resistance in ohms, L is the length in feet, and d is the diameter of the wire in inches. Find the resistance if $k = 0.000025$, $d = 0.015$ in., and $L = 90$ ft.

90. Volume and pressure: $P = \frac{k}{V}$

If temperature remains constant, the pressure of a gas held in a closed container is related to the volume of gas by the formula shown, where P is the pressure in pounds per square inch, V is the volume of gas in cubic inches, and k is a constant that depends on given conditions. Find the pressure exerted by the gas if $k = 440{,}310$ and $V = 22{,}580\ \text{in}^3$.

▶ APPLICATIONS

Translate each key phrase into an algebraic expression, then evaluate as indicated.

91. Cruising speed: A turbo-prop airliner has a cruising speed that is one-half the cruising speed of a 767 jet aircraft. (a) Express the speed of the turbo-prop in terms of the speed of the jet, and (b) determine the speed of the airliner if the cruising speed of the jet is 550 mph.

92. Softball toss: Macklyn can throw a softball two-thirds as far as her father. (a) Express the distance that Macklyn can throw a softball in terms of the distance her father can throw. (b) If her father can throw the ball 210 ft, how far can Macklyn throw the ball?

93. Dimensions of a lawn: The length of a rectangular lawn is 3 ft more than twice its width. (a) Express the length of the lawn in terms of the width. (b) If the width is 52 ft, what is the length?

94. Pitch of a roof: To obtain the proper pitch, the crossbeam for a roof truss must be 2 ft less than three-halves the rafter. (a) Express the length of the crossbeam in terms of the rafter. (b) If the rafter is 18 ft, how long is the crossbeam?

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95. Postage costs: In 2004, a first class stamp cost 22¢ more than it did in 1978. Express the cost of a 2004 stamp in terms of the 1978 cost. If a stamp cost 15¢ in 1978, what was the cost in 2004?

96. Minimum wage: In 2004, the federal minimum wage was \$2.85 per hour more than it was in 1976. Express the 2004 wage in terms of the 1976 wage. If the hourly wage in 1976 was \$2.30, what was it in 2004?

97. Repair costs: The TV repairman charges a flat fee of \$43.50 to come to your house and \$25 per hour for labor. Express the cost of repairing a TV in terms of the time it takes to repair it. If the repair took 1.5 hr, what was the total cost?

98. Repair costs: At the local car dealership, shop charges are \$79.50 to diagnose the problem and \$85 per shop hour for labor. Express the cost of a repair in terms of the labor involved. If a repair takes 3.5 hr, how much will it cost?

▶ EXTENDING THE CONCEPT

99. If C must be a positive odd integer and D must be a negative even integer, then $C^2 + D^2$ must be a:

a. positive odd integer.
b. positive even integer.
c. negative odd integer.
d. negative even integer.
e. Cannot be determined.

100. Historically, several attempts have been made to create metric time using factors of 10, but our current system won out. If 1 day was 10 metric hours, 1 metric hour was 10 metric minutes, and 1 metric minute was 10 metric seconds, what time would it really be if a metric clock read 4:3:5? Assume that each new day starts at midnight.

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R.3 Exponents, Scientific Notation, and a Review of Polynomials

Learning Objectives

In Section R.3 you will review how to:

- ❑ **A.** Apply properties of exponents
- ❑ **B.** Perform operations in scientific notation
- ❑ **C.** Identify and classify polynomial expressions
- ❑ **D.** Add and subtract polynomials
- ❑ **E.** Compute the product of two polynomials
- ❑ **F.** Compute special products: binomial conjugates and binomial squares

In this section, we review basic exponential properties and operations on polynomials. Although there are five to eight exponential properties (depending on how you count them), all can be traced back to the basic definition involving repeated multiplication.

A. The Properties of Exponents

As noted in Section R.1, an exponent tells how many times the base occurs as a factor. For $b \cdot b \cdot b = b^3$, we say b^3 is written in *exponential form*. In some cases, we may refer to b^3 as an **exponential term.**

Exponential Notation

For any positive integer n,

$$b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} \quad \text{and} \quad \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ times}} = b^n$$

The Product and Power Properties

There are two properties that follow immediately from this definition. When b^3 is multiplied by b^2, we have an uninterrupted string of five factors: $b^3 \cdot b^2 = (b \cdot b \cdot b) \cdot (b \cdot b)$, which can be written as b^5. This is an example of the **product property of exponents.**

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WORTHY OF NOTE

In this statement of the product property and the exponential properties that follow, it is assumed that for any expression of the form 0^m, $m > 0$ hence $0^m = 0$.

Product Property Of Exponents

For any base b and positive integers m and n:

$$b^m \cdot b^n = b^{m+n}$$

In words, the property says, *to multiply exponential terms with the* **same base,** *keep the common base and add the exponents*. A special application of the product property uses repeated factors of the *same* exponential term, as in $(x^2)^3$. Using the product property, we have $(x^2)(x^2)(x^2) = x^6$. Notice the same result can be found more quickly by multiplying the inner exponent by the outer exponent: $(x^2)^3 = x^{2 \cdot 3} = x^6$. We generalize this idea to state the **power property of exponents.** In words the property says, *to raise an exponential term to a power, keep the same base and multiply the exponents.*

Power Property of Exponents

For any base b and positive integers m and n:

$$(b^m)^n = b^{m \cdot n}$$

EXAMPLE 1 ► **Multiplying Terms Using Exponential Properties**

Compute each product.

a. $-4x^3 \cdot \frac{1}{2}x^2$ **b.** $(p^3)^2 \cdot (p^4)^2$

Solution ►

a. $-4x^3 \cdot \frac{1}{2}x^2 = (-4 \cdot \frac{1}{2})(x^3 \cdot x^2)$ commutative and associative properties

$= (-2)(x^{3+2})$ product property; simplify

$= -2x^5$ result

b. $(p^3)^2 \cdot (p^4)^2 = p^6 \cdot p^8$ power property

$= p^{6+8}$ product property

$= p^{14}$ result

Now try Exercises 7 through 12 ►

The power property can easily be extended to include more than one factor within the parentheses. This application of the power property is sometimes called the **product to a power property.** We can also raise a quotient of exponential terms to a power. The result is called the **quotient to a power property,** and can be extended to include any number of factors. In words the properties say, to raise a product or quotient of exponential terms to a power, *multiply every exponent inside* the parentheses *by the exponent outside* the parentheses.

Product to a Power Property

For any bases a and b, and positive integers m, n, and p:

$$(a^m b^n)^p = a^{mp} \cdot b^{np}$$

Quotient to a Power Property

For any bases a and $b \neq 0$, and positive integers m, n, and p:

$$\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$$

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EXAMPLE 2 ► Simplifying Terms Using the Power Properties

Simplify using the power property (if possible):

a. $(-3a)^2$ **b.** $-3a^2$ **c.** $\left(\frac{-5a^3}{2b}\right)^2$

Solution ► **a.** $(-3a)^2 = (-3)^2 \cdot (a^1)^2 = 9a^2$ **b.** $-3a^2$ is in simplified form

c. $\left(\frac{-5a^3}{2b}\right)^2 = \frac{(-5)^2(a^3)^2}{2^2b^2} = \frac{25a^6}{4b^2}$

Now try Exercises 13 through 24 ►

WORTHY OF NOTE

Regarding Examples 2(a) and 2(b), note the difference between the expressions $(-3a)^2 = (-3 \cdot a)^2$ and $-3a^2 = -3 \cdot a^2$. In the first, the exponent acts on both the negative 3 *and* the *a*; in the second, the exponent acts on only the *a* and there is no "product to a power."

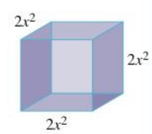
Applications of exponents sometimes involve linking one exponential term with another using a substitution. The result is then simplified using exponential properties.

EXAMPLE 3 ► Applying the Power Property after a Substitution

The formula for the volume of a cube is $V = S^3$, where S is the length of one edge. If the length of each edge is $2x^2$:

a. Find a formula for volume in terms of x.

b. Find the volume if $x = 2$.



Solution ► **a.** $V = S^3$ ($S = 2x^2$)

$= (2x^2)^3$

$= 8x^6$

b. For $V = 8x^6$,

$V = 8(2)^6$ substitute 2 for x

$= 8 \cdot 64$ or 512 $(2)^6 = 64$

The volume of the cube would be 512 units^3.

Now try Exercises 25 and 26 ►

The Quotient Property of Exponents

By combining exponential notation and the property $\frac{x}{x} = 1$ for $x \neq 0$, we note a pattern that helps to simplify a *quotient* of exponential terms. For $\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$ or x^3, the exponent of the final result appears to be the *difference between the exponent in the numerator and the exponent in the denominator*. This seems reasonable since the subtraction would indicate a removal of the factors that reduce to 1. Regardless of how many factors are used, we can generalize the idea and state the **quotient property of exponents.** In words the property says, to divide two exponential terms with the same base, *keep the common base and subtract the exponent of the denominator* from *the exponent of the numerator*.

Quotient Property of Exponents

For any base $b \neq 0$ and positive integers m and n: $\frac{b^m}{b^n} = b^{m-n}$

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Zero and Negative Numbers as Exponents

If the exponent of the denominator is *greater* than the exponent in the numerator, the quotient property yields a negative exponent: $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$. To help understand what a negative exponent *means,* let's look at the expanded form of the expression: $\frac{x^2}{x^5} = \frac{x \cdot x^1}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^3}$. A negative exponent can literally be interpreted as "write the factors as a reciprocal." A good way to remember this is

$$2^{-3} \quad \text{three factors of 2 written as a reciprocal} \quad \frac{2^{-3}}{1} = \frac{1}{2^3} = \frac{1}{8}$$

Since the result would be similar regardless of the base used, we can generalize this idea and state the **property of negative exponents.**

WORTHY OF NOTE

The use of zero as an exponent should not strike you as strange or odd; it's simply a way of saying that *no factors of the base remain,* since all terms have been reduced to 1. For $\frac{2^3}{2^3}$, we have $\frac{8}{8} = 1$, or $\frac{\overset{1}{2} \cdot \overset{1}{2} \cdot \overset{1}{2}}{2 \cdot 2 \cdot 2} = 1$, or $2^{3-3} = 2^0 = 1$.

Property of Negative Exponents

For any base $b \neq 0$ and integer n:

$$\frac{b^{-n}}{1} = \frac{1}{b^n} \qquad \frac{1}{b^{-n}} = \frac{b^n}{1} \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n;\ a \neq 0$$

Finally, when we consider that $\frac{x^3}{x^3} = 1$ by division, and $\frac{x^3}{x^3} = x^{3-3} = x^0$ using the quotient property, we conclude that $x^0 = 1$ as long as $x \neq 0$. We can also generalize this observation and state the meaning of zero as an exponent. In words the property says, *any nonzero quantity raised to an exponent of zero is equal to 1.*

Zero Exponent Property

For any base $b \neq 0$: $b^0 = 1$

EXAMPLE 4 ► Simplifying Expressions Using Exponential Properties

Simplify using exponential properties. Answer using positive exponents only.

a. $\left(\frac{2a^3}{b^2}\right)^{-2}$ **b.** $(3hk^{-2})^3(6h^{-2}k^{-3})^{-2}$

c. $(3x)^0 + 3x^0 + 3^{-2}$ **d.** $\frac{(-2m^2n^3)^5}{(4mn^2)^3}$

Solution ► **a.**

$$\begin{aligned} \left(\frac{2a^3}{b^2}\right)^{-2} &= \left(\frac{b^2}{2a^3}\right)^2 && \text{property of negative exponents} \\ &= \frac{(b^2)^2}{2^2(a^3)^2} && \text{power property} \\ &= \frac{b^4}{4a^6} && \text{result} \end{aligned}$$

b.

$$\begin{aligned} (3hk^{-2})^3(6h^{-2}k^{-3})^{-2} &= (3^3h^3k^{-6})(6^{-2}h^4k^6) && \text{power property} \\ &= 3^3 \cdot 6^{-2} \cdot h^{3+4} \cdot k^{-6+6} && \text{product property} \\ &= \frac{27h^7k^0}{36} && \text{simplify } \left(6^{-2} = \frac{1}{6^2} = \frac{1}{36}\right) \\ &= \frac{3h^7}{4} && \text{result } (k^0 = 1) \end{aligned}$$

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WORTHY OF NOTE

Notice in Example 4(c), we have $(3x)^0 = (3 \cdot x)^0 = 1$, while $3x^0 = 3 \cdot x^0 = 3(1)$. This is another example of operations and grouping symbols working together: $(3x)^0 = 1$ because any *quantity* to the zero power is 1. However, for $3x^0$ there are no grouping symbols, so the exponent 0 acts only on the x and not the 3.

c. $(3x)^0 + 3x^0 + 3^{-2} = 1 + 3(1) + \frac{1}{3^2}$ zero exponent property; property of negative exponents

$= 4 + \frac{1}{9}$ simplify

$= 4\frac{1}{9} = \frac{37}{9}$ result

d. $\frac{(-2m^2n^3)^5}{(4mn^2)^3} = \frac{(-2)^5(m^2)^5(n^3)^5}{4^3m^3(n^2)^3}$ power property

$= \frac{-32m^{10}n^{15}}{64m^3n^6}$ simplify

$= \frac{-1m^7n^9}{2}$ quotient property

$= -\frac{m^7n^9}{2}$ result

Now try Exercises 27 through 66 ►

Summary of Exponential Properties

For real numbers a and b, and integers m, n, and p (excluding 0 raised to a nonpositive power)

Product property: $b^m \cdot b^n = b^{m+n}$

Power property: $(b^m)^n = b^{m \cdot n}$

Product to a power: $(a^m b^n)^p = a^{mp} \cdot b^{np}$

Quotient to a power: $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}\ (b \neq 0)$

Quotient property: $\frac{b^m}{b^n} = b^{m-n}\ (b \neq 0)$

Zero exponents: $b^0 = 1\ (b \neq 0)$

Negative exponents: $\frac{b^{-n}}{1} = \frac{1}{b^n}$, $\frac{1}{b^{-n}} = b^n$, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n\ (a, b \neq 0)$

☑ **A. You've just reviewed how to apply properties of exponents**

B. Exponents and Scientific Notation

In many technical and scientific applications, we encounter numbers that are either extremely large or very, very small. For example, the mass of the moon is over 73 quintillion kilograms (73 followed by 18 zeroes), while the constant for universal gravitation contains 10 zeroes before the first nonzero digit. When computing with numbers of this size, scientific notation has a distinct advantage over the common decimal notation (base-10 place values).

WORTHY OF NOTE

Recall that multiplying by 10's (or multiplying by 10^k, $k > 0$) shifts the decimal to the right k places, making the number larger. Dividing by 10's (or multiplying by 10^{-k}, $k > 0$) shifts the decimal to the left k places, making the number smaller.

Scientific Notation

A non-zero number written in scientific notation has the form

$$N \times 10^k$$

where $1 \leq |N| < 10$ and k is an integer.

To convert a number from decimal notation into scientific notation, we begin by placing the decimal point to the immediate right of the first nonzero digit (creating a number less than 10 but greater than or equal to 1) and multiplying by 10^k. Then we

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determine the power of 10 (the value of k) needed to ensure that the two forms are equivalent. When writing large or small numbers in scientific notation, we sometimes round the value of N to two or three decimal places.

EXAMPLE 5 ▶ Converting from Decimal Notation to Scientific Notation

The mass of the moon is about 73,000,000,000,000,000,000 kg. Write this number in scientific notation.

Solution ▶ Place decimal to the right of first nonzero digit (7) and multiply by 10^k.

$$73{,}000{,}000{,}000{,}000{,}000{,}000 = 7.3 \times 10^k$$

To return the decimal to its original position would require 19 shifts to the *right,* so k must be *positive* 19.

$$73{,}000{,}000{,}000{,}000{,}000{,}000 = 7.3 \times 10^{19}$$

The mass of the moon is 7.3×10^{19} kg.

Now try Exercises 67 and 68 ▶

Converting a number from scientific notation to decimal notation is simply an application of multiplication or division with powers of 10.

EXAMPLE 6 ▶ Converting from Scientific Notation to Decimal Notation

The constant of gravitation is 6.67×10^{-11}. Write this number in common decimal form.

Solution ▶ Since the exponent is *negative* 11, shift the decimal 11 *places to the left,* using placeholder zeroes as needed to return the decimal to its original position:

$$6.67 \times 10^{-11} = 0.000\,000\,000\,066\,7$$

Now try Exercises 69 through 72 ▶

☑ **B.** You've just reviewed how to perform operations in scientific notation

C. Identifying and Classifying Polynomial Expressions

A **monomial** is a term using *only whole number exponents* on variables, with no variables in the denominator. One important characteristic of a monomial is its **degree.** For a monomial in one variable, the degree is the same as the exponent *on the variable*. The degree of a monomial in two or more variables is the sum of exponents occurring on variable factors. A **polynomial** is a monomial or any sum or difference of monomial terms. For instance, $\frac{1}{2}x^2 - 5x + 6$ is a polynomial, while $3n^{-2} + 2n - 7$ is not (the exponent -2 is not a whole number). Identifying polynomials is an important skill because they represent a very different kind of real-world model than nonpolynomials. In addition, there are different **families of polynomials,** with each family having different characteristics. We classify polynomials according to their *degree* and *number of terms*. The **degree of a polynomial** in one variable is the largest exponent occurring on the variable. The degree of a polynomial in more than one variable is the largest sum of exponents in any one term. A polynomial with two terms is called a **binomial** (*bi* means two) and a polynomial with three terms is called a **trinomial** (*tri* means three). There are special names for polynomials with four or more terms, but for these, we simply use the general name *polynomial* (*poly* means many).

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EXAMPLE 7 ▶ Classifying and Describing Polynomials

For each expression:

a. Classify as a monomial, binomial, trinomial, or polynomial.
b. State the degree of the polynomial.
c. Name the coefficient of each term.

Solution ▶

Expression	Classification	Degree	Coefficients
$5x^2y - 2xy$	binomial	three	$5, -2$
$x^2 - 0.81$	binomial	two	$1, -0.81$
$z^3 - 3z^2 + 9z - 27$	polynomial (four terms)	three	$1, -3, 9, -27$
$\frac{-3}{4}x + 5$	binomial	one	$\frac{-3}{4}, 5$
$2x^2 + x - 3$	trinomial	two	$2, 1, -3$

Now try Exercises 73 through 78 ▶

A polynomial expression is in **standard form** when the terms of the polynomial are written in *descending order of degree,* beginning with the highest-degree term. The coefficient of the highest-degree term is called the **leading coefficient.**

EXAMPLE 8 ▶ Writing Polynomials in Standard Form

Write each polynomial in standard form, then identify the leading coefficient.

Solution ▶

Polynomial	Standard Form	Leading Coefficient
$9 - x^2$	$-x^2 + 9$	-1
$5z + 7z^2 + 3z^3 - 27$	$3z^3 + 7z^2 + 5z - 27$	3
$2 + (\frac{-3}{4})x$	$\frac{-3}{4}x + 2$	$\frac{-3}{4}$
$-3 + 2x^2 + x$	$2x^2 + x - 3$	2

✓ **C.** You've just reviewed how to identify and classify polynomial expressions

Now try Exercises 79 through 84 ▶

D. Adding and Subtracting Polynomials

Adding polynomials simply involves using the distributive, commutative, and associative properties to combine like terms (at this point, the properties are usually applied mentally). As with real numbers, the subtraction of polynomials involves adding the opposite of the second polynomial using algebraic addition. This can be viewed as distributing -1 to the second polynomial and combining like terms.

EXAMPLE 9 ▶ Adding and Subtracting Polynomials

Perform the indicated operations:

$(0.7n^3 + 4n^2 + 8) + (0.5n^3 - n^2 - 6n) - (3n^2 + 7n - 10).$

Solution ▶

$0.7n^3 + 4n^2 + 8 + 0.5n^3 - n^2 - 6n - 3n^2 - 7n + 10$ — eliminate parentheses (distributive property)

$= 0.7n^3 + 0.5n^3 + 4n^2 - 1n^2 - 3n^2 - 6n - 7n + 8 + 10$ — use real number properties to collect like terms

$= 1.2n^3 - 13n + 18$ — combine like terms

Now try Exercises 85 through 90 ▶

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Sometimes it's easier to add or subtract polynomials using a vertical format and aligning like terms. Note the use of a placeholder zero in Example 10.

EXAMPLE 10 ► **Subtracting Polynomials Using a Vertical Format**

Compute the difference of $x^3 - 5x + 9$ and $x^3 + 3x^2 + 2x - 8$ using a vertical format.

