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High Frequency Techniques

An Introduction to RF and Microwave Engineering

JOSEPH F. WHITE

HIGH FREQUENCY TECHNIQUES

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to Christopher

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PREFACE

This book is written for the undergraduate or graduate student who wishes to pursue a career in radio-frequency (RF) and microwave engineering. Today's engineer must use the computer as a design tool to be competitive. This text presumes that the student has access to a computer and network simulation software, but the book can be used without them. In either event, this text will prepare the student for the modern engineering environment in which the computer is a tool of daily use.

The computer is used in two ways. First, it performs laborious calculations based on a defined procedure and a set of circuit element values. This is the major use of network simulation, and it is employed throughout this book to show how each network that is described performs over a frequency range. The second way is like the first except that the computer varies the element values either to approach a desired performance goal (optimization) or to show the variation in performance when a quantity of circuits is built using parts whose values vary from piece to piece (yield prediction).

In the second use, the computer is like a thousand monkeys who, it was once postulated, if taught to type, would eventually type all of the world's great literature including an index to the work. But, it was further postulated, they also would type every possible wrong version, incorrect indices included. Today, the engineer's task is to obtain the useful outputs of the computer based on a fundamental understanding of the underlying principles. Within this text, the computer is used as a tool of, not a substitute for, design. This book emphasizes fundamental concepts, engineering techniques, and the regular and intelligent use of the computer as a computational aid.

Within this presentation of theoretical material, computer-generated examples provide insight into the basic performance, bandwidth, and manufacturing yield of RF and microwave networks. This facilitates the evaluation of classical circuit designs and their limitations. However, in modern engineering, rarely is a classical circuit design used in its standard form, although that was necessarily the practice before the availability of personal computers and simulation software. Rather, today the classical design is a point of embarkation from which a specific design is tailored to immediate design needs. The presence of the classical design remains important because it serves as a starting point to define what specifications might reasonably be expected as optimizer goals for the simulation. Effectively, it "gets the thousand monkeys started on the right page."

This book contains a review of wireless history and engineering fundamentals including complex numbers, alternating-current theory, and the logarithmic basis of decibels. All of the text is written in a simple and informal manner so that the presentation of concepts is easy to follow. Many derivations show intermediate steps not usually included in textbooks because the intent is to enlighten, not test, more than need be, the mathematical prowess of the reader. This book also contains exercises that do not have a black or white answer. Exercise questions are asked that require consideration beyond what is covered in the text. This is intentional. It is done to introduce the reader to what happens in the practical realm of engineering.

The reader is cautioned not to interpret the review material and easy readability of this text for a lack of conceptual rigor or thoroughness. As the reader will soon determine, the chapters of this book actually are more encompassing of theoretical concepts and advanced engineering techniques than those of most introductory microwave texts. But the emphasis is on practical technique. For example, the reader will be surprised that, based merely on Q and the complex number conversion between impedance and admittance, a technique called *Q matching* is developed that is familiar to few engineering professionals.

The emphasis of this book is how design challenges would be attacked in a real engineering environment. Some designs, such as distributed filters, are best performed either with proprietary software programs or with the thousand-monkey approach (optimization), but the emphasis is in providing the monkeys with a promising start.

The style of this textbook is derived from a hands-on industrial course that the author has been teaching for some time. In it the student builds on the computer the circuits that are presented, designing them to specifications and verifying how they perform with frequency. This approach quickly builds design confidence in the student. The exercises presented draw from this experience, and they employ the network simulator to reveal both circuit performance and the student's mastery of it. The following paragraphs summarize the major subjects covered.

Chapter 1 contains a review of the origins of wireless transmission. The early and persistent efforts of Guglielmo Marconi in developing radio is an inspiration to engineers today.

Chapter 2 is an engineering review of alternating-current analysis using complex notation (in Appendix B), impedances and decibel, dBm, and dBW measures with the aim of solidifying these basic concepts. Intuitive level proficiency in these fundamentals is as important to microwave and RF engineering as touch typewriting is to efficient writing. Practical realizations of circuit elements are described, including resistors, inductors, and capacitors and their equivalent circuits with parasitic elements. The parasitic reactances of these elements seriously limit their use at high frequencies, and the engineer does well to know these limits and how they come about.

Chapter 3 treats resonators and how their bandwidth is influenced by Q. Based upon the Q ratio of reactance to resistance and the conversion between series and shunt impedances, the scheme called *Q matching* is derived. This enables the engineer to design a *LC* matching network in a few, simply remembered steps.