Solution ►

$$\begin{array}{r} x^3 + \mathbf{0}x^2 - 5x + 9 \\ \underline{-(x^3 + 3x^2 + 2x - 8)} \end{array} \longrightarrow \begin{array}{r} x^3 + \mathbf{0}x^2 - 5x + 9 \\ \underline{-x^3 - 3x^2 - 2x + 8} \\ -3x^2 - 7x + 17 \end{array}$$

The difference is $-3x^2 - 7x + 17$.

D. You've just reviewed how to add and subtract polynomials

Now try Exercises 91 and 92 ►

E. The Product of Two Polynomials

Monomial Times Monomial

The simplest case of polynomial multiplication is the product of monomials shown in Example 1(a). These were computed using exponential properties and the properties of real numbers.

Monomial Times Polynomial

To compute the product of a monomial and a polynomial, we use the distributive property.

EXAMPLE 11 ► **Multiplying a Monomial by a Polynomial**

Find the product: $-2a^2(a^2 - 2a + 1)$.

Solution ►

$$\begin{aligned} -2a^2(a^2 - 2a + 1) &= -2a^2(a^2) - (-2a^2)(2a^1) + (-2a^2)(1) && \text{distribute} \\ &= -2a^4 + 4a^3 - 2a^2 && \text{simplify} \end{aligned}$$

Now try Exercises 93 and 94 ►

Binomial Times Polynomial

For products involving binomials, we still use a version of the distributive property—this time to distribute one polynomial to each term of the other polynomial factor. Note the distribution can be performed either from the left or from the right.

EXAMPLE 12 ► **Multiplying a Binomial by a Polynomial**

Multiply as indicated:

a. $(2z + 1)(z - 2)$ **b.** $(2v - 3)(4v^2 + 6v + 9)$

Solution ►

a.
$$\begin{aligned} (2z + 1)(z - 2) &= 2z(z - 2) + 1(z - 2) && \text{distribute to every term in the first binomial} \\ &= 2z^2 - 4z + 1z - 2 && \text{eliminate parentheses (distribute again)} \\ &= 2z^2 - 3z - 2 && \text{simplify} \end{aligned}$$

b.
$$\begin{aligned} (2v - 3)(4v^2 + 6v + 9) &= 2v(4v^2 + 6v + 9) - 3(4v^2 + 6v + 9) && \text{distribute} \\ &= 8v^3 + 12v^2 + 18v - 12v^2 - 18v - 27 && \text{simplify} \\ &= 8v^3 - 27 && \text{combine like terms} \end{aligned}$$

Now try Exercises 95 through 100 ►

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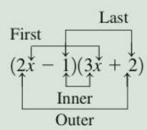
The F-O-I-L Method

By observing the product of two binomials in Example 12(a), we note a pattern that can make the process more efficient. We illustrate here using the product $(2x - 1)(3x + 2)$.

The F-O-I-L Method for Multiplying Binomials

The product of two binomials can quickly be computed by multiplying:



$6x^2 + 4x - 3x - 2$

First Outer Inner Last

and combining like terms

$6x^2 + x - 2$

The first term of the result will always be the product of the first terms from each binomial, and the last term of the result is the product of their last terms. We also note that here, the middle term is found by adding the *outermost product* with the *innermost product*. As you practice with the F-O-I-L process, much of the work can be done mentally and you can often compute the entire product without writing anything down except the answer.

EXAMPLE 13 ▶ Multiplying Binomials Using F-O-I-L

Compute each product mentally:

a. $(5n - 1)(n + 2)$

b. $(2b + 3)(5b - 6)$

✓ **E.** You've just reviewed how to compute the product of two polynomials

Solution ▶

a. $(5n - 1)(n + 2)$: $5n^2 + 9n - 2$ [$10n + (-1n) = 9n$]

product of first two terms; sum of outer and inner; product of last two terms

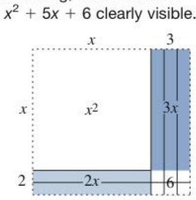
b. $(2b + 3)(5b - 6)$: $10b^2 + 3b - 18$ [$-12b + 15b = 3b$]

product of first two terms; sum of outer and inner; product of last two terms

Now try Exercises 101 through 116 ▶

WORTHY OF NOTE

Consider the product $(x + 3)(x + 2)$ in the context of *area*. If we view $x + 3$ as the length of a rectangle (an unknown length plus 3 units), and $x + 2$ as its width (the same unknown length plus 2 units), a diagram of the total area would look like the following, with the result $x^2 + 5x + 6$ clearly visible.



$(x + 3)(x + 2) = x^2 + 5x + 6$

F. Special Polynomial Products

Certain polynomial products are considered "special" for two reasons: (1) the product follows a predictable pattern, and (2) the result can be used to simplify expressions, graph functions, solve equations, and/or develop other skills.

Binomial Conjugates

Expressions like $x + 7$ and $x - 7$ are called **binomial conjugates.** For any given binomial, its conjugate is found by using the same two terms with the opposite sign between

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them. Example 14 shows that when we multiply a binomial and its conjugate, the "outers" and "inners" sum to zero and the result is a **difference of two squares.**

EXAMPLE 14 ▶ Multiplying Binomial Conjugates

Compute each product mentally:

a. $(x + 7)(x - 7)$ **b.** $(2x - 5y)(2x + 5y)$ **c.** $\left(x + \frac{2}{5}\right)\left(x - \frac{2}{5}\right)$

Solution ▶

$-7x + 7x = 0x$

a. $(x + 7)(x - 7) = x^2 - 49$ difference of squares $(x)^2 - (7)^2$

$10xy + (-10xy) = 0xy$

b. $(2x - 5y)(2x + 5y) = 4x^2 - 25y^2$ difference of squares: $(2x)^2 - (5y)^2$

$-\frac{2}{5}x + \frac{2}{5}x = 0$

c. $\left(x + \frac{2}{5}\right)\left(x - \frac{2}{5}\right) = x^2 - \frac{4}{25}$ difference of squares: $x^2 - \left(\frac{2}{5}\right)^2$

Now try Exercises 117 through 124 ▶

The Product of a Binomial and Its Conjugate

Given any expression that can be written in the form $A + B$, the conjugate of the expression is $A - B$ and their product is a difference of two squares:

$$(A + B)(A - B) = A^2 - B^2$$

Binomial Squares

Expressions like $(x + 7)^2$ are called **binomial squares** and are useful for solving many equations and sketching a number of basic graphs. Note $(x + 7)^2 = (x + 7)(x + 7) = x^2 + 14x + 49$ using the F-O-I-L process. The expression $x^2 + 14x + 49$ is called a **perfect square trinomial** because it is the result of expanding a binomial square. If we write a binomial square in the more general form $(A + B)^2 = (A + B)(A + B)$ and compute the product, we notice a pattern that helps us write the expanded form more quickly.

$$\begin{aligned}(A + B)^2 &= (A + B)(A + B) && \text{repeated multiplication}\\ &= A^2 + AB + AB + B^2 && \text{F-O-I-L}\\ &= A^2 + 2AB + B^2 && \text{simplify (perfect square trinomial)}\end{aligned}$$

The first and last terms of the trinomial are squares of the terms A and B. Also, the middle term of the trinomial is *twice the product of these two terms:* $AB + AB = 2AB$. The F-O-I-L process shows us why. Since the outer and inner products are identical, we always end up with two. A similar result holds for $(A - B)^2$ and the process can be summarized for both cases using the $\pm$ symbol.

LOOKING AHEAD

Although a binomial square can always be found using repeated factors and F-O-I-L, learning to expand them using the pattern is a valuable skill. Binomial squares occur often in a study of algebra and it helps to find the expanded form quickly.

The Square of a Binomial

Given any expression that can be written in the form $(A \pm B)^2$,

1. $(A + B)^2 = A^2 + 2AB + B^2$

2. $(A - B)^2 = A^2 - 2AB + B^2$

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CAUTION ► Note the square of a binomial always results in a trinomial (three terms). Specifically $(A + B)^2 \neq A^2 + B^2$.

EXAMPLE 15 ► Find each binomial square without using F-O-I-L:

a. $(a + 9)^2$ **b.** $(3x - 5)^2$ **c.** $(3 + \sqrt{x})^2$

Solution ►

a. $(a + 9)^2 = a^2 + 2(a \cdot 9) + 9^2$ — $(A + B)^2 = A^2 + 2AB + B^2$
$= a^2 + 18a + 81$ — simplify

b. $(3x - 5)^2 = (3x)^2 - 2(3x \cdot 5) + 5^2$ — $(A - B)^2 = A^2 - 2AB + B^2$
$= 9x^2 - 30x + 25$ — simplify

c. $(3 + \sqrt{x})^2 = 9 + 2(3 \cdot \sqrt{x}) + x$ — $(A + B)^2 = A^2 + 2AB + B^2$
$= 9 + 6\sqrt{x} + x$ — simplify

Now try Exercises 125 through 136 ►

F. You've just reviewed how to compute special products: binomial conjugates and binomial squares

With practice, you will be able to go directly from the binomial square to the resulting trinomial.

R.3 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. The equation $(3x^2)^3 = 27x^6$ is an example of the ________ property of exponents.
2. The equation $(2x^3)^{-2} = \dfrac{1}{4x^6}$ is an example of the property of ________ exponents.
3. The sum of the "outers" and "inners" for $(2x + 5)^2$ is ________, while the sum of outers and inners for $(2x + 5)(2x - 5)$ is ________.
4. The expression $2x^2 - 3x - 10$ can be classified as a ________ of degree ________, with a leading coefficient of ________.
5. Discuss/Explain why one of the following expressions can be simplified further, while the other cannot: (a) $-7n^4 + 3n^2$; (b) $-7n^4 \cdot 3n^2$.
6. Discuss/Explain why the degree of $2x^2y^3$ is greater than the degree of $2x^2 + y^3$. Include additional examples for contrast and comparison.

► DEVELOPING YOUR SKILLS

Determine each product using the product and/or power properties.

7. $\dfrac{2}{3}n^2 \cdot 21n^5$
8. $24g^5 \cdot \dfrac{3}{8}g^9$
9. $(-6p^2q)(2p^3q^3)$
10. $(-1.5vy^2)(-8v^4y)$
11. $(a^2)^4 \cdot (a^3)^2 \cdot b^2 \cdot b^5$
12. $d^2 \cdot d^4 \cdot (c^5)^2 \cdot (c^3)^2$

Simplify each expression using the product to a power property.

13. $(6pq^2)^3$
14. $(-3p^2q)^2$
15. $(3.2hk^2)^3$
16. $(-2.5h^5k)^2$
17. $\left(\dfrac{p}{2q}\right)^2$
18. $\left(\dfrac{b}{3a}\right)^3$

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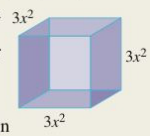
19. $(-0.7c^4)^2(10c^3d^2)^2$ **20.** $(-2.5a^3)^2(3a^2b^2)^3$

21. $\left(\frac{3}{4}x^3y\right)^2$ **22.** $\left(\frac{4}{5}x^3\right)^2$

23. $\left(-\frac{3}{8}x\right)^2\left(16xy^2\right)$ **24.** $\left(\frac{2}{3}m^2n\right)^2 \cdot \left(\frac{1}{2}mn^2\right)$

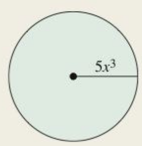
25. Volume of a cube: The formula for the volume of a cube is $V = S^3$, where S is the length of one edge. If the length of each edge is $3x^2$,



a. Find a formula for volume in terms of the variable x.

b. Find the volume of the cube if $x = 2$.

26. Area of a circle: The formula for the area of a circle is $A = \pi r^2$, where r is the length of the radius. If the radius is given as $5x^3$,



a. Find a formula for area in terms of the variable x.

b. Find the area of the circle if $x = 2$.

Simplify using the quotient property or the property of negative exponents. Write answers using positive exponents only.

27. $\frac{-6w^5}{-2w^2}$ **28.** $\frac{8z^7}{16z^5}$

29. $\frac{-12a^3b^5}{4a^2b^4}$ **30.** $\frac{5m^3n^5}{10mn^2}$

31. $\left(\frac{2}{3}\right)^{-3}$ **32.** $\left(\frac{5}{6}\right)^{-1}$

33. $\frac{2}{h^{-3}}$ **34.** $\frac{3}{m^{-2}}$

35. $(-2)^{-3}$ **36.** $(-4)^{-2}$

37. $\left(\frac{-1}{2}\right)^{-3}$ **38.** $\left(\frac{-2}{3}\right)^{-2}$

Simplify each expression using the quotient to a power property.

39. $\left(\frac{2p^4}{q^3}\right)^2$ **40.** $\left(\frac{-5v^4}{7w^3}\right)^2$

41. $\left(\frac{0.2x^2}{0.3y^3}\right)^3$ **42.** $\left(\frac{-0.5a^3}{0.4b^2}\right)^2$

43. $\left(\frac{5m^2n^3}{2r^4}\right)^2$ **44.** $\left(\frac{4p^3}{3x^2y}\right)^3$

45. $\left(\frac{5p^2q^3r^4}{-2pq^2r^4}\right)^2$ **46.** $\left(\frac{9p^3q^2r^3}{12p^5qr^2}\right)^3$

Use properties of exponents to simplify the following. Write the answer using positive exponents only.

47. $\frac{9p^6q^4}{-12p^4q^6}$ **48.** $\frac{5m^5n^2}{10m^5n}$

49. $\frac{20h^{-2}}{12h^5}$ **50.** $\frac{5k^3}{20k^{-2}}$

51. $\frac{(a^2)^3}{a^4 \cdot a^5}$ **52.** $\frac{(5^3)^4}{5^9}$

53. $\left(\frac{a^{-3} \cdot b}{c^{-2}}\right)^{-4}$ **54.** $\frac{(p^{-4}q^8)^2}{p^5q^{-2}}$

55. $\frac{-6(2x^{-3})^2}{10x^{-2}}$ **56.** $\frac{18n^{-3}}{-8(3n^{-2})^3}$

57. $\frac{14a^{-3}bc^0}{-7(3a^2b^{-2}c)^3}$ **58.** $\frac{-3(2x^3y^{-4}z)^2}{18x^{-2}yz^0}$

59. $4^0 + 5^0$ **60.** $(-3)^0 + (-7)^0$

61. $2^{-1} + 5^{-1}$ **62.** $4^{-1} + 8^{-1}$

63. $3^0 + 3^{-1} + 3^{-2}$ **64.** $2^{-2} + 2^{-1} + 2^0$

65. $-5x^0 + (-5x)^0$ **66.** $-2n^0 + (-2n)^0$

Convert the following numbers to scientific notation.

67. In mid-2007, the U.S. Census Bureau estimated the world population at nearly 6,600,000,000 people.

68. The mass of a proton is generally given as 0.000 000 000 000 000 000 000 000 001 670 kg.

Convert the following numbers to decimal notation.

69. As of 2006, the smallest microprocessors in common use measured 6.5×10^{-9} m across.

70. In 2007, the estimated net worth of Bill Gates, the founder of Microsoft, was 5.6×10^{10} dollars.

Compute using scientific notation. Show all work.

71. The average distance between the Earth and the planet Jupiter is 465,000,000 mi. How many hours would it take a satellite to reach the planet if it traveled an average speed of 17,500 mi per hour? How many days? Round to the nearest whole.

72. In fiscal terms, a nation's debt-per-capita is the ratio of its total debt to its total population. In the year 2007, the total U.S. debt was estimated at $9,010,000,000,000, while the population was estimated at 303,000,000. What was the U.S. debt-per-capita ratio for 2007? Round to the nearest whole dollar.

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Identify each expression as a polynomial or nonpolynomial (if a nonpolynomial, state why); classify each as a monomial, binomial, trinomial, or none of these; and state the degree of the polynomial.

73. $-35w^3 + 2w^2 + (-12w) + 14$

74. $-2x^3 + \frac{2}{3}x^2 - 12x + 1.2$

75. $5n^{-2} + 4n + \sqrt{17}$

76. $\frac{4}{r^3} + 2.7r^2 + r + 1$

77. $p^3 - \frac{2}{5}$

78. $q^3 + 2q^{-2} - 5q$

Write the polynomial in standard form and name the leading coefficient.

79. $7w + 8.2 - w^3 - 3w^2$

80. $-2k^2 - 12 - k$

81. $c^3 + 6 + 2c^2 - 3c$

82. $-3v^3 + 14 + 2v^2 + (-12v)$

83. $12 - \frac{2}{3}x^2$

84. $8 + 2n^2 + 7n$

Find the indicated sum or difference.

85. $(3p^3 - 4p^2 + 2p - 7) + (p^2 - 2p - 5)$

86. $(5q^2 - 3q + 4) + (-3q^2 + 3q - 4)$

87. $(5.75b^2 + 2.6b - 1.9) + (2.1b^2 - 3.2b)$

88. $(0.4n^2 + 5n - 0.5) + (0.3n^2 - 2n + 0.75)$

89. $(\frac{3}{4}x^2 - 5x + 2) - (\frac{1}{2}x^2 + 3x - 4)$

90. $(\frac{5}{9}n^2 + 4n - \frac{1}{2}) - (\frac{2}{3}n^2 - 2n + \frac{3}{4})$

91. Subtract $q^5 + 2q^4 + q^2 + 2q$ from $q^6 + 2q^5 + q^4 + 2q^3$ using a vertical format.

92. Find $x^4 + 2x^3 + x^2 + 2x$ decreased by $x^4 - 3x^3 + 4x^2 - 3x$ using a vertical format.

Compute each product.

93. $-3x(x^2 - x - 6)$

94. $-2v^2(v^2 + 2v - 15)$

95. $(3r - 5)(r - 2)$

96. $(s - 3)(5s + 4)$

97. $(x - 3)(x^2 + 3x + 9)$

98. $(z + 5)(z^2 - 5z + 25)$

99. $(b^2 - 3b - 28)(b + 2)$

100. $(2h^2 - 3h + 8)(h - 1)$

101. $(7v - 4)(3v - 5)$

102. $(6w - 1)(2w + 5)$

103. $(3 - m)(3 + m)$

104. $(5 + n)(5 - n)$

105. $(p - 2.5)(p + 3.6)$

106. $(q - 4.9)(q + 1.2)$

107. $(x + \frac{1}{2})(x + \frac{1}{4})$

108. $(z + \frac{1}{3})(z + \frac{5}{6})$

109. $(m + \frac{3}{4})(m - \frac{3}{4})$

110. $(n - \frac{2}{5})(n + \frac{2}{5})$

111. $(3x - 2y)(2x + 5y)$

112. $(6a + b)(a + 3b)$

113. $(4c + d)(3c + 5d)$

114. $(5x + 3y)(2x - 3y)$

115. $(2x^2 + 5)(x^2 - 3)$

116. $(3y^2 - 2)(2y^2 + 1)$

For each binomial, determine its conjugate and then find the product of the binomial with its conjugate.

117. $4m - 3$

118. $6n + 5$

119. $7x - 10$

120. $c + 3$

121. $6 + 5k$

122. $11 - 3r$

123. $x + \sqrt{6}$

124. $p - \sqrt{2}$

Find each binomial square.

125. $(x + 4)^2$

126. $(a - 3)^2$

127. $(4g + 3)^2$

128. $(5x - 3)^2$

129. $(4p - 3q)^2$

130. $(5c + 6d)^2$

131. $(4 - \sqrt{x})^2$

132. $(\sqrt{x} + 7)^2$

Compute each product.

133. $(x - 3)(y + 2)$

134. $(a + 3)(b - 5)$

135. $(k - 5)(k + 6)(k + 2)$

136. $(a + 6)(a - 1)(a + 5)$

▶ WORKING WITH FORMULAS

137. Medication in the bloodstream: $M = 0.5t^4 + 3t^3 - 97t^2 + 348t$

If 400 mg of a pain medication are taken orally, the number of milligrams in the bloodstream is modeled by the formula shown, where M is the number of milligrams and t is the time in hours, $0 \le t < 5$. Construct a table of values for $t = 1$ through 5, then answer the following.

a. How many milligrams are in the bloodstream after 2 hr? After 3 hr?

b. Based on parts a and b, would you expect the number of milligrams in the bloodstream after 4 hr to be less or more than in part b? Why?

c. Approximately how many hours until the medication wears off (the number of milligrams in the bloodstream is 0)?

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138. Amount of a mortgage payment:

$$M = \frac{A\left(\frac{r}{12}\right)\left(1 + \frac{r}{12}\right)^n}{\left(1 + \frac{r}{12}\right)^n - 1}$$

The monthly mortgage payment required to pay off (or amortize) a loan is given by the formula shown, where M is the monthly payment, A is the original amount of the loan, r is the annual interest rate, and n is the term of the loan in months. Find the monthly payment (to the nearest cent) required to purchase a \$198,000 home, if the interest rate is 6.5% and the home is financed over 30 yr.

▶ APPLICATIONS

139. Attraction between particles: In electrical theory, the force of attraction between two particles P and Q with opposite charges is modeled by $F = \frac{kPQ}{d^2}$, where d is the distance between them and k is a constant that depends on certain conditions. This is known as Coulomb's law. Rewrite the formula using a negative exponent.

140. Intensity of light: The intensity of illumination from a light source depends on the distance from the source according to $I = \frac{k}{d^2}$, where I is the intensity measured in footcandles, d is the distance from the source in feet, and k is a constant that depends on the conditions. Rewrite the formula using a negative exponent.

141. Rewriting an expression: In advanced mathematics, negative exponents are widely used because they are easier to work with than rational expressions. Rewrite the expression $\frac{5}{x^3} + \frac{3}{x^2} + \frac{2}{x^1} + 4$ using negative exponents.

142. Swimming pool hours: A swimming pool opens at 8 A.M. and closes at 6 P.M. In summertime, the

number of people in the pool at any time can be approximated by the formula $S(t) = -t^2 + 10t$, where S is the number of swimmers and t is the number of hours the pool has been open (8 A.M.: $t = 0$, 9 A.M.: $t = 1$, 10 A.M.: $t = 2$, etc.).

a. How many swimmers are in the pool at 6 P.M.? Why?

b. Between what times would you expect the largest number of swimmers?

c. Approximately how many swimmers are in the pool at 3 P.M.?

d. Create a table of values for $t = 1, 2, 3, 4, \ldots$ and check your answer to part b.

143. Maximizing revenue: A sporting goods store finds that if they price their video games at \$20, they make 200 sales per day. For each decrease of \$1, 20 additional video games are sold. This means the store's revenue can be modeled by the formula $R = (20 - 1x)(200 + 20x)$, where x is the number of \$1 decreases. Multiply out the binomials and use a table of values to determine what price will give the most revenue.

144. Maximizing revenue: Due to past experience, a jeweler knows that if they price jade rings at \$60, they will sell 120 each day. For each decrease of \$2, five additional sales will be made. This means the jeweler's revenue can be modeled by the formula $R = (60 - 2x)(120 + 5x)$, where x is the number of \$2 decreases. Multiply out the binomials and use a table of values to determine what price will give the most revenue.

▶ EXTENDING THE CONCEPT

145. If $(3x^2 + kx + 1) - (kx^2 + 5x - 7) + (2x^2 - 4x - k) = -x^2 - 3x + 2$, what is the value of k?

146. If $\left(2x + \frac{1}{2x}\right)^2 = 5$, then the expression $4x^2 + \frac{1}{4x^2}$ is equal to what number?

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R.4 Factoring Polynomials

Learning Objectives

In Section R.4 you will review:

- ☐ **A.** Factoring out the greatest common factor
- ☐ **B.** Common binomial factors and factoring by grouping
- ☐ **C.** Factoring quadratic polynomials
- ☐ **D.** Factoring special forms and quadratic forms

It is often said that knowing which tool to use is just as important as knowing how to use the tool. In this section, we review the tools needed to factor an expression, an important part of solving polynomial equations. This section will also help us decide which factoring tool is appropriate when many different factorable expressions are presented.

A. The Greatest Common Factor

To **factor** an expression means to *rewrite the expression as an equivalent product*. The distributive property is an example of factoring in action. To factor $2x^2 + 6x$, we might first rewrite each term using the common factor $2x$: $2x^2 + 6x = 2x \cdot x + 2x \cdot 3$, then apply the distributive property to obtain $2x(x + 3)$. We commonly say that we have *factored out 2x*. The **greatest common factor** (or GCF) is the largest factor common to *all* terms in the polynomial.

EXAMPLE 1 ► **Factoring Polynomials**

Factor each polynomial:

a. $12x^2 + 18xy - 30y$ **b.** $x^5 + x^2$

Solution ► **a.** 6 is common to all three terms:

$$12x^2 + 18xy - 30y \quad \text{mentally: } 6 \cdot 2x^2 + 6 \cdot 3xy - 6 \cdot 5y$$
$$= 6(2x^2 + 3xy - 5y)$$

b. x^2 is common to both terms:

$$x^5 + x^2 \quad \text{mentally: } x^2 \cdot x^3 + x^2 \cdot 1$$
$$= x^2(x^3 + 1)$$

Now try Exercises 7 and 8 ►

☑ **A.** You've just reviewed how to factor out the greatest common factor

B. Common Binomial Factors and Factoring by Grouping

If the terms of a polynomial have a **common *binomial* factor,** it can also be factored out using the distributive property.

EXAMPLE 2 ► **Factoring Out a Common Binomial Factor**

Factor:

a. $(x + 3)x^2 + (x + 3)5$ **b.** $x^2(x - 2) - 3(x - 2)$

Solution ► **a.** $(x + 3)x^2 + (x + 3)5$
$= (x + 3)(x^2 + 5)$

b. $x^2(x - 2) - 3(x - 2)$
$= (x - 2)(x^2 - 3)$

Now try Exercises 9 and 10 ►

One application of removing a binomial factor involves **factoring by grouping.** At first glance, the expression $x^3 + 2x^2 + 3x + 6$ appears unfactorable. But by grouping the terms (applying the associative property), we can remove a monomial factor from each subgroup, which then reveals a common binomial factor.

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EXAMPLE 3 ► **Factoring by Grouping**

Factor $3t^3 + 15t^2 - 6t - 30$.

Solution ► Notice that all four terms have a common factor of 3. Begin by factoring it out.

$$3t^3 + 15t^2 - 6t - 30 \quad \text{original polynomial}$$
$$= 3(t^3 + 5t^2 - 2t - 10) \quad \text{factor out 3}$$
$$= 3(\underline{t^3 + 5t^2} - \underline{2t - 10}) \quad \text{group remaining terms}$$
$$= 3[t^2(t + 5) - 2(t + 5)] \quad \text{factor common } \textit{monomial}$$
$$= 3(t + 5)(t^2 - 2) \quad \text{factor common } \textit{binomial}$$

Now try Exercises 11 and 12 ►

When asked to factor an expression, first look for common factors. The resulting expression will be easier to work with and help ensure the final answer is written in **completely factored form.** If a four-term polynomial cannot be factored as written, try rearranging the terms to find a combination that enables factoring by grouping.

☑ **B.** You've just reviewed how to factor by grouping

C. Factoring Quadratic Polynomials

A quadratic polynomial is one that can be written in the form $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$. One common form of factoring involves quadratic trinomials such as $x^2 + 7x + 10$ and $2x^2 - 13x + 15$. While we know $(x + 5)(x + 2) = x^2 + 7x + 10$ and $(2x - 3)(x - 5) = 2x^2 - 13x + 15$ using F-O-I-L, how can we factor these trinomials without seeing the original problem in advance? First, it helps to place the trinomials in two families—those with a leading coefficient of 1 and those with a leading coefficient other than 1.

WORTHY OF NOTE

Similarly, a cubic polynomial is one of the form $ax^3 + bx^2 + cx + d$. It's helpful to note that a cubic polynomial can be factored by grouping only when $ad = bc$, where a, b, c, and d are the coefficients shown. This is easily seen in Example 3, where $(3)(-30) = (15)(-6)$ gives $-90 = -90$✓.

$ax^2 + bx + c$, where $a = 1$

When $a = 1$, the only factor pair for x^2 (other than $1 \cdot x^2$) is $x \cdot x$ and the first term in each binomial will be x: $(x \quad)(x \quad)$. The following observation helps guide us to the complete factorization. Consider the product $(x + b)(x + a)$:

$$(x + b)(x + a) = x^2 + ax + bx + ab \quad \text{F-O-I-L}$$
$$= x^2 + (a + b)x + ab \quad \text{distributive property}$$

Note the last term is the product ab (the *lasts*), while the coefficient of the middle term is $a + b$ (the sum of the *outers* and *inners*). Since the last term of $x^2 - 8x + 7$ is 7 and the coefficient of the middle term is -8, we are seeking two numbers with a product of positive 7 and a sum of negative 8. The numbers are -7 and -1, so the factored form is $(x - 7)(x - 1)$. It is also helpful to note that if the constant term is positive, the binomials will have *like* signs, since only *the product of like signs is positive*. If the constant term is negative, the binomials will have *unlike* signs, since only *the product of unlike signs is negative*. This means we can use the sign of the linear term (the term with degree 1) to guide our choice of factors.

Factoring Trinomials with a Leading Coefficient of 1

If the constant term is positive, the binomials will have *like* signs:

$$(x + \quad)(x + \quad) \text{ or } (x - \quad)(x - \quad),$$

to match the sign of the linear (middle) term.

If the constant term is negative, the binomials will have *unlike* signs:

$$(x + \quad)(x - \quad),$$

with the larger factor placed in the binomial whose sign *matches* the linear (middle) term.

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EXAMPLE 4 ► Factoring Trinomials

Factor these expressions:

a. $-x^2 + 11x - 24$ **b.** $x^2 - 10 - 3x$

Solution ► **a.** First rewrite the trinomial in standard form as $-1(x^2 - 11x + 24)$. For $x^2 - 11x + 24$, the constant term is positive so the binomials will have like signs. Since the linear term is negative,

$$-1(x^2 - 11x + 24) = -1(x - \quad)(x - \quad) \quad \text{like signs, both negative}$$
$$= -1(x - 8)(x - 3) \quad (-8)(-3) = 24; -8 + (-3) = -11$$

b. First rewrite the trinomial in standard form as $x^2 - 3x - 10$. The constant term is negative so the binomials will have unlike signs. Since the linear term is negative,

$$x^2 - 3x - 10 = (x + \quad)(x - \quad) \quad \text{unlike signs, one positive and one negative}$$
$$= (x + 2)(x - 5) \quad 5 > 2\text{, 5 is placed in the second binomial; } (2)(-5) = -10; 2 + (-5) = -3$$

Now try Exercises 13 and 14 ►

Sometimes we encounter **prime polynomials,** or polynomials that cannot be factored. For $x^2 + 9x + 15$, the factor pairs of 15 are $1 \cdot 15$ and $3 \cdot 5$, with neither pair having a sum of +9. We conclude that $x^2 + 9x + 15$ is prime.

$ax^2 + bx + c$, where $a \neq 1$

If the leading coefficient is not one, the possible combinations of outers and inners are more numerous. Furthermore, their sum will change depending on the position of the possible factors. Note that $(2x + 3)(x + 9) = 2x^2 + 21x + 27$ and $(2x + 9)(x + 3) = 2x^2 + 15x + 27$ result in a different middle term, even though identical numbers were used.

To factor $2x^2 - 13x + 15$, note the constant term is positive so the binomials *must have like signs*. The negative linear term indicates these signs will be negative. We then list possible factors for the first and last terms of each binomial, then sum the outer and inner products.

Possible First and Last Terms for $2x^2$ and 15	Sum of Outers and Inners
1. $(2x - 1)(x - 15)$	$-30x - 1x = -31x$
2. $(2x - 15)(x - 1)$	$-2x - 15x = -17x$
3. $(2x - 3)(x - 5)$	$-10x - 3x = -13x$ ←
4. $(2x - 5)(x - 3)$	$-6x - 5x = -11x$

As you can see, only possibility 3 yields a linear term of $-13x$, and the correct factorization is then $(2x - 3)(x - 5)$. With practice, this **trial-and-error** process can be completed very quickly.

If the constant term is negative, the number of possibilities can be reduced by finding a factor pair with a sum *or* difference equal to the *absolute value* of the linear coefficient, as we can then arrange the sign of each binomial to obtain the needed result (see Example 5).

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EXAMPLE 5 ► **Factoring a Trinomial Using Trial and Error**

Factor $6z^2 - 11z - 35$.

Solution ► Note the constant term is negative (binomials will have unlike signs), $|-11| = 11$, and the factors of 35 are $1 \cdot 35$ and $5 \cdot 7$. Two possible first terms are: $(6z \quad)(z \quad)$ and $(3z \quad)(2z \quad)$, and we begin with 5 and 7 as factors of 35.

$(6z \quad)(z \quad)$	Outers/Inners		$(3z \quad)(2z \quad)$	Outers/Inners	
	Sum	Diff		Sum	Diff
1. $(6z \quad 5)(z \quad 7)$	$47z$	$37z$	3. $(3z \quad 5)(2z \quad 7)$	$31z$	$11z$ ←
2. $(6z \quad 7)(z \quad 5)$	$37z$	$23z$	4. $(3z \quad 7)(2z \quad 5)$	$29z$	$1z$

Since possibility 3 yields the linear term of $11z$, we need not consider other factors of 35 and write the factored form as $6z^2 - 11z - 35 = (3z \quad 5)(2z \quad 7)$. The signs can then be arranged to obtain a middle term of $-11z$: $(3z + 5)(2z - 7)$, $-21z + 10z = -11z$ ✓.

Now try Exercises 15 and 16 ►

☑ **C. You've just reviewed how to factor quadratic polynomials**

WORTHY OF NOTE

In an attempt to factor a *sum* of two perfect squares, say $v^2 + 49$, let's list all possible binomial factors. These are (1) $(v + 7)(v + 7)$, (2) $(v - 7)(v - 7)$, and (3) $(v + 7)(v - 7)$. Note that (1) and (2) are the binomial squares $(v + 7)^2$ and $(v - 7)^2$, with each product resulting in a "middle" term, whereas (3) is a binomial times its conjugate, resulting in a *difference* of squares: $v^2 - 49$. With all possibilities exhausted, we conclude that *the sum of two squares is prime!*

D. Factoring Special Forms and Quadratic Forms

Next we consider methods to factor each of the special products we encountered in Section R.3.

The Difference of Two Squares

Multiplying and factoring are inverse processes. Since $(x - 7)(x + 7) = x^2 - 49$, we know that $x^2 - 49 = (x - 7)(x + 7)$. In words, *the difference of two squares will factor into a binomial and its conjugate.* To find the terms of the factored form, rewrite each term in the original expression as a square: $(\quad)^2$.

Factoring the Difference of Two Perfect Squares

Given any expression that can be written in the form $A^2 - B^2$,

$$A^2 - B^2 = (A + B)(A - B)$$

Note that the *sum* of two perfect squares $A^2 + B^2$ *cannot be factored* using real numbers (the expression is prime). As a reminder, always check for a common factor first and be sure to write all results in completely factored form. See Example 6(c).

EXAMPLE 6 ► **Factoring the Difference of Two Perfect Squares**

Factor each expression completely.

a. $4w^2 - 81$ **b.** $v^2 + 49$ **c.** $-3n^2 + 48$ **d.** $z^4 - \frac{1}{81}$ **e.** $x^2 - 7$

Solution ►

a. $4w^2 - 81 = (2w)^2 - 9^2$ write as a difference of squares

$= (2w + 9)(2w - 9)$ $A^2 - B^2 = (A + B)(A - B)$

b. $v^2 + 49$ is prime.

c. $-3n^2 + 48 = -3(n^2 - 16)$ factor out -3

$= -3[n^2 - (4)^2]$ write as a difference of squares

$= -3(n + 4)(n - 4)$ $A^2 - B^2 = (A + B)(A - B)$

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d. $z^4 - \frac{1}{81} = (z^2)^2 - (\frac{1}{9})^2$ — write as a difference of squares

$= (z^2 + \frac{1}{9})(z^2 - \frac{1}{9})$ — $A^2 - B^2 = (A + B)(A - B)$

$= [z^2 - (\frac{1}{3})^2](z^2 + \frac{1}{9})$ — write as a difference of squares

$= (z + \frac{1}{3})(z - \frac{1}{3})(z^2 + \frac{1}{9})$ — result

e. $x^2 - 7 = (x)^2 - (\sqrt{7})^2$ — write as a difference of squares

$= (x + \sqrt{7})(x - \sqrt{7})$ — $A^2 - B^2 = (A + B)(A - B)$

Now try Exercises 17 and 18 ►

Perfect Square Trinomials

Since $(x + 7)^2 = x^2 + 14x + 49$, we know that $x^2 + 14x + 49 = (x + 7)^2$. In words, *a perfect square trinomial will factor into a binomial square.* To use this idea effectively, we must learn to *identify* perfect square trinomials. Note that the first and last terms of $x^2 + 14x + 49$ are *the squares* of x and 7, and the middle term is *twice the product of these two terms*: $2(7x) = 14x$. These are the characteristics of a perfect square trinomial.

Factoring Perfect Square Trinomials

Given any expression that can be written in the form $A^2 \pm 2AB + B^2$,

1. $A^2 + 2AB + B^2 = (A + B)^2$
2. $A^2 - 2AB + B^2 = (A - B)^2$

EXAMPLE 7 ► **Factoring a Perfect Square Trinomial**

Factor $12m^3 - 12m^2 + 3m$.

Solution ► $12m^3 - 12m^2 + 3m$ — check for common factors: GCF $= 3m$

$= 3m(4m^2 - 4m + 1)$ — factor out $3m$

For the remaining trinomial $4m^2 - 4m + 1\ldots$

1. Are the first and last terms perfect squares?

$$4m^2 = (2m)^2 \text{ and } 1 = (1)^2 \checkmark \quad \text{Yes.}$$

2. Is the linear term twice the product of $2m$ and 1?

$$2 \cdot 2m \cdot 1 = 4m \checkmark \quad \text{Yes.}$$

Factor as a binomial square: $4m^2 - 4m + 1 = (2m - 1)^2$

This shows $12m^3 - 12m^2 + 3m = 3m(2m - 1)^2$.

Now try Exercises 19 and 20 ►

CAUTION ► As shown in Example 7, be sure to include the GCF in your final answer. It is a common error to "leave the GCF behind."

In actual practice, the tests for a perfect square trinomial are performed mentally, with only the factored form being written down.

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Sum or Difference of Two Perfect Cubes

Recall that the *difference* of two perfect squares is factorable, but the *sum* of two perfect squares is prime. In contrast, *both the sum and difference of two perfect* cubes *are factorable*. For either $A^3 + B^3$ or $A^3 - B^3$ we have the following:

1. Each will factor into the product of a binomial and a trinomial: $(\quad)(\quad)$ binomial, trinomial
2. The terms of the binomial are the quantities being cubed: $(A \quad B)(\quad)$
3. The terms of the trinomial are the square of A, the product AB, and the square of B, respectively: $(A \quad B)(A^2 \quad AB \quad B^2)$
4. The binomial takes the same sign as the original expression: $(A \pm B)(A^2 \quad AB \quad B^2)$
5. The middle term of the trinomial takes the opposite sign of the original exercise (the last term is always positive): $(A \pm B)(A^2 \mp AB + B^2)$

Factoring the Sum or Difference of Two Perfect Cubes: $A^3 \pm B^3$

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$
$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

EXAMPLE 8 ► Factoring the Sum and Difference of Two Perfect Cubes

Factor completely:

a. $x^3 + 125$ **b.** $-5m^3n + 40n^4$

Solution ►

a. $x^3 + 125 = x^3 + 5^3$ — write terms as perfect cubes

Use $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$ — factoring template

$x^3 + 5^3 = (x + 5)(x^2 - 5x + 25)$ — $A \to x$ and $B \to 5$

b. $-5m^3n + 40n^4 = -5n(m^3 - 8n^3)$ — check for common factors (GCF $= -5n$)

$= -5n[m^3 - (2n)^3]$ — write terms as perfect cubes

Use $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ — factoring template

$m^3 - (2n)^3 = (m - 2n)[m^2 + m(2n) + (2n)^2]$ — $A \to m$ and $B \to 2n$

$= (m - 2n)(m^2 + 2mn + 4n^2)$ — simplify

$\Rightarrow -5m^3n + 40n^4 = -5n(m - 2n)(m^2 + 2mn + 4n^2)$. — factored form

The results for parts (a) and (b) can be checked using multiplication.

Now try Exercises 21 and 22 ►

Using *u*-Substitution to Factor Quadratic Forms

For any quadratic expression $ax^2 + bx + c$ in standard form, the degree of the leading term is twice the degree of the middle term. Generally, a trinomial is in **quadratic form** if it can be written as $a(\_\_)^2 + b(\_\_) + c$, where the parentheses "hold" the same factors. The equation $x^4 - 13x^2 + 36 = 0$ is in quadratic form since $(x^2)^2 - 13(x^2) + 36 = 0$. In many cases, we can factor these expressions using a **placeholder substitution** that transforms these expressions into a more recognizable form. In a study of algebra, the letter "u" often plays this role. If we let u represent x^2, the expression $(x^2)^2 - 13(x^2) + 36$ becomes $u^2 - 13u + 36$, which can be factored into $(u - 9)(u - 4)$. After "unsubstituting" (replace u with x^2), we have $(x^2 - 9)(x^2 - 4) = (x + 3)(x - 3)(x + 2)(x - 2)$.