Chapter 4 introduces distributed circuits based on transmission lines and their properties. This is the beginning of microwave design theory. Important ideas such as wavelength, voltage standing-wave ratio (VSWR), reflections, return loss, mismatch loss, and mismatch error are presented. These are followed by slotted line measurements and the derivation of the telegrapher and transmission line equations. Phase and group velocity concepts and reflection coefficient related to impedance and distributed matching are introduced. The transmission line impedance transformation equation is derived and applied to special cases of easy applicability. Fano's limit is presented. It is an important restriction on the capacity for matching over a frequency band and was derived in terms of reflection coefficient.

Chapter 5 is devoted to the basis and use of the Smith chart, the *sine qua non* for microwave engineers. The Smith chart affords a window into the workings of transmission lines, rendering their very complex impedance transformation behavior clearly understandable with a single diagram. This presentation reveals how the function of the Smith chart in handling impedance transformation arises out of the constant magnitude of the reflection coefficient along a lossless line, that the chart is merely the reflection coefficient plane, a principle often overlooked. Navigating the chart using impedance, admittance, reflection coefficient, and *Q* circles is presented. Matching to complex load impedances, estimating VSWR bandwidth, and developing equivalent circuits are among the illustrated techniques.

Chapter 6 is a presentation of matrix algebra and definitions for the *Z*, *Y*, *ABCD*, *S*, and *T* matrices. Matrix use underlies most circuit derivations and measurement techniques. This chapter demonstrates how and when to use the different matrices and their limitations. For example, it shows how to employ the *ABCD* matrix to derive remarkably general equivalent circuits in just a few steps, such as the lumped equivalent circuit of a transmission line and a perfectly matched, variable attenuator.

Chapter 7 is a very broad presentation of electromagnetic (EM) field theory tailored to the needs of the microwave and RF engineer. It begins with the physics and the defining experiments that led to the formulation of Maxwell's equations, which are then used to derive fundamental results throughout the chapter. This includes the famous wave equation, from which Maxwell was first led to conclude that light and electromagnetic fields were one and the same.

Throughout this book, techniques are introduced as needed. This is particularly true in this chapter. Vector mathematics are presented including the gradient, dot product, cross product, divergence, curl, and Laplacian as they are required to describe EM field properties and relationships. This direct applicability of the vector operations helps to promote a physical understanding of them as well as the electromagnetic field relationships they are used to describe.

The depth of Chapter 7 is unusual for an introductory text. It extends from the most basic of concepts to quite advanced applications. Skin effect, intrinsic impedance of conductors, Poynting's theorem, wave polarization, the derivation of coaxial transmission line and rectangular waveguide propagating fields, Fourier series and Green's functions, higher order modes in circuits, vector potential, antennas, and radio system path loss are developed in mathematical detail.

Under the best of circumstances, field theory is difficult to master. To accommodate this wide range of electromagnetic topics, the mathematical derivations are uncommonly complete, including many intermediate steps often omitted but necessary for efficient reading and more rapid understanding of the principles.

Chapter 7 concludes with an important use of the computer to perform EM field simulation of distributed circuits. This is shown to provide greater design accuracy than can be obtained with conventional, ideal distributed models.

Chapter 8 treats directional couplers, an important ingredient of microwave measurements and systems. This chapter shows how couplers are analyzed and used. It introduces the *even- and odd-mode analysis* method, which is demonstrated by an analysis of the backward wave coupler. The results, rarely found so thoroughly described in any reference, describe an astounding device. The backward wave coupler has perfect match, infinite isolation, and exactly 90° phase split *at all frequencies*. Cohn's reentrant geometry, used to achieve a 3-dB backward wave, 5-to-1 bandwidth coupler is presented. The uses of couplers as power dividers, reflection phase shifter networks, and as impedance measuring elements in network analyzers are also discussed.

Chapter 9 shows the reader how to design filters beginning with low-pass prototypes having maximally flat (Butterworth), equal-ripple (Chebyshev), and near constant delay (Bessel) characteristics. The classic techniques for scaling these filters to high-pass, bandpass, and bandstop filters are provided. The effect of filter Q on insertion loss is demonstrated. The elliptic filter, having equal stopband ripple, is introduced. Identical resonator filters using top coupling are described as a means to extend the practical frequency range of lumped-element designs.

Half-wave transmission line resonators are used to introduce distributed filters. The Richards transformation and Kuroda's identities are presented as a means of translating lumped-element designs to distributed filters. Mumford's quarter-wave stub filters are presented and shown to be a suitable basis for simulation software optimization of equal-ripple and other passband filters. Kuroda's identities are presented in terms of transmission lines rather than the customary, but vague, "unit elements," simplifying their adoption. This permits students to understand and use Kuroda's identities immediately, even proving their validity as one of the exercises.