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EXAMPLE 9 ▶ Factoring a Quadratic Form

Write in completely factored form: $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$.

Solution ▶ Expanding the binomials would produce a fourth-degree polynomial that would be very difficult to factor. Instead we note the expression is in *quadratic form*. Letting u represent $x^2 - 2x$ (the variable part of the "middle" term), $(x^2 - 2x)^2 - 2(x^2 - 2x) - 3$ becomes $u^2 - 2u - 3$.

$$u^2 - 2u - 3 = (u - 3)(u + 1) \quad \text{factor}$$

To finish up, write the expression in terms of x, substituting $x^2 - 2x$ for u.

$$= (x^2 - 2x - 3)(x^2 - 2x + 1) \quad \text{substitute } x^2 - 2x \text{ for } u$$

The resulting trinomials can be further factored.

$$= (x - 3)(x + 1)(x - 1)^2 \quad x^2 - 2x + 1 = (x - 1)^2$$

Now try Exercises 23 and 24 ▶

D. You've just reviewed how to factor special forms and quadratic forms

It is well known that information is retained longer and used more effectively when it's placed in an organized form. The "factoring flowchart" provided in Figure R.5 offers a streamlined and systematic approach to factoring and the concepts involved. However, with some practice the process tends to "flow" more naturally than following a chart, with many of the decisions becoming automatic.

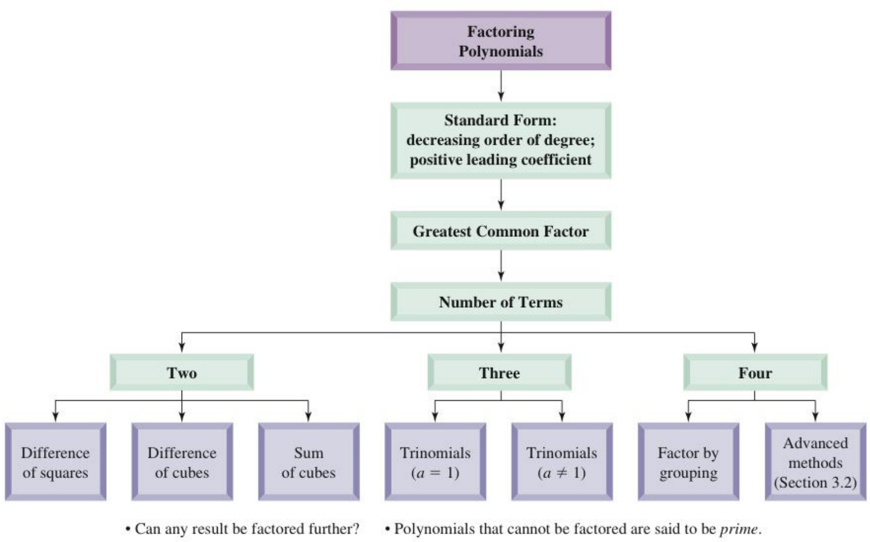

Figure R.5

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R.4 EXERCISES

▶ CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. To factor an expression means to rewrite the expression as an equivalent ________.

2. If a polynomial will not factor, it is said to be a(n) ________ polynomial.

3. The difference of two perfect squares always factors into the product of a(n) ________ and its ________.

4. The expression $x^2 + 6x + 9$ is said to be a(n) ________ ________ trinomial, since its factored form is a perfect (binomial) square.

5. Discuss/Explain why $4x^2 - 36 = (2x - 6)(2x + 6)$ is not written in completely factored form, then rewrite it so it is factored completely.

6. Discuss/Explain why $a^3 + b^3$ is factorable, but $a^2 + b^2$ is not. Demonstrate by writing $x^3 + 64$ in factored form, and by exhausting all possibilities for $x^2 + 64$ to show it is prime.

▶ DEVELOPING YOUR SKILLS

Factor each expression using the method indicated.

Greatest Common Factor

7. a. $-17x^2 + 51$ **b.** $21b^3 - 14b^2 + 56b$ **c.** $-3a^4 + 9a^2 - 6a^3$

8. a. $-13n^2 - 52$ **b.** $9p^2 + 27p^3 - 18p^4$ **c.** $-6g^5 + 12g^4 - 9g^3$

Common Binomial Factor

9. a. $2a(a + 2) + 3(a + 2)$ **b.** $(b^2 + 3)3b + (b^2 + 3)2$ **c.** $4m(n + 7) - 11(n + 7)$

10. a. $5x(x - 3) - 2(x - 3)$ **b.** $(v - 5)2v + (v - 5)3$ **c.** $3p(q^2 + 5) + 7(q^2 + 5)$

Grouping

11. a. $9q^3 + 6q^2 + 15q + 10$ **b.** $h^5 - 12h^4 - 3h + 36$ **c.** $k^5 - 7k^3 - 5k^2 + 35$

12. a. $6h^3 - 9h^2 - 2h + 3$ **b.** $4k^3 + 6k^2 - 2k - 3$ **c.** $3x^2 - xy - 6x + 2y$

Trinomial Factoring where $|a| = 1$

13. a. $-p^2 + 5p + 14$ **b.** $q^2 - 4q - 45$ **c.** $n^2 + 20 - 9n$

14. a. $-m^2 + 13m - 42$ **b.** $x^2 + 12 + 13x$ **c.** $v^2 + 10v + 15$

Trinomial Factoring where $a \neq 1$

15. a. $3p^2 - 13p - 10$ **b.** $4q^2 + 7q - 15$ **c.** $10u^2 - 19u - 15$

16. a. $6v^2 + v - 35$ **b.** $20x^2 + 53x + 18$ **c.** $15z^2 - 22z - 48$

Difference of Perfect Squares

17. a. $4s^2 - 25$ **b.** $9x^2 - 49$ **c.** $50x^2 - 72$ **d.** $121h^2 - 144$ **e.** $b^2 - 5$

18. a. $9v^2 - \frac{1}{25}$ **b.** $25w^2 - \frac{1}{49}$ **c.** $v^4 - 1$ **d.** $16z^4 - 81$ **e.** $x^2 - 17$

Perfect Square Trinomials

19. a. $a^2 - 6a + 9$ **b.** $b^2 + 10b + 25$ **c.** $4m^2 - 20m + 25$ **d.** $9n^2 - 42n + 49$

20. a. $x^2 + 12x + 36$ **b.** $z^2 - 18z + 81$ **c.** $25p^2 - 60p + 36$ **d.** $16q^2 + 40q + 25$

Sum/Difference of Perfect Cubes

21. a. $8p^3 - 27$ **b.** $m^3 + \frac{1}{8}$ **c.** $g^3 - 0.027$ **d.** $-2t^4 + 54t$

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22. **a.** $27q^3 - 125$ **b.** $n^3 + \frac{8}{27}$
c. $b^3 - 0.125$ **d.** $3r^4 - 24r$

***u*-Substitution**

23. **a.** $x^4 - 10x^2 + 9$ **b.** $x^4 + 13x^2 + 36$
c. $x^6 - 7x^3 - 8$

24. **a.** $x^6 - 26x^3 - 27$
b. $3(n + 5)^2 + (2n + 10) - 21$
c. $2(z + 3)^2 + (3z + 9) - 54$

25. Completely factor each of the following (recall that "1" is its own perfect square and perfect cube).
a. $n^2 - 1$ **b.** $n^3 - 1$
c. $n^3 + 1$ **d.** $28x^3 - 7x$

26. Carefully factor each of the following trinomials, if possible. Note differences and similarities.
a. $x^2 - x + 6$ **b.** $x^2 + x - 6$
c. $x^2 + x + 6$ **d.** $x^2 - x - 6$
e. $x^2 - 5x + 6$ **f.** $x^2 + 5x - 6$

Factor each expression completely, if possible. Rewrite the expression in standard form (factor out "−1" if needed) and factor out the GCF if one exists. If you believe the expression will not factor, write "prime."

27. $a^2 + 7a + 10$ **28.** $b^2 + 9b + 20$

29. $2x^2 - 24x + 40$ **30.** $10z^2 - 140z + 450$

31. $64 - 9m^2$ **32.** $25 - 16n^2$

33. $-9r + r^2 + 18$ **34.** $28 + s^2 - 11s$

35. $2h^2 + 7h + 6$ **36.** $3k^2 + 10k + 8$

37. $9k^2 - 24k + 16$ **38.** $4p^2 - 20p + 25$

39. $-6x^3 + 39x^2 - 63x$ **40.** $-28z^3 + 16z^2 + 80z$

41. $12m^2 - 40m + 4m^3$ **42.** $-30n - 4n^2 + 2n^3$

43. $a^2 - 7a - 60$ **44.** $b^2 - 9b - 36$

45. $8x^3 - 125$ **46.** $27r^3 + 64$

47. $m^2 + 9m - 24$ **48.** $n^2 - 14n - 36$

49. $x^3 - 5x^2 - 9x + 45$ **50.** $x^3 + 3x^2 - 4x - 12$

51. Match each expression with the description that fits *best.*
____ **a.** prime polynomial
____ **b.** standard trinomial $a = 1$
____ **c.** perfect square trinomial
____ **d.** difference of cubes
____ **e.** binomial square
____ **f.** sum of cubes
____ **g.** binomial conjugates
____ **h.** difference of squares
____ **i.** standard trinomial $a \neq 1$
A. $x^3 + 27$ **B.** $(x + 3)^2$
C. $x^2 - 10x + 25$ **D.** $x^2 - 144$
E. $x^2 - 3x - 10$ **F.** $8s^3 - 125t^3$
G. $2x^2 - x - 3$ **H.** $x^2 + 9$
I. $(x - 7)$ and $(x + 7)$

52. Match each polynomial to its factored form. Two of them are prime.
____ **a.** $4x^2 - 9$
____ **b.** $4x^2 - 28x + 49$
____ **c.** $x^3 - 125$
____ **d.** $8x^3 + 27$
____ **e.** $x^2 - 3x - 10$
____ **f.** $x^2 + 3x + 10$
____ **g.** $2x^2 - x - 3$
____ **h.** $2x^2 + x - 3$
____ **i.** $x^2 + 25$
A. $(x - 5)(x^2 + 5x + 25)$
B. $(2x - 3)(x + 1)$
C. $(2x + 3)(2x - 3)$
D. $(2x - 7)^2$
E. prime trinomial
F. prime binomial
G. $(2x + 3)(x - 1)$
H. $(2x + 3)(4x^2 - 6x + 9)$
I. $(x - 5)(x + 2)$

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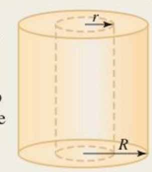
▶ WORKING WITH FORMULAS

53. Surface area of a cylinder: $2\pi r^2 + 2\pi rh$

The surface area of a cylinder is given by the formula shown, where h is the height of the cylinder and r is the radius. Factor out the GCF and use the result to find the surface area of a cylinder where $r = 35$ cm and $h = 65$ cm. Answer in exact form and in approximate form rounded to the nearest whole number.

54. Volume of a cylindrical shell: $\pi R^2h - \pi r^2h$

The volume of a cylindrical shell (a larger cylinder with a smaller cylinder removed) can be found using the formula shown, where R is the radius of the larger cylinder and r is the radius of the smaller. Factor the expression completely and use the result to find the volume of a shell where $R = 9$ cm, $r = 3$ cm, and $h = 10$ cm (use $\pi \approx 3.14$).

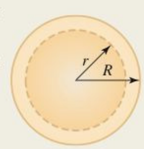


▶ APPLICATIONS

In many cases, factoring an expression can make it easier to evaluate as in the following applications.

55. Conical shells: The volume of a conical shell (like the shell of an ice cream cone) is given by the formula $V = \frac{1}{3}\pi R^2h - \frac{1}{3}\pi r^2h$, where R is the outer radius and r is the inner radius of the cone. Write the formula in completely factored form, then find the volume of a shell when $R = 5.1$ cm, $r = 4.9$ cm, and $h = 9$ cm. Answer in exact form and in approximate form rounded to the nearest tenth.

56. Spherical shells: The volume of a spherical shell (like the outer shell of a cherry cordial) is given by the formula $V = \frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$, where R is the outer radius and r is the inner radius of the shell. Write the right-hand side in completely factored form, then find the volume of a shell where $R = 1.8$ cm and $r = 1.5$ cm.



57. Volume of a box: The volume of a rectangular box x inches in height is given by the relationship $V = x^3 + 8x^2 + 15x$. Factor the right-hand side to determine: (a) The number of inches that the width exceeds the height, (b) the number of inches the length exceeds the height, and (c) the volume given the height is 2 ft.

58. Shipping textbooks: A publisher ships paperback books stacked x copies high in a box. The total number of books shipped per box is given by the relationship $B = x^3 - 13x^2 + 42x$. Factor the right-hand side to determine (a) how many more or fewer books fit the width of the box (than the height), (b) how many more or fewer books fit the length of the box (than the height), and (c) the number of books shipped per box if they are stacked 10 high in the box.

59. Space-Time relationships: Due to the work of Albert Einstein and other physicists who labored on space-time relationships, it is known that the faster an object moves the shorter it appears to become. This phenomenon is modeled by the **Lorentz transformation** $L = L_0\sqrt{1 - \left(\frac{v}{c}\right)^2}$, where L_0 is the length of the object at rest, L is the relative length when the object is moving at velocity v, and c is the speed of light. Factor the radicand and use the result to determine the relative length of a 12-in. ruler if it is shot past a stationary observer at 0.75 times the speed of light ($v = 0.75c$).

60. Tubular fluid flow: As a fluid flows through a tube, it is flowing faster at the center of the tube than at the sides, where the tube exerts a backward drag. **Poiseuille's law** gives the velocity of the flow at any point of the cross section: $v = \frac{G}{4\eta}(R^2 - r^2)$, where R is the inner radius of the tube, r is the distance from the center of the tube to a point in the flow, G represents what is called the pressure gradient, and η is a constant that depends on the viscosity of the fluid. Factor the right-hand side and find v given $R = 0.5$ cm, $r = 0.3$ cm, $G = 15$, and $\eta = 0.25$.

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▶ EXTENDING THE CONCEPT

61. Factor out a constant that leaves integer coefficients for each term:
a. $\frac{1}{2}x^4 + \frac{1}{8}x^3 - \frac{3}{4}x^2 + 4$
b. $\frac{2}{3}b^5 - \frac{1}{6}b^3 + \frac{4}{9}b^2 - 1$

62. If $x = 2$ is substituted into $2x^3 + hx + 8$, the result is zero. What is the value of h?

63. Factor the expression: $192x^3 - 164x^2 - 270x$.

64. As an alternative to evaluating polynomials by direct substitution, **nested factoring** can be used. The method has the advantage of using only products and sums—no powers. For $P = x^3 + 3x^2 + 1x + 5$, we begin by grouping all variable terms and factoring x: $P = [x^3 + 3x^2 + 1x] + 5 = x[x^2 + 3x + 1] + 5$. Then we group the inner terms with x and factor again: $P = x[x^2 + 3x + 1] + 5 = x[x(x + 3) + 1] + 5$. The expression can now be evaluated using any input and the order of operations. If $x = 2$, we quickly find that $P = 27$. Use this method to evaluate $H = x^3 + 2x^2 + 5x - 9$ for $x = -3$.

Factor each expression completely.

65. $x^4 - 81$

66. $16n^4 - 1$

67. $p^6 - 1$

68. $m^6 - 64$

69. $q^4 - 28q^2 + 75$

70. $a^4 - 18a^2 + 32$

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R.5 Rational Expressions

Learning Objectives

In Section R.5 you will learn how to:

- ❑ **A.** Write a rational expression in simplest form
- ❑ **B.** Multiply and divide rational expressions
- ❑ **C.** Add and subtract rational expressions
- ❑ **D.** Simplify compound fractions
- ❑ **E.** Rewrite formulas and algebraic models

A rational number is one that can be written as the quotient of two integers. Similarly, a *rational expression* is one that can be written as the quotient of two polynomials. We can apply the skills developed in a study of fractions (how to reduce, add, subtract, multiply, and divide) to **rational expressions,** sometimes called **algebraic fractions.**

A. Writing a Rational Expression in Simplest Form

A rational expression is in **simplest form** when the numerator and denominator have no common factors (other than 1). After factoring the numerator and denominator, we apply the **fundamental property of rational expressions.**

Fundamental Property of Rational Expressions

If P, Q, and R are polynomials, with $Q, R \neq 0$,

$$(1)\ \frac{P \cdot R}{Q \cdot R} = \frac{P}{Q} \quad \text{and} \quad (2)\ \frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$$

In words, the property says (1) a rational expression can be simplified by canceling common factors in the numerator and denominator, and (2) an equivalent expression can be formed by multiplying numerator and denominator by the same nonzero polynomial.

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EXAMPLE 1 ► **Simplifying a Rational Expression**

Write the expression in simplest form: $\frac{x^2 - 1}{x^2 - 3x + 2}$.

Solution ►

$$\frac{x^2 - 1}{x^2 - 3x + 2} = \frac{(x-1)(x+1)}{(x-1)(x-2)} \quad \text{factor numerator and denominator}$$

$$= \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} \quad \text{cancel common factors}$$

$$= \frac{x+1}{x-2} \quad \text{simplest form}$$

Now try Exercises 7 through 10 ►

WORTHY OF NOTE

If we view *a* and *b* as two points on the number line, we note that they are the same distance apart, regardless of the order they are subtracted. This tells us the numerator and denominator will have the same absolute value but be opposite in sign, giving a value of −1 (check using a few test values).

When simplifying rational expressions, we sometimes encounter expressions of the form $\frac{a-b}{b-a}$. If we factor −1 from the numerator, we see that $\frac{a-b}{b-a} = \frac{-1\cancel{(b-a)}}{\cancel{b-a}} = -1$.

CAUTION ► When reducing rational numbers or expressions, only common *factors* can be reduced. It is incorrect to reduce (or divide out) individual terms: $\frac{-6 + 4\sqrt{3}}{2} \neq -3 + 4\sqrt{3}$, and $\frac{x+1}{x+2} \neq \frac{1}{2}$ (except for $x = 0$)

EXAMPLE 2 ► **Simplifying a Rational Expression**

Write the expression in simplest form: $\frac{(6 - 2x)}{x^2 - 9}$.

Solution ►

$$\frac{(6-2x)}{x^2 - 9} = \frac{2(3-x)}{(x-3)(x+3)} \quad \text{factor numerator and denominator}$$

$$= \frac{(2)(-1)}{x+3} \quad \text{reduce: } \frac{(3-x)}{(x-3)} = -1$$

$$= \frac{-2}{x+3} \quad \text{simplest form}$$

Now try Exercises 11 through 16 ►

☑ **A.** You've just reviewed how to write a rational expression in simplest form

B. Multiplication and Division of Rational Expressions

Operations on rational expressions use the factoring skills reviewed earlier, along with much of what we know about rational numbers.

Multiplying Rational Expressions

Given that *P*, *Q*, *R*, and *S* are polynomials with $Q, S \neq 0$,

$$\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$$

1. Factor all numerators and denominators completely.
2. Reduce common factors.
3. Multiply numerator × numerator and denominator × denominator.

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EXAMPLE 3 ► **Multiplying Rational Expressions**

Compute the product: $\frac{2a+2}{3a-3a^2}\cdot\frac{3a^2-a-2}{9a^2-4}$.

Solution ►

$$\frac{2a+2}{3a-3a^2}\cdot\frac{3a^2-a-2}{9a^2-4} = \frac{2(a+1)}{3a(1-a)}\cdot\frac{(3a+2)(a-1)}{(3a-2)(3a+2)} \quad \text{factor}$$

$$= \frac{2(a+1)}{3a(\cancel{1-a})}\cdot\frac{(\cancel{3a+2})\overset{(-1)}{(\cancel{a-1})}}{(3a-2)(\cancel{3a+2})} \quad \text{reduce: } \frac{a-1}{1-a} = -1$$

$$= \frac{-2(a+1)}{3a(3a-2)} \quad \text{simplest form}$$

Now try Exercises 17 through 20 ►

To divide fractions, we multiply the first expression by the *reciprocal of the second*. The quotient of two rational expressions is computed in the same way.

Dividing Rational Expressions

Given that P, Q, R, and S are polynomials with $Q, R, S \neq 0$,

$$\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q}\cdot\frac{S}{R} = \frac{PS}{QR}$$

Invert the divisor and multiply.