Chapter 9 is concluded with a treatment of manufacturing yield illustrated using a filter circuit. A special method of integrating the Gaussian, or normal curve, is presented showing how the "one-sigma" specification is used to determine component and circuit yield. The evaluation of the yield of a practical filter circuit using the network simulator is presented. In this process specifications are applied to the circuit and its performance analyzed assuming it is fabricated using a random sample (*Monte Carlo analysis*) of normally distributed components. The resulting yield from 500 circuits so "built" is determined, showing how the effects of component tolerances and specifications on production yields can be determined even before any materials are procured or assembled.

Chapter 10 is applied to transistor amplifiers. The key to amplifier design is the stabilizing and matching of the transistor to its source and load environment, but this must be performed by taking the whole frequency range over which the device has gain into account, a massive calculation task if performed manually. This is handled using S parameters and the network simulator as a design tool. Constant gain and noise figure circles on the Smith chart are described and their design use demonstrated with actual transistor parameters.

The principal design methods including unilateral gain, operating gain, available gain, simultaneously matched, and low noise amplifier techniques are described and demonstrated with available transistor S parameters. Special amplifier topics are presented, including unilateral figure of merit, nonlinear effects, gain saturation, third-order intercept, spurious free dynamic range, and noise limits. The effects of VSWR interaction with cascaded amplifier stages are demonstrated and the use of negative feedback to reduce the VSWR interaction and to design well-matched, broadband amplifiers is shown.

The intent of including so much theoretical and practical material in this text is to provide an immediate familiarity with a wide variety of circuits, their capabilities and limitations, and the means to design them. This permits the engineer to proceed directly to a practical circuit design without the daunting task of researching the material in multiple library references. These topics are illustrated with recommendations on how to use computer optimizations intelligently to direct the computer to search for *circuits whose performance is realistically achievable*.

One could spend years in the microwave engineering practice and not gain experience with this broad a spectrum of topics. The student who reads this book and completes its exercises, in my experience, will be unusually well qualified to embark on a microwave and RF engineering career.

Comments and corrections from readers are welcome.

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The *Smith chart* symbolized on the cover and employed within this text is reproduced through the courtesy of Anita Smith, owner of Analog Instrument Company, Box 950, New Providence, New Jersey 07974. I am happy to acknowledge the late Phillip Smith for this remarkable tool, arguably the most profound insight of the microwave field. Numerous Smith chart matching solutions were performed using the software program *WinSmith* available from Noble Publishing Co., Norcross, Georgia 30071.

All of the circuit simulations have been performed using the Genesys software suite provided through the courtesy of Randall Rhea, founder of Eagleware Inc, Norcross, Georgia 30071. My thanks also go to the members of the Eagleware on-line support team, whose assistance improved the many simulation examples that appear in this text.

My gratitude to Dr. Les Besser who encouraged me to begin microwave teaching and shared with me many RF and microwave facts and design methods. I also thank Gerald DiPiazza for his patience and help in critical field theory development in this text.

I gratefully acknowledge Dr. Peter Rizzi, my colleague and friend, who patiently read the manuscript and made numerous suggestions to improve its readability, usefulness, and accuracy. He directly contributed the portions on noise and noise temperature. Dr. Rizzi is the author of *Microwave Engineering and Passive Circuits*, an important, widely used text that is referenced extensively in these notes. He is a professor of microwaves who is loved by his students. No one but I can appreciate the magnitude of his contributions.

Anyone who has written a book knows how much patience his spouse requires. My thanks and love to Eloise.

THE AUTHOR

Joseph White is an instructor and consultant in the RF and microwave community, also known as the “wireless” industry.

He received the BS EE degree from Case Institute of Technology, the MS EE degree from Northeastern University and the Ph.D. degree from the Electrical Engineering Department of Rensselaer Polytechnic Institute with specialty in electrophysics and engaged in semiconductor engineering at M/A-

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He also wrote *Microwave Semiconductor Engineering*, a textbook in its third printing since 1977.

He has taught courses on RF and microwave engineering at both the introductory and advanced engineering levels. He has lectured in the United States and internationally on microwave subjects for more than 30 years.

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Questions, corrections and comments about this book are welcome. Please e-mail them to the author at jfwhite@ieee.org.