EXAMPLE 4 ► **Dividing Rational Expressions**

Compute the quotient $\frac{4m^3-12m^2+9m}{m^2-49} \div \frac{10m^2-15m}{m^2+4m-21}$.

Solution ►

$$\frac{4m^3-12m^2+9m}{m^2-49} \div \frac{10m^2-15m}{m^2+4m-21}$$

$$= \frac{4m^3-12m^2+9m}{m^2-49}\cdot\frac{m^2+4m-21}{10m^2-15m} \quad \text{invert and multiply}$$

$$= \frac{m(4m^2-12m+9)}{(m+7)(m-7)}\cdot\frac{(m+7)(m-3)}{5m(2m-3)} \quad \text{factor}$$

$$= \frac{\cancel{m}(2m-3)\cancel{(2m-3)}}{\cancel{(m+7)}(m-7)}\cdot\frac{\cancel{(m+7)}(m-3)}{5\cancel{m}\cancel{(2m-3)}} \quad \text{factor and reduce}$$

$$= \frac{(2m-3)(m-3)}{5(m-7)} \quad \text{lowest terms}$$

Note that we sometimes refer to simplest form as *lowest terms*.

Now try Exercises 21 through 42 ►

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CAUTION ► For products like $\frac{(w+7)(w-7)}{(w-7)(w-2)} \cdot \frac{(w-2)}{(w+7)}$, it is a common mistake to think that all factors "cancel," leaving an answer of zero. Actually, all factors *reduce to 1,* and the result is a value of 1 for all inputs where the product is defined.

$$\frac{\overset{1}{\cancel{(w+7)}}\overset{1}{\cancel{(w-7)}}}{\underset{1}{\cancel{(w-7)}}\underset{1}{\cancel{(w-2)}}} \cdot \frac{\overset{1}{\cancel{(w-2)}}}{\underset{1}{\cancel{(w+7)}}} = 1$$

B. You've just reviewed how to multiply and divide rational expressions

C. Addition and Subtraction of Rational Expressions

Recall that the addition and subtraction of *fractions* requires finding the lowest common denominator (LCD) and building equivalent fractions. The sum or difference of the numerators is then placed over this denominator. The procedure for the addition and subtraction of *rational expressions* is very much the same.

Addition and Subtraction of Rational Expressions

1. Find the LCD of all rational expressions.
2. Build equivalent expressions using the LCD.
3. Add or subtract numerators as indicated.
4. Write the result in lowest terms.

EXAMPLE 5 ► **Adding and Subtracting Rational Expressions**

Compute as indicated:

a. $\frac{7}{10x} + \frac{3}{25x^2}$ **b.** $\frac{10x}{x^2-9} - \frac{5}{x-3}$

Solution ► **a.** The LCD for $10x$ and $25x^2$ is $50x^2$. *find the LCD*

$$\frac{7}{10x} + \frac{3}{25x^2} = \frac{7}{10x}\cdot\frac{(5x)}{(5x)} + \frac{3}{25x^2}\cdot\frac{(2)}{(2)}$$ *write equivalent expressions*

$$= \frac{35x}{50x^2} + \frac{6}{50x^2}$$ *simplify*

$$= \frac{35x+6}{50x^2}$$ *add the numerators and write the result over the LCD*

The result is in simplest form.

b. The LCD for $x^2 - 9$ and $x - 3$ is $(x-3)(x+3)$. *find the LCD*

$$\frac{10x}{x^2-9} - \frac{5}{x-3} = \frac{10x}{(x-3)(x+3)} - \frac{5}{x-3}\cdot\frac{(x+3)}{(x+3)}$$ *write equivalent expressions*

$$= \frac{10x - 5(x+3)}{(x-3)(x+3)}$$ *subtract numerators, write the result over the LCD*

$$= \frac{10x - 5x - 15}{(x-3)(x+3)}$$ *distribute*

$$= \frac{5x - 15}{(x-3)(x+3)}$$ *combine like terms*

$$= \frac{5\overset{1}{\cancel{(x-3)}}}{\underset{1}{\cancel{(x-3)}}(x+3)} = \frac{5}{x+3}$$ *factor and reduce*

Now try Exercises 43 through 48 ►

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EXAMPLE 6 ► **Adding and Subtracting Rational Expressions**

Perform the operations indicated:

a. $\frac{5}{n+2} - \frac{n-3}{n^2-4}$ **b.** $\frac{b^2}{4a^2} - \frac{c}{a}$

Solution ► **a.** The LCD for $n + 2$ and $n^2 - 4$ is $(n + 2)(n - 2)$.

$$\frac{5}{n+2} - \frac{n-3}{n^2-4} = \frac{5}{(n+2)} \cdot \frac{(n-2)}{(n-2)} - \frac{n-3}{(n+2)(n-2)}$$ write equivalent expressions

$$= \frac{5(n-2) - (n-3)}{(n+2)(n-2)}$$ subtract numerators, write the result over the LCD

$$= \frac{5n - 10 - n + 3}{(n+2)(n-2)}$$ distribute

$$= \frac{4n - 7}{(n+2)(n-2)}$$ result

b. The LCD for a and $4a^2$ is $4a^2$: $\frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{(4a)}{(4a)}$ write equivalent expressions

$$= \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$$ simplify

$$= \frac{b^2 - 4ac}{4a^2}$$ subtract numerators, write the result over the LCD

☑ **C.** You've just reviewed how to add and subtract rational expressions

Now try Exercises 49 through 64 ►

CAUTION ► When the second term in a subtraction has a binomial numerator as in Example 6(a), be sure the subtraction *is applied to both terms*. It is a common error to write $\frac{5(n-2)}{(n+2)(n-2)} - \frac{n-3}{(n+2)(n-2)} = \frac{5n - 10 - n - 3}{(n+2)(n-2)}$ ✗ in which the subtraction is applied to the first term only. This is incorrect!

D. Simplifying Compound Fractions

Rational expressions whose numerator or denominator contain a fraction are called **compound fractions.** The expression $\frac{\frac{2}{3m} - \frac{3}{2}}{\frac{3}{4m} - \frac{1}{3m^2}}$ is a compound fraction with a numerator of $\frac{2}{3m} - \frac{3}{2}$ and a denominator of $\frac{3}{4m} - \frac{1}{3m^2}$. The two methods commonly used to simplify compound fractions are summarized in the following boxes.

Simplifying Compound Fractions (Method I)

1. Add/subtract fractions in the numerator, writing them as a single expression.
2. Add/subtract fractions in the denominator, also writing them as a single expression.
3. Multiply the numerator by the reciprocal of the denominator and simplify if possible.

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Simplifying Compound Fractions (Method II)

1. Find the LCD of all fractions in the numerator and denominator.
2. Multiply the numerator and denominator by this LCD and simplify.
3. Simplify further if possible.

Method II is illustrated in Example 7.

EXAMPLE 7 ► Simplifying a Compound Fraction

Simplify the compound fraction:

$$\frac{\frac{2}{3m}-\frac{3}{2}}{\frac{3}{4m}-\frac{1}{3m^2}}$$

Solution ► The LCD for all fractions is $12m^2$.

$$\frac{\frac{2}{3m}-\frac{3}{2}}{\frac{3}{4m}-\frac{1}{3m^2}} = \frac{\left(\frac{2}{3m}-\frac{3}{2}\right)\left(\frac{12m^2}{1}\right)}{\left(\frac{3}{4m}-\frac{1}{3m^2}\right)\left(\frac{12m^2}{1}\right)}$$ multiply numerator and denominator by $12m^2 = \frac{12m^2}{1}$

$$= \frac{\left(\frac{2}{3m}\right)\left(\frac{12m^2}{1}\right)-\left(\frac{3}{2}\right)\left(\frac{12m^2}{1}\right)}{\left(\frac{3}{4m}\right)\left(\frac{12m^2}{1}\right)-\left(\frac{1}{3m^2}\right)\left(\frac{12m^2}{1}\right)}$$ distribute

$$= \frac{8m-18m^2}{9m-4}$$ simplify

$$= \frac{2m(4-9m)}{9m-4} = -2m$$ factor and write in lowest terms

☑ **D.** You've just reviewed how to simplify compound fractions

Now try Exercises 65 through 74 ►

E. Rewriting Formulas and Algebraic Models

In many fields of study, formulas and algebraic models involve rational expressions and we often need to write them in an alternative form.

EXAMPLE 8 ► Rewriting a Formula

In an electrical circuit with two resistors in parallel, the total resistance R is related to resistors R_1 and R_2 by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. Rewrite the right-hand side as a single term.

Solution ►

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$$ LCD for the right-hand side is R_1R_2

$$=\frac{R_2}{R_1R_2}+\frac{R_1}{R_1R_2}$$ build equivalent expressions using LCD

$$=\frac{R_2+R_1}{R_1R_2}$$ write as a single expression

Now try Exercises 75 and 76 ►

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EXAMPLE 9 ► **Simplifying an Algebraic Model**

When studying rational expressions and rates of change, we encounter the expression $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$. Simplify the compound fraction.

Solution ► Using Method I gives:

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{h} \quad \text{LCD for the numerator is } x(x+h)$$

$$= \frac{\frac{x-(x+h)}{x(x+h)}}{h} \quad \text{write numerator as a single expression}$$

$$= \frac{\frac{-h}{x(x+h)}}{h} \quad \text{simplify}$$

$$= \frac{-h}{x(x+h)} \cdot \frac{1}{h} \quad \text{invert and multiply}$$

$$= \frac{-1}{x(x+h)} \quad \text{result}$$

E. You've just reviewed how to rewrite formulas and algebraic models

Now try Exercises 77 through 80 ►

R.5 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. In simplest form, $(a - b)/(a - b)$ is equal to ____, while $(a - b)/(b - a)$ is equal to ____.
2. A rational expression is in ________ ____ when the numerator and denominator have no common factors, other than ____.
3. As with numeric fractions, algebraic fractions require a ________ __________ for addition and subtraction.
4. Since $x^2 + 9$ is prime, the expression $(x^2 + 9)/(x + 3)$ is already written in ________ ____.

State T or F and discuss/explain your response.

5. $\dfrac{x}{x+3} - \dfrac{x+1}{x+3} = \dfrac{1}{x+3}$
6. $\dfrac{\cancel{(x+3)}\cancel{(x-2)}}{\cancel{(x-2)}\cancel{(x+3)}} = 0$

► DEVELOPING YOUR SKILLS

Reduce to lowest terms.

7. a. $\dfrac{a-7}{-3a+21}$ b. $\dfrac{2x+6}{4x^2-8x}$
8. a. $\dfrac{x-4}{-7x+28}$ b. $\dfrac{3x-18}{6x^2-12x}$
9. a. $\dfrac{x^2-5x-14}{x^2+6x-7}$ b. $\dfrac{a^2+3a-28}{a^2-49}$
10. a. $\dfrac{r^2+3r-10}{r^2+r-6}$ b. $\dfrac{m^2+3m-4}{m^2-4m}$

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11. a. $\frac{x-7}{7-x}$ **b.** $\frac{5-x}{x-5}$

12. a. $\frac{v^2-3v-28}{49-v^2}$ **b.** $\frac{u^2-10u+25}{25-u^2}$

13. a. $\frac{-12a^3b^5}{4a^2b^{-4}}$ **b.** $\frac{7x+21}{63}$

c. $\frac{y^2-9}{3-y}$ **d.** $\frac{m^3n-m^3}{m^4-m^4n}$

14. a. $\frac{5m^{-3}n^5}{-10mn^2}$ **b.** $\frac{-5v+20}{25}$

c. $\frac{n^2-4}{2-n}$ **d.** $\frac{w^4-w^4v}{w^3v-w^3}$

15. a. $\frac{2n^3+n^2-3n}{n^3-n^2}$ **b.** $\frac{6x^2+x-15}{4x^2-9}$

c. $\frac{x^3+8}{x^2-2x+4}$ **d.** $\frac{mn^2+n^2-4m-4}{mn+n+2m+2}$

16. a. $\frac{x^3+4x^2-5x}{x^3-x}$ **b.** $\frac{5p^2-14p-3}{5p^2+11p+2}$

c. $\frac{12y^2-13y+3}{27y^3-1}$ **d.** $\frac{ax^2-5x^2-3a+15}{ax-5x+5a-25}$

Compute as indicated. Write final results in lowest terms.

17. $\frac{a^2-4a+4}{a^2-9}\cdot\frac{a^2-2a-3}{a^2-4}$

18. $\frac{b^2+5b-24}{b^2-6b+9}\cdot\frac{b}{b^2-64}$

19. $\frac{x^2-7x-18}{x^2-6x-27}\cdot\frac{2x^2+7x+3}{2x^2+5x+2}$

20. $\frac{6v^2+23v+21}{4v^2-4v-15}\cdot\frac{4v^2-25}{3v+7}$

21. $\frac{p^3-64}{p^3-p^2}\div\frac{p^2+4p+16}{p^2-5p+4}$

22. $\frac{a^2+3a-28}{a^2+5a-14}\div\frac{a^3-4a^2}{a^3-8}$

23. $\frac{3x-9}{4x+12}\div\frac{3-x}{5x+15}$

24. $\frac{5b-10}{7b-28}\div\frac{2-b}{5b-20}$

25. $\frac{a^2+a}{a^2-3a}\cdot\frac{3a-9}{2a+2}$

26. $\frac{p^2-36}{2p}\cdot\frac{4p^2}{2p^2+12p}$

27. $\frac{8}{a^2-25}\cdot(a^2-2a-35)$

28. $(m^2-16)\cdot\frac{m^2-5m}{m^2-m-20}$

29. $\frac{xy-3x+2y-6}{x^2-3x-10}\div\frac{xy-3x}{xy-5y}$

30. $\frac{2a-ab+7b-14}{b^2-14b+49}\div\frac{ab-2a}{ab-7a}$

31. $\frac{m^2+2m-8}{m^2-2m}\div\frac{m^2-16}{m^2}$

32. $\frac{18-6x}{x^2-25}\div\frac{2x^2-18}{x^3-2x^2-25x+50}$

33. $\frac{y+3}{3y^2+9y}\cdot\frac{y^2+7y+12}{y^2-16}\div\frac{y^2+4y}{y^2-4y}$

34. $\frac{x^2+4x-5}{x^2-5x-14}\div\frac{x^2-1}{x^2-4}\cdot\frac{x+1}{x+5}$

35. $\frac{x^2-0.49}{x^2+0.5x-0.14}\div\frac{x^2-0.10x+0.21}{x^2-0.09}$

36. $\frac{x^2-0.25}{x^2+0.1x-0.2}\div\frac{x^2-0.8x+0.15}{x^2-0.16}$

37. $\frac{n^2-\frac{4}{9}}{n^2-\frac{13}{15}n+\frac{2}{15}}\div\frac{n^2+\frac{4}{3}n+\frac{4}{9}}{n^2-\frac{1}{25}}$

38. $\frac{q^2-\frac{9}{25}}{q^2-\frac{1}{10}q-\frac{3}{10}}\div\frac{q^2+\frac{17}{20}q+\frac{3}{20}}{q^2-\frac{1}{16}}$

39. $\frac{3a^3-24a^2-12a+96}{a^2-11a+24}\div\frac{6a^2-24}{3a^3-81}$

40. $\frac{p^3+p^2-49p-49}{p^2+6p-7}\div\frac{p^2+p+1}{p^3-1}$

41. $\frac{4n^2-1}{12n^2-5n-3}\cdot\frac{6n^2+5n+1}{2n^2+n}\cdot\frac{12n^2-17n+6}{6n^2-7n+2}$

42. $\left(\frac{4x^2-25}{x^2-11x+30}\div\frac{2x^2-x-15}{x^2-9x+18}\right)\frac{4x^2+25x-21}{12x^2-5x-3}$

Compute as indicated. Write answers in lowest terms [recall that $a-b=-1(b-a)$].

43. $\frac{3}{8x^2}+\frac{5}{2x}$ **44.** $\frac{15}{16y}-\frac{7}{2y^2}$

45. $\frac{7}{4x^2y^3}-\frac{1}{8xy^4}$ **46.** $\frac{3}{6a^3b}+\frac{5}{9ab^3}$

47. $\frac{4p}{p^2-36}-\frac{2}{p-6}$ **48.** $\frac{3q}{q^2-49}-\frac{3}{2q-14}$

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49. $\frac{m}{m^2 - 16} + \frac{4}{4 - m}$ **50.** $\frac{2}{4 - p^2} + \frac{p}{p - 2}$

51. $\frac{2}{m - 7} - 5$ **52.** $\frac{4}{x - 1} - 9$

53. $\frac{y + 1}{y^2 + y - 30} - \frac{2}{y + 6}$

54. $\frac{4n}{n^2 - 5n} - \frac{3}{4n - 20}$

55. $\frac{1}{a + 4} + \frac{a}{a^2 - a - 20}$

56. $\frac{2x - 1}{x^2 + 3x - 4} - \frac{x - 5}{x^2 + 3x - 4}$

57. $\frac{3y - 4}{y^2 + 2y + 1} - \frac{2y - 5}{y^2 + 2y + 1}$

58. $\frac{-2}{3a + 12} - \frac{7}{a^2 + 4a}$

59. $\frac{2}{m^2 - 9} + \frac{m - 5}{m^2 + 6m + 9}$

60. $\frac{m + 2}{m^2 - 25} - \frac{m + 6}{m^2 - 10m + 25}$

61. $\frac{y + 2}{5y^2 + 11y + 2} + \frac{5}{y^2 + y - 6}$

62. $\frac{m - 4}{3m^2 - 11m + 6} + \frac{m}{2m^2 - m - 15}$

Write each term as a rational expression. Then compute the sum or difference indicated.

63. a. $p^{-2} - 5p^{-1}$ **b.** $x^{-2} + 2x^{-3}$

64. a. $3a^{-1} + (2a)^{-1}$ **b.** $2y^{-1} - (3y)^{-1}$

Simplify each compound rational expression. Use either method.

65. $\dfrac{\frac{5}{a} - \frac{1}{4}}{\frac{25}{a^2} - \frac{1}{16}}$ **66.** $\dfrac{\frac{8}{x^3} - \frac{1}{27}}{\frac{2}{x} - \frac{1}{3}}$

67. $\dfrac{p + \frac{1}{p - 2}}{1 + \frac{1}{p - 2}}$ **68.** $\dfrac{1 + \frac{3}{y - 6}}{y + \frac{9}{y - 6}}$

69. $\dfrac{\frac{2}{3 - x} + \frac{3}{x - 3}}{\frac{4}{x} + \frac{5}{x - 3}}$ **70.** $\dfrac{\frac{1}{y - 5} - \frac{2}{5 - y}}{\frac{3}{y - 5} - \frac{2}{y}}$

71. $\dfrac{\frac{2}{y^2 - y - 20}}{\frac{3}{y + 4} - \frac{4}{y - 5}}$ **72.** $\dfrac{\frac{2}{x^2 - 3x - 10}}{\frac{6}{x + 2} - \frac{4}{x - 5}}$

Rewrite each expression as a compound fraction. Then simplify using either method.

73. a. $\frac{1 + 3m^{-1}}{1 - 3m^{-1}}$ **b.** $\frac{1 + 2x^{-2}}{1 - 2x^{-2}}$

74. a. $\frac{4 - 9a^{-2}}{3a^{-2}}$ **b.** $\frac{3 + 2n^{-1}}{5n^{-2}}$

Rewrite each expression as a single term.

75. $\frac{1}{f_1} + \frac{1}{f_2}$ **76.** $\frac{1}{w} + \frac{1}{x} - \frac{1}{y}$

77. $\dfrac{\frac{a}{x + h} - \frac{a}{x}}{h}$ **78.** $\dfrac{\frac{a}{h - x} - \frac{a}{-x}}{h}$

79. $\dfrac{\frac{1}{2(x + h)^2} - \frac{1}{2x^2}}{h}$ **80.** $\dfrac{\frac{a}{(x + h)^2} - \frac{a}{x^2}}{h}$

▶ WORKING WITH FORMULAS

81. Cost to seize illegal drugs: $C = \frac{450P}{100 - P}$

The cost C, in millions of dollars, for a government to find and seize $P\%$ $(0 \le P < 100)$ of a certain illegal drug is modeled by the rational equation shown. Complete the table (round to the nearest dollar) and answer the following questions.

a. What is the cost of seizing 40% of the drugs? Estimate the cost at 85%.

b. Why does cost increase dramatically the closer you get to 100%?

c. Will 100% of the drugs ever be seized?

P	$\frac{450P}{100 - P}$
40	
60	
80	
90	
93	
95	
98	
100	

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82. Chemicals in the bloodstream: $C = \dfrac{200H^2}{H^3 + 40}$

Rational equations are often used to model chemical concentrations in the bloodstream. The percent concentration C of a certain drug H hours after injection into muscle tissue can be modeled by the equation shown ($H \geq 0$). Complete the table (round to the nearest tenth of a percent) and answer the following questions.

a. What is the percent concentration of the drug 3 hr after injection?

b. Why is the concentration virtually equal at $H = 4$ and $H = 5$?

c. Why does the concentration begin to decrease?

d. How long will it take for the concentration to become less than 10%?

H	$\dfrac{200H^2}{H^3 + 40}$
0	
1	
2	
3	
4	
5	
6	
7	

▶ APPLICATIONS

83. Stock prices: When a hot new stock hits the market, its price will often rise dramatically and then taper off over time. The equation $P = \dfrac{50(7d^2 + 10)}{d^3 + 50}$ models the price of stock XYZ d days after it has "hit the market." Create a table of values showing the price of the stock for the first 10 days and comment on what you notice. Find the opening price of the stock—does the stock ever return to its original price?

84. Population growth: The Department of Wildlife introduces 60 elk into a new game reserve. It is projected that the size of the herd will grow according to the equation $N = \dfrac{10(6 + 3t)}{1 + 0.05t}$, where N is the number of elk and t is the time in years. Approximate the population of elk after 14 yr.

85. Typing speed: The number of words per minute that a beginner can type is approximately by the equation $N = \dfrac{60t - 120}{t}$, where N is the number of words per minute after t weeks, $2 < t < 12$. Use a table to determine how many weeks it takes for a student to be typing an average of forty-five words per minute.

86. Memory retention: A group of students is asked to memorize 50 Russian words that are unfamiliar to them. The number N of these words that the average student remembers D days later is modeled by the equation $N = \dfrac{5D + 35}{D}$ ($D \geq 1$). How many words are remembered after (a) 1 day? (b) 5 days? (c) 12 days? (d) 35 days? (e) 100 days? According to this model, is there a certain number of words that the average student never forgets? How many?

▶ EXTENDING THE CONCEPT

87. One of these expressions is *not* equal to the others. Identify which and explain why.

a. $\dfrac{20n}{10n}$ **b.** $20 \cdot n \div 10 \cdot n$

c. $20n \cdot \dfrac{1}{10n}$ **d.** $\dfrac{20}{10} \cdot \dfrac{n}{n}$

88. The average of A and B is x. The average of C, D, and E is y. The average of A, B, C, D, and E is

a. $\dfrac{3x + 2y}{5}$ **b.** $\dfrac{2x + 3y}{5}$

c. $\dfrac{2(x + y)}{5}$ **d.** $\dfrac{3(x + y)}{5}$

89. Given the rational numbers $\frac{2}{5}$ and $\frac{3}{4}$, what is the reciprocal of the sum of their reciprocals? Given that $\frac{a}{b}$ and $\frac{c}{d}$ are *any* two numbers—what is the reciprocal of the sum of their reciprocals?

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R.6 Radicals and Rational Exponents

Learning Objectives

In Section R.6 you will learn how to:

- ❑ **A.** Simplify radical expressions of the form $\sqrt[n]{a^n}$
- ❑ **B.** Rewrite and simplify radical expressions using rational exponents
- ❑ **C.** Use properties of radicals to simplify radical expressions
- ❑ **D.** Add and subtract radical expressions
- ❑ **E.** Multiply and divide radical expressions; write a radical expression in simplest form
- ❑ **F.** Evaluate formulas involving radicals

Square roots and cube roots come from a much larger family called **radical expressions.** Expressions containing radicals can be found in virtually every field of mathematical study, and are an invaluable tool for modeling many real-world phenomena.

A. Simplifying Radical Expressions of the Form $\sqrt[n]{a^n}$

In Section R.1 we noted $\sqrt{a} = b$ only if $b^2 = a$. The expression $\sqrt{-16}$ does not represent a real number because there is no number b such that $b^2 = -16$, showing $\sqrt{a}$ is a real number only if $a \geq 0$. Of particular interest to us now is an inverse operation for a^2. In other words, what operation can be applied to a^2 to return a? Consider the following.

EXAMPLE 1 ► Evaluating a Radical Expression

Evaluate $\sqrt{a^2}$ for the values given:

a. $a = 3$ **b.** $a = 5$ **c.** $a = -6$

Solution ► **a.** $\sqrt{3^2} = \sqrt{9} = 3$ **b.** $\sqrt{5^2} = \sqrt{25} = 5$ **c.** $\sqrt{(-6)^2} = \sqrt{36} = 6$

Now try Exercises 7 and 8 ►

The pattern seemed to indicate that $\sqrt{a^2} = a$ and that our search for an inverse operation was complete—until Example 1(c), where we found that $\sqrt{(-6)^2} \neq -6$. Using the absolute value concept, we can repair this apparent discrepancy and state a general rule for simplifying these expressions: $\sqrt{a^2} = |a|$. For expressions like $\sqrt{49x^2}$ and $\sqrt{y^6}$, the radicands can be rewritten as perfect squares and simplified in the same manner: $\sqrt{49x^2} = \sqrt{(7x)^2} = 7|x|$ and $\sqrt{y^6} = \sqrt{(y^3)^2} = |y^3|$.

The Square Root of a^2: $\sqrt{a^2}$

For any real number a, $\sqrt{a^2} = |a|$.

EXAMPLE 2 ► Simplifying Square Root Expressions

Simplify each expression.

a. $\sqrt{169x^2}$ **b.** $\sqrt{x^2 - 10x + 25}$

Solution ► **a.** $\sqrt{169x^2} = |13x|$

$= 13|x|$ since x could be negative

b. $\sqrt{x^2 - 10x + 25} = \sqrt{(x - 5)^2}$

$= |x - 5|$ since $x - 5$ could be negative

Now try Exercises 9 and 10 ►

CAUTION ► In Section R.3, we noted that $(A + B)^2 \neq A^2 + B^2$, indicating that you cannot square the individual terms in a sum (the square of a binomial results in a perfect square trinomial). In a similar way, $\sqrt{A^2 + B^2} \neq A + B$, and you cannot take the square root of individual terms. There is a big difference between the expressions $\sqrt{A^2 + B^2}$ and $\sqrt{(A + B)^2} = |A + B|$. Try evaluating each when $A = 3$ and $B = 4$.

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To investigate expressions like $\sqrt[3]{x^3}$, note the radicand in both $\sqrt[3]{8}$ and $\sqrt[3]{-64}$ can be written as a perfect cube. From our earlier definition of cube roots we know $\sqrt[3]{8} = \sqrt[3]{(2)^3} = 2$, $\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$, and that every real number has only one real cube root. For this reason, absolute value notation is not used or needed when taking cube roots.

The Cube Root of a^3: $\sqrt[3]{a^3}$

For any real number a, $\sqrt[3]{a^3} = a$.

EXAMPLE 3 ► Simplifying Cube Root Expressions

Simplify each expression.

a. $\sqrt[3]{-27x^3}$ **b.** $\sqrt[3]{-64n^6}$

Solution ► **a.** $\sqrt[3]{-27x^3} = \sqrt[3]{(-3x)^3} = -3x$ **b.** $\sqrt[3]{-64n^6} = \sqrt[3]{(-4n^2)^3} = -4n^2$

Now try Exercises 11 and 12 ►

WORTHY OF NOTE

Just as $\sqrt[2]{-16}$ is not a real number, $\sqrt[4]{-16}$ or $\sqrt[6]{-16}$ do not represent real numbers. An even number of repeated factors is always positive!

We can extend these ideas to fourth roots, fifth roots, and so on. For example, the fifth root of a is b only if $b^5 = a$. In symbols, $\sqrt[5]{a} = b$ implies $b^5 = a$. Since an odd number of negative factors is always negative: $(-2)^5 = -32$, and an even number of negative factors is always positive: $(-2)^4 = 16$, we must take the index into account when evaluating expressions like $\sqrt[n]{a^n}$. If n is even and the radicand is unknown, absolute value notation must be used.

The nth Root of a^n: $\sqrt[n]{a^n}$

For any real number a,

1. $\sqrt[n]{a^n} = |a|$ when n is even. **2.** $\sqrt[n]{a^n} = a$ when n is odd.

EXAMPLE 4 ► Simplifying Radical Expressions

Simplify each expression.

a. $\sqrt[4]{81}$ **b.** $\sqrt[4]{-81}$ **c.** $\sqrt[5]{32}$ **d.** $\sqrt[5]{-32}$
e. $\sqrt[4]{16m^4}$ **f.** $\sqrt[5]{32p^5}$ **g.** $\sqrt[6]{(m+5)^6}$ **h.** $\sqrt[7]{(x-2)^7}$

Solution ►

a. $\sqrt[4]{81} = 3$ **b.** $\sqrt[4]{-81}$ is not a real number
c. $\sqrt[5]{32} = 2$ **d.** $\sqrt[5]{-32} = -2$
e. $\sqrt[4]{16m^4} = \sqrt[4]{(2m)^4} = |2m|$ or $2|m|$ **f.** $\sqrt[5]{32p^5} = \sqrt[5]{(2p)^5} = 2p$
g. $\sqrt[6]{(m+5)^6} = |m+5|$ **h.** $\sqrt[7]{(x-2)^7} = x - 2$

A. You've just reviewed how to simplify radical expressions of the form $\sqrt[n]{a^n}$

Now try Exercises 13 and 14 ►

B. Radical Expressions and Rational Exponents

As an alternative to radical notation, a rational (fractional) exponent can be used, along with the power property of exponents. For $\sqrt[3]{a^3} = a$, notice that an exponent of one-third can replace the cube root notation and produce the same result: $\sqrt[3]{a^3} = (a^3)^{\frac{1}{3}} = a^{\frac{3}{3}} = a$. In the same way, an exponent of one-half can replace the square root notation: $\sqrt{a^2} = (a^2)^{\frac{1}{2}} = a^{\frac{2}{2}} = |a|$. In general, we have the following:

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Rational Exponents

If a is a real number and n is an integer greater than 1,

then $\sqrt[n]{a} = \sqrt[n]{a^1} = a^{\frac{1}{n}}$

provided $\sqrt[n]{a}$ represents a real number.

EXAMPLE 5 ► Simplifying Radical Expressions Using Rational Exponents

Simplify by rewriting each radicand as a perfect nth power and converting to rational exponent notation.

a. $\sqrt[3]{-125}$ **b.** $-\sqrt[4]{16x^{20}}$ **c.** $\sqrt[4]{-81}$ **d.** $\sqrt[3]{\frac{8w^3}{27}}$

Solution ► **a.** $\sqrt[3]{-125} = \sqrt[3]{(-5)^3}$
$= [(-5)^3]^{\frac{1}{3}}$
$= (-5)^{\frac{3}{3}}$
$= -5$

b. $-\sqrt[4]{16x^{20}} = -\sqrt[4]{(2x^5)^4}$
$= -[(2x^5)^4]^{\frac{1}{4}}$
$= -(2x^5)^{\frac{4}{4}}$
$= -2|x|^5$

c. $\sqrt[4]{-81} = (-81)^{\frac{1}{4}}$
is not a real number

d. $\sqrt[3]{\frac{8w^3}{27}} = \sqrt[3]{\left(\frac{2w}{3}\right)^3}$
$= \left[\left(\frac{2w}{3}\right)^3\right]^{\frac{1}{3}}$
$= \left(\frac{2w}{3}\right)^{\frac{3}{3}}$
$= \frac{2w}{3}$

Now try Exercises 15 and 16 ►

WORTHY OF NOTE

Any rational number can be decomposed into the product of a unit fraction and an integer: $\frac{m}{n} = \frac{1}{n} \cdot m$.

When a rational exponent is used, as in $\sqrt[n]{a} = \sqrt[n]{a^1} = a^{\frac{1}{n}}$, the denominator of the exponent represents the index number, while the numerator of the exponent represents the original power on a. *This is true even when the exponent on a is something other than one!* In other words, the radical expression $\sqrt[4]{16^3}$ can be rewritten as $(16^3)^{\frac{1}{4}} = (16^{\frac{3}{1}})^{\frac{1}{4}}$ or $16^{\frac{3}{4}}$. This is further illustrated in Figure R.6 where we see the rational exponent has the form, "power over root." To evaluate this expression without the aid of a calculator, we use the commutative property to rewrite $(16^{\frac{3}{1}})^{\frac{1}{4}}$ as $(16^{\frac{1}{4}})^{\frac{3}{1}}$ and begin with the fourth root of 16: $(16^{\frac{1}{4}})^{\frac{3}{1}} = 2^3 = 8$.

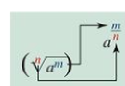

Figure R.6

In general, if m and n have no common factors (other than 1) the expression $a^{\frac{m}{n}}$ can be interpreted in the following two ways.

Rational Exponents

If $\frac{m}{n}$ is a rational number expressed in lowest terms with $n \geq 2$, then

(1) $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ or (2) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

(compute $\sqrt[n]{a}$, then take the mth power) (compute a^m, then take the nth root)

provided $\sqrt[n]{a}$ represents a real number.

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EXAMPLE 6 ► **Simplifying Expressions with Rational Exponents**

Find the value of each expression without a calculator, by rewriting the exponent as the product of a unit fraction and an integer.

a. $27^{\frac{2}{3}}$ **b.** $(-8)^{\frac{4}{3}}$ **c.** $\left(\frac{4x^6}{9}\right)^{\frac{5}{2}}$

Solution ►

a. $27^{\frac{2}{3}} = 27^{\frac{1}{3}\cdot 2} = \left(27^{\frac{1}{3}}\right)^2 = 3^2$ or 9

b. $(-8)^{\frac{4}{3}} = (-8)^{\frac{1}{3}\cdot 4} = [(-8)^{\frac{1}{3}}]^4 = [-2]^4 = 16$

c. $\left(\frac{4x^6}{9}\right)^{\frac{5}{2}} = \left(\frac{4x^6}{9}\right)^{\frac{1}{2}\cdot 5} = \left[\left(\frac{4x^6}{9}\right)^{\frac{1}{2}}\right]^5 = \left[\frac{2|x|^3}{3}\right]^5 = \frac{32|x|^{15}}{243}$

Now try Exercises 17 and 18 ►

WORTHY OF NOTE

While the expression $(-8)^{\frac{1}{3}} = \sqrt[3]{-8}$ represents the real number -2, the expression $(-8)^{\frac{2}{6}} = (\sqrt[6]{-8})^2$ is not a real number, even though $\frac{1}{3} = \frac{2}{6}$. Note that the second exponent is not in lowest terms.

Expressions with rational exponents are generally easier to evaluate if we compute the root first, then apply the exponent. Computing the root first also helps us determine whether or not an expression represents a real number.

EXAMPLE 7 ► **Simplifying Expressions with Rational Exponents**

Simplify each expression, if possible.

a. $-49^{\frac{3}{2}}$ **b.** $(-49)^{\frac{3}{2}}$ **c.** $(-8)^{\frac{2}{3}}$ **d.** $-8^{-\frac{2}{3}}$

Solution ►

a. $-49^{\frac{3}{2}} = -\left(49^{\frac{1}{2}}\right)^3 = -(\sqrt{49})^3 = -(7)^3$ or -343

b. $(-49)^{\frac{3}{2}} = [(-49)^{\frac{1}{2}}]^3, = (\sqrt{-49})^3$ not a real number

c. $(-8)^{\frac{2}{3}} = [(-8)^{\frac{1}{3}}]^2 = (\sqrt[3]{-8})^2 = (-2)^2$ or 4

d. $-8^{-\frac{2}{3}} = -\left(8^{\frac{1}{3}}\right)^{-2} = -(\sqrt[3]{8})^{-2} = -2^{-2}$ or $-\frac{1}{4}$

Now try Exercises 19 through 22 ►

☑ **B. You've just reviewed how to rewrite and simplify radical expressions using rational exponents**

C. Using Properties of Radicals to Simplify Radical Expressions

The properties used to simplify radical expressions are closely connected to the properties of exponents. For instance, the product to a power property holds even when n is a rational number. This means $(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$ and $(4 \cdot 25)^{\frac{1}{2}} = 4^{\frac{1}{2}} \cdot 25^{\frac{1}{2}}$. When the second statement is expressed in radical form, we have $\sqrt{4 \cdot 25} = \sqrt{4} \cdot \sqrt{25}$, with both forms having a value of 10. This suggests the **product property of radicals,** which can be extended to include cube roots, fourth roots, and so on.

Product Property of Radicals

If $\sqrt[n]{A}$ and $\sqrt[n]{B}$ represent real-valued expressions, then

$$\sqrt[n]{AB} = \sqrt[n]{A} \cdot \sqrt[n]{B} \quad \text{and} \quad \sqrt[n]{A} \cdot \sqrt[n]{B} = \sqrt[n]{AB}.$$

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CAUTION Note that this property applies only to a *product* of two terms, not to a sum or difference. In other words, while $\sqrt{9x^2} = |3x|$, $\sqrt{9 + x^2} \neq |3 + x|$!

One application of the product property is to simplify radical expressions. In general, the expression $\sqrt[n]{a}$ is in simplified form if a has no factors (other than 1) that are perfect nth roots.

EXAMPLE 8 ▶ Simplifying Radical Expressions

Write each expression in simplest form using the product property.

a. $\sqrt{18}$ **b.** $5\sqrt[3]{125x^4}$ **c.** $\dfrac{-4 + \sqrt{20}}{2}$

Solution ▶ **a.** $\sqrt{18} = \sqrt{9 \cdot 2}$
$= \sqrt{9}\sqrt{2}$
$= 3\sqrt{2}$

b. $5\sqrt[3]{125x^4} = 5 \cdot \sqrt[3]{125 \cdot x^4}$
These steps can be done mentally. $\begin{cases} = 5 \cdot \sqrt[3]{125} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x^1} \\ = 5 \cdot 5 \cdot x \cdot \sqrt[3]{x} \end{cases}$
$= 25x\sqrt[3]{x}$

c. $\dfrac{-4 + \sqrt{20}}{2} = \dfrac{-4 + \sqrt{4 \cdot 5}}{2}$ look for perfect square factors of 20
$= \dfrac{-4 + 2\sqrt{5}}{2}$ product property
$= \dfrac{2(-2 + \sqrt{5})}{2}$ factor and reduce
$= -2 + \sqrt{5}$ result

Now try Exercises 23 and 24 ▶

WORTHY OF NOTE

For expressions like those in Example 8(c), students must resist the "temptation" to reduce individual terms as in $\dfrac{-4 + \sqrt{20}}{2} \not\rightarrow -2 + \sqrt{20}$. Remember, only *factors* can be reduced.

When radicals are *combined* using the product property, the result may contain a perfect nth root, which should be simplified. Note that the *index numbers must be the same* in order to use this property.

EXAMPLE 9 ▶ Simplifying Radical Expressions

Combine factors using the product property and simplify: $1.2\sqrt[3]{16n^4}\,\sqrt[3]{4n^5}$.

Solution ▶ $1.2\sqrt[3]{16n^4}\,\sqrt[3]{4n^5} = 1.2\sqrt[3]{64 \cdot n^9}$ product property

Since the index is 3, we look for perfect cube factors in the radicand.

$= 1.2\sqrt[3]{64}\,\sqrt[3]{n^9}$ product property
$= 1.2\sqrt[3]{64}\,\sqrt[3]{(n^3)^3}$ rewrite n^9 as a perfect cube
$= 1.2(4)n^3$ simplify
$= 4.8n^3$ result

Now try Exercises 25 and 26 ▶

WORTHY OF NOTE

Rational exponents also could have been used to simplify the expression from Example 9, since $1.2\sqrt[3]{64}\,\sqrt[3]{n^9} = 1.2(4)n^{\frac{9}{3}} = 4.8n^3$. Also see Example 11.

The **quotient property of radicals** can also be established using exponential properties. The fact that $\dfrac{\sqrt{100}}{\sqrt{25}} = \sqrt{\dfrac{100}{25}} = 2$ suggests the following:

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Quotient Property of Radicals

If $\sqrt[n]{A}$ and $\sqrt[n]{B}$ represent real-valued expressions with $B \neq 0$, then

$$\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}} \quad \text{and} \quad \frac{\sqrt[n]{A}}{\sqrt[n]{B}} = \sqrt[n]{\frac{A}{B}}.$$

Many times the product and quotient properties must work together to simplify a radical expression, as shown in Example 10.