CHAPTER 1

Introduction

1.1 BEGINNING OF WIRELESS

> WIRELESS TELEGRAPHY—At a time when relations are strained between Spain and this country, nothing could be more welcome than a practical method of carrying on electrical communication between distant points on land, and between ships at sea, without any prearranged connection between the two points. During the last year Guglielmo Marconi, an Italian student, developed a system of wireless telegraphy able to transmit intelligible Morse signals to a distance of over ten miles. It has been left, however, for an American inventor to design an apparatus suitable to the requirements of wireless telegraphy in this country. After months of experimenting, Mr. W. J. Clarke, of the United States Electrical Supply Company, has designed a complete wireless telegraphy apparatus that will probably come rapidly into use.
>
> —Scientific American April, 1898

This announcement appeared near the beginning of radio technology. Webster's dictionary [1] lists over 150 definitions that begin with the word *radio*, the first being:

> 1a. ... the transmission and reception of electric impulses or signals by means of electromagnetic waves without a connecting wire (includes wireless, television and radar).

This remains today the real definition of *wireless* and, equivalently, *radio*. Today the uses of radio communication include not only the broadcast of sound through amplitude modulation (AM) and frequency modulation (FM) radio and video through television, but also a broad collection of radio applications, cordless telephones, cell phones, TV, and VCR remotes, automobile remote door locks, garage door openers, and so on.

There is some question about who actually invented radio as a communica-

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tive method. Mahlon Loomis, a dentist, experimented with wireless telegraphy using wires supported by kites and a galvanometer to sense the changes in current flow in a second wire when the ground connection of the first was interrupted. He received a patent in 1873 for this system [2].

James Clerk Maxwell [3], more about Maxwell's equations later, predicted the propagation of electromagnetic waves through a vacuum in about 1862. Nathan Stubblefield, a Kentucky farmer and sometimes telephone repairman, demonstrated wireless telephony as early as 1892, but to only one man, and in 1902 to a group [2].

Alexander Popov is said to have "utilized his equipment to obtain information for a study of atmospheric electricity ... On 7 May 1895, in a lecture before the Russian Physicist Society of St. Petersburg, he stated he had transmitted and received signals at an intervening distance of 600 yards" [4]. In 1888 Heinrich Hertz conducted an experimental demonstration in a classroom at Karlsruhe Polytechnic in Berlin of the generation and detection of the propagating electromagnetic waves predicted by Maxwell [2].

Sir Oliver Lodge, a professor at Liverpool University was experimenting with wireless telegraphy in 1888, and he patented a system in 1897. Marconi purchased his patent in 1911 [2].

In the public mind Guglielmo Marconi enjoys the most credit for "inventing" radio. He was awarded patents for it; therefore, the Patent Office believed that he had made radio-related inventions. However, the U.S. Navy report [4] states

> Marconi can scarcely be called an inventor. His contribution was more in the fields of applied research and engineering development. He possessed a very practical business acumen, and he was not hampered by the same driving urge to do fundamental research, which had caused Lodge and Popoff to procrastinate in the development of a commercial radio system.

This is perhaps the most accurate description of Marconi's role in developing radio technology, a new communication medium. Nikola Tesla had earlier patents, although the focus of his work appears to have been directed to the transmission of power rather than to communication via radio waves. Tesla, well known for his *Tesla coil* that generated high voltages, actually detected signals consisting of noise bursts, resulting from the large atmospheric electrical discharges he originated, that had traveled completely around the earth. In 1943 the U.S. Supreme Court ruled that Marconi's patents were invalid due to Tesla's prior descriptions, but by that time both Marconi and Tesla were deceased [2].

From its beginnings around 1900, radio moved out to fill many communicative voids. In 1962 George Southworth, a well-known researcher in the field of microwaves, wrote a book about his 40 years of experience in the field [5, p. 1]. He begins:

> One of the more spectacular technical developments of our age has been radio. Beginning about the turn of the century with ship-to-shore telegraphy, radio has been extended through the years to intercontinental telegraphy, to broadcasting, to radio astronomy and to satellite communications.

Today, after an additional 40 years, Southworth could make a much longer list of radio applications. It would include garage door openers, global positioning satellites, cellular telephones, wireless computer networks, and radar applications such as speed measurement, ship and aircraft guidance, military surveillance, weapon directing, air traffic control, and automobile anticollision systems. The frequency spectrum for practical wireless devices has expanded as well. Amplitude modulated radio begins at 535 kHz and television remote controls extend into the infrared.

The advance of wireless applications is not complete and probably never will be. Certainly the last decade has seen an explosive growth in applications. And the quantities of systems has been extraordinary, too. Witness the adoption of the cellular telephone, which today rivals the wired telephone in numbers of applications.