EXAMPLE 10 ► Simplifying Radical Expressions

Simplify each expression:

a. $\dfrac{\sqrt{18a^5}}{\sqrt{2a}}$ b. $\sqrt[3]{\dfrac{81}{125x^3}}$

Solution ►

a. $\dfrac{\sqrt{18a^5}}{\sqrt{2a}} = \sqrt{\dfrac{18a^5}{2a}}$
$= \sqrt{9a^4}$
$= 3a^2$

b. $\sqrt[3]{\dfrac{81}{125x^3}} = \dfrac{\sqrt[3]{81}}{\sqrt[3]{125x^3}}$
$= \dfrac{\sqrt[3]{27 \cdot 3}}{5x}$
$= \dfrac{3\sqrt[3]{3}}{5x}$

Now try Exercises 27 and 28 ►

Radical expressions can also be simplified using rational exponents.

EXAMPLE 11 ► Using Rational Exponents to Simplify Radical Expressions

Simplify using rational exponents:

a. $\sqrt{36p^4q^5}$ b. $v\sqrt[3]{v^4}$ c. $\sqrt[3]{\sqrt{x}}$ d. $\sqrt[3]{m}\sqrt{m}$

Solution ►

a. $\sqrt{36p^4q^5} = (36p^4q^5)^{\frac{1}{2}}$
$= 36^{\frac{1}{2}}p^{\frac{4}{2}}q^{\frac{5}{2}}$
$= 6p^2q^{(\frac{4}{2}+\frac{1}{2})}$
$= 6p^2q^2q^{\frac{1}{2}}$
$= 6p^2q^2\sqrt{q}$

b. $v\sqrt[3]{v^4} = v^1 \cdot v^{\frac{4}{3}}$
$= v^{\frac{3}{3}} \cdot v^{\frac{4}{3}}$
$= v^{\frac{7}{3}}$
$= v^{\frac{6}{3}}v^{\frac{1}{3}}$
$= v^2\sqrt[3]{v}$

c. $\sqrt[3]{\sqrt{x}} = \sqrt[3]{x^{\frac{1}{2}}}$
$= (x^{\frac{1}{2}})^{\frac{1}{3}}$
$= x^{\frac{1}{2}\cdot\frac{1}{3}}$
$= x^{\frac{1}{6}}$ or $\sqrt[6]{x}$

d. $\sqrt[3]{m}\sqrt{m} = m^{\frac{1}{3}}m^{\frac{1}{2}}$
$= m^{\frac{1}{3}+\frac{1}{2}}$
$= m^{\frac{5}{6}}$
$= \sqrt[6]{m^5}$

☑ **C.** You've just reviewed how to use properties of radicals to simplify radical expressions

Now try Exercises 29 and 30 ►

D. Addition and Subtraction of Radical Expressions

Since $3x$ and $5x$ are like terms, we know $3x + 5x = 8x$. If $x = \sqrt[3]{7}$, the sum becomes $3\sqrt[3]{7} + 5\sqrt[3]{7} = 8\sqrt[3]{7}$, illustrating how *like* radical expressions can be combined. Like radicals are those that have *the same index and radicand.* In some cases, we can identify like radicals only after radical terms have been simplified.

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EXAMPLE 12 ► **Adding and Subtracting Radical Expressions**

Simplify and combine (if possible).

a. $\sqrt{45} + 2\sqrt{20}$ **b.** $\sqrt[3]{16x^5} - x\sqrt[3]{54x^2}$

Solution ►

a. $\sqrt{45} + 2\sqrt{20} = 3\sqrt{5} + 2(2\sqrt{5})$ simplify radicals: $\sqrt{45} = \sqrt{9 \cdot 5}$; $\sqrt{20} = \sqrt{4 \cdot 5}$

$= 3\sqrt{5} + 4\sqrt{5}$ like radicals

$= 7\sqrt{5}$ result

b. $\sqrt[3]{16x^5} - x\sqrt[3]{54x^2} = \sqrt[3]{8 \cdot 2 \cdot x^3 \cdot x^2} - x\sqrt[3]{27 \cdot 2 \cdot x^2}$

$= 2x\sqrt[3]{2x^2} - 3x\sqrt[3]{2x^2}$ simplify radicals

$= -x\sqrt[3]{2x^2}$ result

☑ **D.** You've just reviewed how to add and subtract radical expressions

Now try Exercises 31 through 34 ►

E. Multiplication and Division of Radical Expressions; Radical Expressions in Simplest Form

Multiplying radical expressions is simply an extension of our earlier work. The multiplication can take various forms, from the distributive property to any of the special products reviewed in Section R.3. For instance, $(A \pm B)^2 = A^2 \pm 2AB + B^2$, even if A or B is a radical term.

EXAMPLE 13 ► **Multiplying Radical Expressions**

Compute each product and simplify.

a. $5\sqrt{3}(\sqrt{6} - 4\sqrt{3})$ **b.** $(2\sqrt{2} + 6\sqrt{3})(3\sqrt{10} + \sqrt{15})$

c. $(x + \sqrt{7})(x - \sqrt{7})$ **d.** $(3 - \sqrt{2})^2$

Solution ►

a. $5\sqrt{3}(\sqrt{6} - 4\sqrt{3}) = 5\sqrt{18} - 20(\sqrt{3})^2$ distribute; $(\sqrt{3})^2 = 3$

$= 5(3)\sqrt{2} - (20)(3)$ simplify: $\sqrt{18} = 3\sqrt{2}$

$= 15\sqrt{2} - 60$ result

b. $(2\sqrt{2} + 6\sqrt{3})(3\sqrt{10} + \sqrt{15}) = 6\sqrt{20} + 2\sqrt{30} + 18\sqrt{30} + 6\sqrt{45}$ F-O-I-L

$= 12\sqrt{5} + 20\sqrt{30} + 18\sqrt{5}$ extract roots *and* simplify

$= 30\sqrt{5} + 20\sqrt{30}$ result

c. $(x + \sqrt{7})(x - \sqrt{7}) = x^2 - (\sqrt{7})^2$ $(A + B)(A - B) = A^2 - B^2$

$= x^2 - 7$ result

d. $(3 - \sqrt{2})^2 = (3)^2 - 2(3)(\sqrt{2}) + (\sqrt{2})^2$ $(A + B)^2 = A^2 + 2AB + B^2$

$= 9 - 6\sqrt{2} + 2$ simplify each term

$= 11 - 6\sqrt{2}$ result

LOOKING AHEAD

Notice that the answer for Example 13(c) contains no radical terms, since the outer and inner products sum to zero. This result will be used to simplify certain radical expressions in this section and later in Chapter 1.

Now try Exercises 35 through 38 ►

One application of products and powers of radical expressions is to evaluate certain quadratic expressions, as illustrated in Example 14.

EXAMPLE 14 ► **Evaluating a Quadratic Expression**

Show that when $x^2 - 4x + 1$ is evaluated at $x = 2 + \sqrt{3}$, the result is zero.

Solution ►

$x^2 - 4x + 1$ original expression

$(2 + \sqrt{3})^2 - 4(2 + \sqrt{3}) + 1$ substitute $2 + \sqrt{3}$ for x

$4 + 4\sqrt{3} + 3 - 8 - 4\sqrt{3} + 1$ multiply

$(4 + 3 - 8 + 1) + (4\sqrt{3} - 4\sqrt{3})$ commutative and associative properties

0 ✓

Now try Exercises 39 through 42 ►

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When we applied the quotient property in Example 10, we obtained a denominator free of radicals. Sometimes the denominator is not automatically free of radicals, and the need to write radical expressions in *simplest form* comes into play. This process is called **rationalizing the denominator.**

Radical Expressions in Simplest Form

A radical expression is in simplest form if:

1. The radicand has no perfect nth root factors.
2. The radicand contains no fractions.
3. No radicals occur in a denominator.

As with other types of simplification, the desired form can be achieved in various ways. If the denominator is a single radical term, we multiply the numerator and denominator by the factors required to eliminate the radical in the denominator [see Examples 15(a) and 15(b)]. If the radicand is a rational expression, it is generally easier to build an equivalent fraction *within the radical* having perfect nth root factors in the denominator [see Example 15(c)].

EXAMPLE 15 ► Simplifying Radical Expressions

Simplify by rationalizing the denominator. Assume $a, x \neq 0$.

a. $\dfrac{2}{5\sqrt{3}}$ **b.** $\dfrac{-7}{\sqrt[3]{x}}$ **c.** $\sqrt[3]{\dfrac{3}{4a^4}}$

Solution ►

a. $\dfrac{2}{5\sqrt{3}} = \dfrac{2}{5\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}}$ multiply numerator and denominator by $\sqrt{3}$

$= \dfrac{2\sqrt{3}}{5(\sqrt{3})^2} = \dfrac{2\sqrt{3}}{15}$ simplify—denominator is now rational

b. $\dfrac{-7}{\sqrt[3]{x}} = \dfrac{-7(\sqrt[3]{x})(\sqrt[3]{x})}{\sqrt[3]{x}(\sqrt[3]{x})(\sqrt[3]{x})}$ multiply using two additional factors of $\sqrt[3]{x}$

$= \dfrac{-7\sqrt[3]{x^2}}{\sqrt[3]{x^3}}$ product property

$= \dfrac{-7\sqrt[3]{x^2}}{x}$ $\sqrt[3]{x^3} = x$

c. $\sqrt[3]{\dfrac{3}{4a^4}} = \sqrt[3]{\dfrac{3}{4a^4} \cdot \dfrac{2a^2}{2a^2}}$ $4 \cdot 2 = 8$ is the smallest perfect cube with 4 as a factor; $a^4 \cdot a^2 = a^6$ is the smallest perfect cube with a^4 as a factor

$= \sqrt[3]{\dfrac{6a^2}{8a^6}}$ the denominator is now a perfect cube—simplify

$= \dfrac{\sqrt[3]{6a^2}}{2a^2}$ result

Now try Exercises 43 and 44 ►

In some applications, the denominator may be a sum or difference containing a radical term. In this case, the methods from Example 15 are ineffective, and instead we multiply by a conjugate since $(A + B)(A - B) = A^2 - B^2$. If either A or B is a square root, the result will be a denominator free of radicals.

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EXAMPLE 16 ► **Simplifying Radical Expressions Using a Conjugate**

Simplify the expression by rationalizing the denominator. Write the answer in exact form and approximate form rounded to three decimal places. $\dfrac{2+\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution ►

$$\frac{2+\sqrt{3}}{\sqrt{6}-\sqrt{2}} = \frac{2+\sqrt{3}}{\sqrt{6}-\sqrt{2}}\cdot\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} \quad \text{multiply by the conjugate of the denominator}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}+\sqrt{18}+\sqrt{6}}{(\sqrt{6})^2-(\sqrt{2})^2} \quad \text{FOIL; difference of squares}$$

$$= \frac{3\sqrt{6}+2\sqrt{2}+3\sqrt{2}}{6-2} \quad \text{simplify}$$

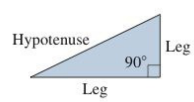
$$= \frac{3\sqrt{6}+5\sqrt{2}}{4} \quad \text{exact form}$$

$$\approx 3.605 \quad \text{approximate form}$$

E. You've just reviewed how to multiply and divide radical expressions and write a radical expression in simplest form

Now try Exercises 45 through 48 ►

F. Formulas and Radicals

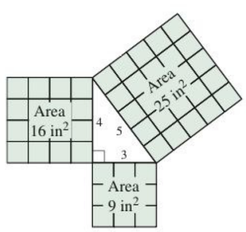


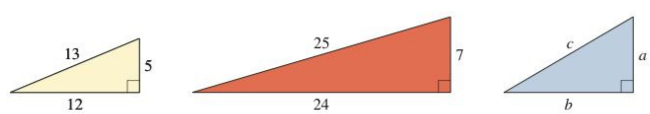
A right triangle is one that has a 90° angle. The longest side (opposite the right angle) is called the **hypotenuse,** while the other two sides are simply called "legs." The **Pythagorean theorem** is a formula that says if you add the square of each leg, the result will be equal to the square of the hypotenuse. Furthermore, we note the converse of this theorem is also true.

Pythagorean Theorem

1. For any right triangle with legs a and b and hypotenuse c, $a^2 + b^2 = c^2$
2. For any triangle with sides a, b, and c, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.

A geometric interpretation of the theorem is given in the figure, which shows $3^2 + 4^2 = 5^2$.





$5^2 + 12^2 = 13^2$; $25 + 144 = 169$✓ | $7^2 + 24^2 = 25^2$; $49 + 576 = 625$✓ | $a^2 + b^2 = c^2$ general case

EXAMPLE 17 ► **Applying the Pythagorean Theorem**

An extension ladder is placed 9 ft from the base of a building in an effort to reach a third-story window that is 27 ft high. What is the minimum length of the ladder required? Answer in exact form using radicals, and approximate form by rounding to one decimal place.

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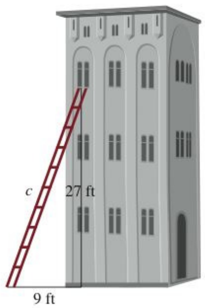

Solution ► We can assume the building makes a 90° angle with the ground, and use the Pythagorean theorem to find the required length. Let c represent this length.

$$c^2 = a^2 + b^2 \quad \text{Pythagorean theorem}$$
$$c^2 = (9)^2 + (27)^2 \quad \text{substitute 9 for } a \text{ and 27 for } b$$
$$c^2 = 81 + 729 \quad 9^2 = 81,\ 27^2 = 729$$
$$c^2 = 810 \quad \text{add}$$
$$c = \sqrt{810} \quad \text{definition of square root; } c > 0$$
$$c = 9\sqrt{10} \quad \text{exact form: } \sqrt{810} = \sqrt{81 \cdot 10} = 9\sqrt{10}$$
$$c \approx 28.5 \text{ ft} \quad \text{approximate form}$$

The ladder must be at least 28.5 ft tall.

Now try Exercises 51 and 52 ►

F. You've just reviewed how to evaluate formulas involving radicals

R.6 EXERCISES

► CONCEPTS AND VOCABULARY

Fill in each blank with the appropriate word or phrase. Carefully reread the section, if necessary.

1. $\sqrt[n]{a^n} = |a|$ if $n > 0$ is a(n) ________ integer.

2. The conjugate of $2 - \sqrt{3}$ is ________.

3. By decomposing the rational exponent, we can rewrite $16^{\frac{3}{4}}$ as $(16^{\frac{?}{?}})^{?}$.

4. $(x^{\frac{3}{2}})^{\frac{2}{3}} = x^{\frac{3}{2} \cdot \frac{2}{3}} = x^1$ is an example of the ________ property of exponents.

5. Discuss/Explain what it means when we say an expression like $\sqrt{A}$ has been written in simplest form.

6. Discuss/Explain why it would be easier to simplify the expression given using rational exponents rather than radicals:

$$\frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}}$$

► DEVELOPING YOUR SKILLS

Evaluate the expression $\sqrt{x^2}$ for the values given.

7. a. $x = 9$ **b.** $x = -10$

8. a. $x = 7$ **b.** $x = -8$

Simplify each expression, assuming that variables can represent any real number.

9. a. $\sqrt{49p^2}$ **b.** $\sqrt{(x - 3)^2}$
c. $\sqrt{81m^4}$ **d.** $\sqrt{x^2 - 6x + 9}$

10. a. $\sqrt{25n^2}$ **b.** $\sqrt{(y + 2)^2}$
c. $\sqrt{v^{10}}$ **d.** $\sqrt{4a^2 + 12a + 9}$

11. a. $\sqrt[3]{64}$ **b.** $\sqrt[3]{-125x^3}$
c. $\sqrt[3]{216z^{12}}$ **d.** $\sqrt[3]{\frac{v^3}{-8}}$

12. a. $\sqrt[3]{-8}$ **b.** $\sqrt[3]{-125p^3}$
c. $\sqrt[3]{27q^9}$ **d.** $\sqrt[3]{\frac{w^3}{-64}}$

13. a. $\sqrt[6]{64}$ **b.** $\sqrt[6]{-64}$
c. $\sqrt[5]{243x^{10}}$ **d.** $\sqrt[5]{-243x^5}$
e. $\sqrt[5]{(k - 3)^5}$ **f.** $\sqrt[6]{(h + 2)^6}$

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14. **a.** $\sqrt[4]{216}$ **b.** $\sqrt[4]{-216}$
c. $\sqrt[5]{1024z^{15}}$ **d.** $\sqrt[5]{-1024z^{20}}$
e. $\sqrt[5]{(q-9)^5}$ **f.** $\sqrt[6]{(p+4)^6}$

15. **a.** $\sqrt[3]{-125}$ **b.** $-\sqrt[4]{81n^{12}}$
c. $\sqrt{-36}$ **d.** $\sqrt{\frac{49v^{10}}{36}}$

16. **a.** $\sqrt[3]{-216}$ **b.** $-\sqrt[4]{16m^{24}}$
c. $\sqrt{-121}$ **d.** $\sqrt{\frac{25x^6}{4}}$

17. **a.** $8^{\frac{2}{3}}$ **b.** $\left(\frac{16}{25}\right)^{\frac{3}{2}}$
c. $\left(\frac{4}{25}\right)^{-\frac{3}{2}}$ **d.** $\left(\frac{-27p^6}{8q^3}\right)^{\frac{2}{3}}$

18. **a.** $9^{\frac{3}{2}}$ **b.** $\left(\frac{4}{9}\right)^{\frac{3}{2}}$
c. $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$ **d.** $\left(\frac{-125v^9}{27w^6}\right)^{\frac{2}{3}}$

19. **a.** $-144^{\frac{3}{2}}$ **b.** $\left(-\frac{4}{25}\right)^{\frac{3}{2}}$
c. $(-27)^{-\frac{2}{3}}$ **d.** $-\left(\frac{27x^3}{64}\right)^{-\frac{4}{3}}$

20. **a.** $-100^{\frac{3}{2}}$ **b.** $\left(-\frac{49}{36}\right)^{\frac{3}{2}}$
c. $(-125)^{-\frac{2}{3}}$ **d.** $-\left(\frac{x^9}{8}\right)^{-\frac{4}{3}}$

Use properties of exponents to simplify. Answer in exponential form without negative exponents.

21. **a.** $\left(2n^2p^{-\frac{2}{5}}\right)^5$ **b.** $\left(\frac{8y^{\frac{3}{4}}}{64y^{\frac{3}{2}}}\right)^{\frac{1}{3}}$

22. **a.** $\left(\frac{24x^{\frac{3}{8}}}{4x^{\frac{1}{2}}}\right)^2$ **b.** $\left(2x^{-\frac{1}{4}}y^{\frac{3}{4}}\right)^4$

Simplify each expression. Assume all variables represent non-negative real numbers.

23. **a.** $\sqrt{18m^2}$ **b.** $-2\sqrt[3]{-125p^3q^7}$
c. $\frac{3}{8}\sqrt[3]{64m^3n^5}$ **d.** $\sqrt{32p^3q^6}$
e. $\frac{-6+\sqrt{28}}{2}$ **f.** $\frac{27-\sqrt{72}}{6}$

24. **a.** $\sqrt{8x^6}$ **b.** $3\sqrt[3]{128a^4b^2}$
c. $\frac{2}{9}\sqrt[3]{27a^2b^6}$ **d.** $\sqrt{54m^6n^8}$
e. $\frac{12-\sqrt{48}}{8}$ **f.** $\frac{-20+\sqrt{32}}{4}$

25. **a.** $2.5\sqrt{18a}\sqrt{2a^3}$ **b.** $-\frac{2}{3}\sqrt{3b}\sqrt{12b^2}$
c. $\sqrt{\frac{x^3y}{3}}\sqrt{\frac{4x^5y}{12y}}$ **d.** $\sqrt[3]{9v^2u}\sqrt[3]{3u^5v^2}$

26. **a.** $5.1\sqrt{2p}\sqrt{32p^5}$ **b.** $-\frac{4}{5}\sqrt{5q}\sqrt{20q^3}$
c. $\sqrt{\frac{ab^2}{3}}\sqrt{\frac{25ab^4}{27}}$ **d.** $\sqrt[3]{5cd^2}\sqrt[3]{25cd}$

27. **a.** $\frac{\sqrt{8m^5}}{\sqrt{2m}}$ **b.** $\frac{\sqrt[3]{108n^4}}{\sqrt[3]{4n}}$
c. $\sqrt{\frac{45}{16x^2}}$ **d.** $12\sqrt[3]{\frac{81}{8z^9}}$

28. **a.** $\frac{\sqrt{27y^7}}{\sqrt{3y}}$ **b.** $\frac{\sqrt[3]{72b^5}}{\sqrt[3]{3b^2}}$
c. $\sqrt{\frac{20}{4x^4}}$ **d.** $-9\sqrt[3]{\frac{125}{27x^6}}$

29. **a.** $\sqrt[5]{32x^{10}y^{15}}$ **b.** $x\sqrt[4]{x^5}$
c. $\sqrt[4]{\sqrt[3]{b}}$ **d.** $\frac{\sqrt[3]{6}}{\sqrt{6}}$
e. $\sqrt{b}\sqrt[4]{b}$

30. **a.** $\sqrt[4]{81a^{12}b^{16}}$ **b.** $a\sqrt[5]{a^6}$
c. $\sqrt{\sqrt[4]{a}}$ **d.** $\frac{\sqrt[3]{3}}{\sqrt[4]{3}}$
e. $\sqrt[3]{c}\sqrt[4]{c}$

Simplify and add (if possible).

31. **a.** $12\sqrt{72}-9\sqrt{98}$
b. $8\sqrt{48}-3\sqrt{108}$
c. $7\sqrt{18m}-\sqrt{50m}$
d. $2\sqrt{28p}-3\sqrt{63p}$

32. **a.** $-3\sqrt{80}+2\sqrt{125}$
b. $5\sqrt{12}+2\sqrt{27}$
c. $3\sqrt{12x}-5\sqrt{75x}$
d. $3\sqrt{40q}+9\sqrt{10q}$

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33. a. $3x\sqrt[3]{54x} - 5\sqrt[3]{16x^4}$
b. $\sqrt{4} + \sqrt{3x} - \sqrt{12x} + \sqrt{45}$
c. $\sqrt{72x^3} + \sqrt{50} - \sqrt{7x} + \sqrt{27}$

34. a. $5\sqrt[3]{54m^3} - 2m\sqrt[3]{16m^3}$
b. $\sqrt{10b} + \sqrt{200b} - \sqrt{20} + \sqrt{40}$
c. $\sqrt{75r^3} + \sqrt{32} - \sqrt{27r} + \sqrt{38}$

Compute each product and simplify the result.

35. a. $(7\sqrt{2})^2$ **b.** $\sqrt{3}(\sqrt{5} + \sqrt{7})$
c. $(n + \sqrt{5})(n - \sqrt{5})$ **d.** $(6 - \sqrt{3})^2$

36. a. $(0.3\sqrt{5})^2$ **b.** $\sqrt{5}(\sqrt{6} - \sqrt{2})$
c. $(4 + \sqrt{3})(4 - \sqrt{3})$ **d.** $(2 + \sqrt{5})^2$

37. a. $(3 + 2\sqrt{7})(3 - 2\sqrt{7})$
b. $(\sqrt{5} - \sqrt{14})(\sqrt{2} + \sqrt{13})$
c. $(2\sqrt{2} + 6\sqrt{6})(3\sqrt{10} + \sqrt{7})$

38. a. $(5 + 4\sqrt{10})(1 - 2\sqrt{10})$
b. $(\sqrt{3} + \sqrt{2})(\sqrt{10} + \sqrt{11})$
c. $(3\sqrt{5} + 4\sqrt{2})(\sqrt{15} + \sqrt{6})$

Use a substitution to verify the solutions to the quadratic equation given.

39. $x^2 - 4x + 1 = 0$
a. $x = 2 + \sqrt{3}$ **b.** $x = 2 - \sqrt{3}$

40. $x^2 - 10x + 18 = 0$
a. $x = 5 - \sqrt{7}$ **b.** $x = 5 + \sqrt{7}$

41. $x^2 + 2x - 9 = 0$
a. $x = -1 + \sqrt{10}$ **b.** $x = -1 - \sqrt{10}$

42. $x^2 - 14x + 29 = 0$
a. $x = 7 - 2\sqrt{5}$ **b.** $x = 7 + 2\sqrt{5}$

Rationalize each expression by building perfect *n*th root factors for each denominator. Assume all variables represent positive quantities.

43. a. $\frac{3}{\sqrt{12}}$ **b.** $\sqrt{\frac{20}{27x^3}}$
c. $\sqrt{\frac{27}{50b}}$ **d.** $\sqrt[3]{\frac{1}{4p}}$ **e.** $\frac{5}{\sqrt[3]{a}}$

44. a. $\frac{-4}{\sqrt{20}}$ **b.** $\sqrt{\frac{125}{12n^3}}$
c. $\sqrt{\frac{5}{12x}}$ **d.** $\sqrt[3]{\frac{3}{2m^2}}$ **e.** $\frac{-8}{3\sqrt[3]{5}}$

Simplify the following expressions by rationalizing the denominators. Where possible, state results in exact form and approximate form, rounded to hundredths.

45. a. $\frac{8}{3 + \sqrt{11}}$ **b.** $\frac{6}{\sqrt{x} - \sqrt{2}}$

46. a. $\frac{7}{\sqrt{7} + 3}$ **b.** $\frac{12}{\sqrt{x} + \sqrt{3}}$

47. a. $\frac{\sqrt{10} - 3}{\sqrt{3} + \sqrt{2}}$ **b.** $\frac{7 + \sqrt{6}}{3 - 3\sqrt{2}}$

48. a. $\frac{1 + \sqrt{2}}{\sqrt{6} + \sqrt{14}}$ **b.** $\frac{1 + \sqrt{6}}{5 + 2\sqrt{3}}$

▶ WORKING WITH FORMULAS

49. Fish length to weight relationship: $L = 1.13(W)^{\frac{1}{3}}$

The length to weight relationship of a female Pacific halibut can be approximated by the formula shown, where *W* is the weight in pounds and *L* is the length in feet. A fisherman lands a halibut that weighs 400 lb. Approximate the length of the fish (round to two decimal places).

50. Timing a falling object: $t = \frac{\sqrt{s}}{4}$

The time it takes an object to fall a certain distance is given by the formula shown, where *t* is the time in seconds and *s* is the distance the object has fallen. Approximate the time it takes an object to hit the ground, if it is dropped from the top of a building that is 80 ft in height (round to hundredths).

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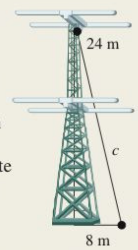
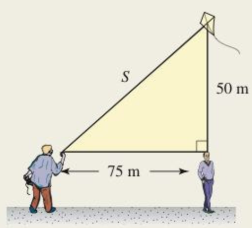
▶ APPLICATIONS

51. Length of a cable: A radio tower is secured by cables that are anchored in the ground 8 m from its base. If the cables are attached to the tower 24 m above the ground, what is the length of each cable? Answer in (a) exact form using radicals, and (b) approximate form by rounding to one decimal place.



52. Height of a kite: Benjamin Franklin is flying his kite in a storm once again. John Adams has walked to a position directly under the kite and is 75 m from Ben. If the kite is 50 m above John Adams' head, how much string S has Ben let out? Answer in (a) exact form using radicals, and (b) approximate form by rounding to one decimal place.



The time T (in days) required for a planet to make one revolution around the sun is modeled by the function $T = 0.407R^{\frac{3}{2}}$, where R is the maximum radius of the planet's orbit (in millions of miles). This is known as *Kepler's third law of planetary motion*. Use the equation given to approximate the number of days required for one complete orbit of each planet, given its maximum orbital radius.

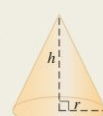
53. a. Earth: 93 million mi
b. Mars: 142 million mi
c. Mercury: 36 million mi

54. a. Venus: 67 million mi
b. Jupiter: 480 million mi
c. Saturn: 890 million mi

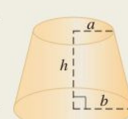
55. Accident investigation: After an accident, police officers will try to determine the approximate velocity V that a car was traveling using the formula $V = 2\sqrt{6L}$, where L is the length of the skid marks in feet and V is the velocity in miles per hour. (a) If the skid marks were 54 ft long, how fast was the car traveling? (b) Approximate the speed of the car if the skid marks were 90 ft long.

56. Wind-powered energy: If a wind-powered generator is delivering P units of power, the velocity V of the wind (in miles per hour) can be determined using $V = \sqrt[3]{\frac{P}{k}}$, where k is a constant that depends on the size and efficiency of the generator. Rationalize the radical expression and use the new version to find the velocity of the wind if $k = 0.004$ and the generator is putting out 13.5 units of power.

57. Surface area: The lateral surface area (surface area excluding the base) S of a cone is given by the formula $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height of the cone. Find the surface area of a cone that has a radius of 6 m and a height of 10 m. Answer in simplest form.



58. Surface area: The lateral surface area S of a frustum (a truncated cone) is given by the formula $S = \pi(a + b)\sqrt{h^2 + (b - a)^2}$, where a is the radius of the upper base, b is the radius of the lower base, and h is the height. Find the surface area of a frustum where $a = 6$ m, $b = 8$ m, and $h = 10$ m. Answer in simplest form.



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The expression $x^2 - 7$ is not factorable using *integer values*. But the expression *can be written* in the form $x^2 - (\sqrt{7})^2$, enabling us to factor it as a "binomial" and its conjugate: $(x + \sqrt{7})(x - \sqrt{7})$. Use this idea to factor the following expressions.

59. a. $x^2 - 5$ **b.** $n^2 - 19$

60. a. $4v^2 - 11$ **b.** $9w^2 - 11$

▶ EXTENDING THE CONCEPT

61. The following terms $\sqrt{3x} + \sqrt{9x} + \sqrt{27x} + \ldots$ form a pattern that continues until the sixth term is found. (a) Compute the sum of all six terms; (b) develop a system (investigate the pattern further) that will enable you to find the sum of 12 such terms *without actually writing out the terms*.

62. Find a quick way to simplify the expression without the aid of a calculator.

$$\left(\left(\left(\left(\left(3^{\frac{5}{6}}\right)^{\frac{3}{2}}\right)^{\frac{4}{5}}\right)^{\frac{3}{4}}\right)^{\frac{2}{5}}\right)^{\frac{10}{3}}$$

63. If $\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)^2 = \frac{9}{2}$, find the value of $x^{\frac{1}{2}} + x^{-\frac{1}{2}}$.

64. Rewrite by rationalizing the *numerator:*

$$\frac{\sqrt{x+h} - \sqrt{x}}{h}$$

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OVERVIEW OF CHAPTER R

Important Definitions, Properties, Formulas, and Relationships

R.1 Notation and Relations

concept	notation	description	example
• Set notation:	$\{members\}$	braces enclose the members of a set	set of even whole numbers $A = \{0, 2, 4, 6, 8, \ldots\}$
• Is an element of	$\in$	indicates membership in a set	$14 \in A$
• Empty set	$\varnothing$ or $\{\ \}$	a set having no elements	odd numbers in A
• Is a proper subset of	$\subset$	indicates the elements of one set are entirely contained in another	$S = \{0, 6, 12, 18, 24, \ldots\}$ $S \subset A$
• Defining a set	$\{x \mid x \ldots\}$	the set of all x, *such that* $x \ldots$	$S = \{x \mid x = 6n \text{ for } n \in \mathbb{W}\}$

R.1 Sets of Numbers

- Natural: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$
- Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- Irrational: $\mathbb{H} =$ {numbers with a nonterminating, nonrepeating decimal form}
- Whole: $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$
- Rational: $\mathbb{Q} = \left\{\frac{p}{q}, \text{ where } p, q \in \mathbb{Z};\ q \neq 0\right\}$
- Real: $\mathbb{R} =$ {all rational and irrational numbers}

R.1 Absolute Value of a Number

$$|n| = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$$

R.1 Distance between *a* and *b* on a number line

$|a - b|$ or $|b - a|$

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R.2 Properties of Real Numbers: For real numbers a, b, and c,

Commutative Property
- Addition: $a + b = b + a$
- Multiplication: $a \cdot b = b \cdot a$

Identities
- Additive: $0 + a = a$
- Multiplicative: $1 \cdot a = a$

Associative Property
- Addition: $(a + b) + c = a + (b + c)$
- Multiplication: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

Inverses
- Additive: $a + (-a) = 0$
- Multiplicative: $\frac{p}{q} \cdot \frac{q}{p} = 1; p, q \neq 0$

R.3 Properties of Exponents: For real numbers a and b, and integers m, n, and p (excluding 0 raised to a nonpositive power),

- Product property: $b^m \cdot b^n = b^{m+n}$
- Product to a power: $(a^m b^n)^p = a^{mp} \cdot b^{np}$
- Quotient property: $\frac{b^m}{b^n} = b^{m-n}$ $(b \neq 0)$
- Negative exponents: $b^{-n} = \frac{1}{b^n}; \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $(a, b \neq 0)$
- Power property: $(b^m)^n = b^{mn}$
- Quotient to a power: $\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}$ $(b \neq 0)$
- Zero exponents: $b^0 = 1$ $(b \neq 0)$
- Scientific notation: $N \times 10^k$; $1 \leq |N| < 10, k \in \mathbb{Z}$

R.3 Special Products

- $(A + B)(A - B) = A^2 - B^2$
- $(A - B)(A^2 + AB + B^2) = A^3 - B^3$
- $(A + B)^2 = A^2 + 2AB + B^2$; $(A - B)^2 = A^2 - 2AB + B^2$
- $(A + B)(A^2 - AB + B^2) = A^3 + B^3$

R.4 Special Factorizations

- $A^2 - B^2 = (A + B)(A - B)$
- $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
- $A^2 \pm 2AB + B^2 = (A \pm B)^2$
- $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

R.5 Rational Expressions: For polynomials P, Q, R, and S with no denominator of zero,

- Lowest terms: $\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q}$
- Multiplication: $\frac{P}{Q} \cdot \frac{R}{S} = \frac{P \cdot R}{Q \cdot S} = \frac{PR}{QS}$
- Addition: $\frac{P}{R} + \frac{Q}{R} = \frac{P + Q}{R}$
- Addition/subtraction with unlike denominators:
  1. Find the LCD of all rational expressions.
  2. Build equivalent expressions using LCD.
  3. Add/subtract numerators as indicated.
  4. Write the result in lowest terms.
- Equivalence: $\frac{P}{Q} = \frac{P \cdot R}{Q \cdot R}$
- Division: $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R} = \frac{PS}{QR}$
- Subtraction: $\frac{P}{R} - \frac{Q}{R} = \frac{P - Q}{R}$

R.6 Properties of Radicals

- $\sqrt{a}$ is a real number only for $a \geq 0$
- $\sqrt[n]{a} = b$, only if $b^n = a$
- For any real number a, $\sqrt[n]{a^n} = |a|$ when n is even
- $\sqrt{a} = b$, only if $b^2 = a$
- If n is even, $\sqrt[n]{a}$ represents a real number only if $a \geq 0$
- For any real number a, $\sqrt[n]{a^n} = a$ when n is odd

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- If a is a real number and n is an integer greater than 1, then $\sqrt[n]{a} = a^{\frac{1}{n}}$ provided $\sqrt[n]{a}$ represents a real number
- If $\sqrt[n]{A}$ and $\sqrt[n]{B}$ represent real numbers, $\sqrt[n]{AB} = \sqrt[n]{A} \cdot \sqrt[n]{B}$
- If $\frac{m}{n}$ is a rational number written in lowest terms with $n \geq 2$, then $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ provided $\sqrt[n]{a}$ represents a real number.
- If $\sqrt[n]{A}$ and $\sqrt[n]{B}$ represent real numbers and $B \neq 0$, $\sqrt[n]{\frac{A}{B}} = \frac{\sqrt[n]{A}}{\sqrt[n]{B}}$
- A radical expression is in simplest form when:
  1. the radicand has no factors that are perfect nth roots,
  2. the radicand contains no fractions, and
  3. no radicals occur in a denominator.

R.6 Pythagorean Theorem

- For any right triangle with legs a and b and hypotenuse c: $a^2 + b^2 = c^2$.
- For any triangle with sides a, b, and c, if $a^2 + b^2 = c^2$, then the triangle is a right triangle.

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PRACTICE TEST

1. State true or false. If false, state why.
   a. $\mathbb{H} \subset \mathbb{R}$ b. $\mathbb{N} \subset \mathbb{Q}$
   c. $\sqrt{2} \in \mathbb{Q}$ d. $\frac{1}{2} \notin \mathbb{W}$
2. State the value of each expression.
   a. $\sqrt{121}$ b. $\sqrt[3]{-125}$
   c. $\sqrt{-36}$ d. $\sqrt{400}$
3. Evaluate each expression:
   a. $\frac{7}{8} - \left(-\frac{1}{4}\right)$ b. $-\frac{1}{3} - \frac{5}{6}$
   c. $-0.7 + 1.2$ d. $1.3 + (-5.9)$
4. Evaluate each expression:
   a. $(-4)\left(-2\frac{1}{3}\right)$ b. $(-0.6)(-1.5)$
   c. $\frac{-2.8}{-0.7}$ d. $4.2 \div (-0.6)$
5. Evaluate using a calculator: $2000\left(1 + \frac{0.08}{12}\right)^{12 \cdot 10}$
6. State the value of each expression, if possible.
   a. $0 \div 6$ b. $6 \div 0$
7. State the number of terms in each expression and identify the coefficient of each.
   a. $-2v^2 + 6v + 5$ b. $\frac{c + 2}{3} + c$
8. Evaluate each expression given $x = -0.5$ and $y = -2$. Round to hundredths as needed.
   a. $2x - 3y^2$ b. $\sqrt{2} - x(4 - x^2) + \frac{y}{x}$
9. Translate each phrase into an algebraic expression.
   a. Nine less than twice a number is subtracted from the number cubed.
   b. Three times the square of half a number is subtracted from twice the number.
10. Create a mathematical model using descriptive variables.
    a. The radius of the planet Jupiter is approximately 119 mi less than 11 times the radius of the Earth. Express the radius of Jupiter in terms of the Earth's radius.
    b. Last year, Video Venue Inc. earned \$1.2 million more than four times what it earned this year. Express last year's earnings of Video Venue Inc. in terms of this year's earnings.
11. Simplify by combining like terms.
    a. $8v^2 + 4v - 7 + v^2 - v$
    b. $-4(3b - 2) + 5b$
    c. $4x - (x - 2x^2) + x(3 - x)$
12. Factor each expression completely.
    a. $9x^2 - 16$ b. $4v^3 - 12v^2 + 9v$
    c. $x^3 + 5x^2 - 9x - 45$
13. Simplify using the properties of exponents.
    a. $\frac{5}{b^{-3}}$ b. $(-2a^3)^2(a^2b^4)^3$
    c. $\left(\frac{m^2}{2n}\right)^3$ d. $\left(\frac{5p^2q^3r^4}{-2pq^2r^4}\right)^2$

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14. Simplify using the properties of exponents.

a. $\dfrac{-12a^3b^5}{3a^2b^4}$

b. $(3.2 \times 10^{-17}) \times (2.0 \times 10^{15})$

c. $\left(\dfrac{a^{-3} \cdot b}{c^{-2}}\right)^{-4}$ **d.** $-7x^0 + (-7x)^0$

15. Compute each product.

a. $(3x^2 + 5y)(3x^2 - 5y)$

b. $(2a + 3b)^2$

16. Add or subtract as indicated.

a. $(-5a^3 + 4a^2 - 3) + (7a^4 + 4a^2 - 3a - 15)$

b. $(2x^2 + 4x - 9) - (7x^4 - 2x^2 - x - 9)$

Simplify or compute as indicated.

17. a. $\dfrac{x - 5}{5 - x}$ **b.** $\dfrac{4 - n^2}{n^2 - 4n + 4}$

c. $\dfrac{x^3 - 27}{x^2 + 3x + 9}$ **d.** $\dfrac{3x^2 - 13x - 10}{9x^2 - 4}$

e. $\dfrac{x^2 - 25}{3x^2 - 11x - 4} \div \dfrac{x^2 + x - 20}{x^2 - 8x + 16}$

f. $\dfrac{m + 3}{m^2 + m - 12} - \dfrac{2}{5(m + 4)}$

18. a. $\sqrt{(x + 11)^2}$ **b.** $\sqrt[3]{\dfrac{-8}{27v^3}}$

c. $\left(\dfrac{25}{16}\right)^{\frac{-3}{2}}$ **d.** $\dfrac{-4 + \sqrt{32}}{8}$

e. $7\sqrt{40} - \sqrt{90}$ **f.** $(x + \sqrt{5})(x - \sqrt{5})$

g. $\sqrt{\dfrac{2}{5x}}$ **h.** $\dfrac{8}{\sqrt{6} - \sqrt{2}}$

19. Maximizing revenue: Due to past experience, the manager of a video store knows that if a popular video game is priced at $30, the store will sell 40 each day. For each decrease of $0.50, one additional sale will be made. The formula for the store's revenue is then $R = (30 - 0.5x)(40 + x)$, where x represents the number of times the price is decreased. Multiply the binomials and use a table of values to determine (a) the number of 50¢ decreases that will give the most revenue and (b) the maximum amount of revenue.

20. Diagonal of a rectangular prism: Use the Pythagorean theorem to determine the length of the diagonal of the rectangular prism shown in the figure. (*Hint:* First find the diagonal of the base.)