Sending signals over telegraph wires formed the basis for the early wireless technology to follow. Using the Current International Morse code characters for the early Morse code message transmitted over the first telegraph wires, the first message inaugurating service between Baltimore and Washington, D.C., in 1843, would have looked like

.-- .... .- -   .... .- - ....   --. --- -.. .-- .-. --- ..- --. .... -   ..--..
W h a t   h a t h   G o d w r o u g h t   ?

Most of the full code cipher is shown in Figure 1.1-1. Morse code remains useful, although fewer individuals can interpret it on the fly. A distress signal using the code in Figure 1.1-1 can be sent using a transmitting radio or even a flashlight. Marconi's early wireless transmissions used pulse code modulation,

A .-	*K* -.-	*U* ..-	*5*	,	*(COMMA)*	--..--
B -...	*L* .-..	*V* ...-	*6* -....	.	*(PERIOD)*	.-.-.-
C -.-.	*M* --	*W* .--	*7* --...	?		..--..
D -..	*N* -.	*X* -..-	*8* ---..	;		-.-.-.
E .	*O* ---	*Y* -.--	*9* ----.	:		---...
F ..-.	*P* .--.	*Z* --..	*0*-----	’	*(APOSTROPHE)*	.----.
G --.	*Q* --.-	*1* .----		-	*(HYPHEN)*	-....-
H	*R* .-.	*2* ..---		/	*(slash)*	-..-.
I ..	*S* ...	*3* ...--		*(or)*	*PARENTHESIS*	-.--.-
J .---	*T* -	*4* ....-		______	*UNDERLINE*	..--.-

Figure 1.1-1 International Morse Code remains a standard for distress signals, S.O.S. is (... --- ...) *(English Characters, [1]). Derived from the work of Samuel Morse (1791–1872).*

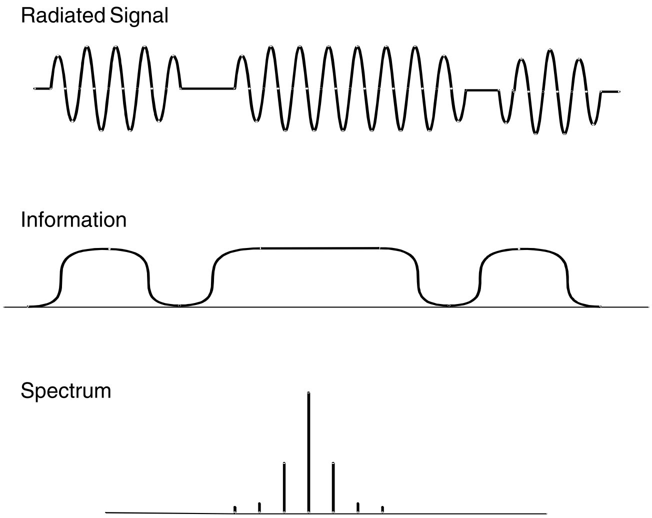

Figure 1.1-2 Modulation format for Morse code, illustrated for letter *R*. Today, pulse shaping, as suggested above, would be employed to reduce transmission spectrum, but Marconi's spark gap transmitter doubtless spanned an enormously wide bandwidth.

dots and dashes achieved by keying the transmitter on and off. Some nautical buoys are identifiable by the Morse letter that their lights flash.

Today, Marconi would need a transmitting license, and were he to continue with his prior transmission technique, his license almost certainly would be suspended due to the broad spectrum of his transmissions (Fig. 1.1-2). His RF source was a spark gap oscillator (Fig. 1.1-3), likely occupying a very broad transmission bandwidth. Powered by a several horsepower generator, the operating transmitter was audible without a radio receiver for several miles.

Marconi had his pivotal triumph in December, 1901, when the Morse character "s" was received at St. John's, Newfoundland (Figs. 1.1-4 and 1.1-5). It was transmitted from Poldhu, Cornwall England, 1800 miles across the Atlantic Ocean [5, p. 13; 6, p. 4]. From the South Wellfleet station, Marconi, himself, transmitted the first trans-Atlantic message on January 17, 1903, a communication from the president of the United States to the king of England.

1.2 CURRENT RADIO SPECTRUM

Today's radio spectrum is very crowded. Obtaining a commercial license to radiate carries the obligation to use bandwidth efficiently, using as little bandwidth as practical to convey the information to be transmitted (Tables 1.2-1 and 1.2-2).

Just the frequency allocations for the United States alone cannot be placed in a table of reasonable size. They occupy numerous pages of the *Rules and*

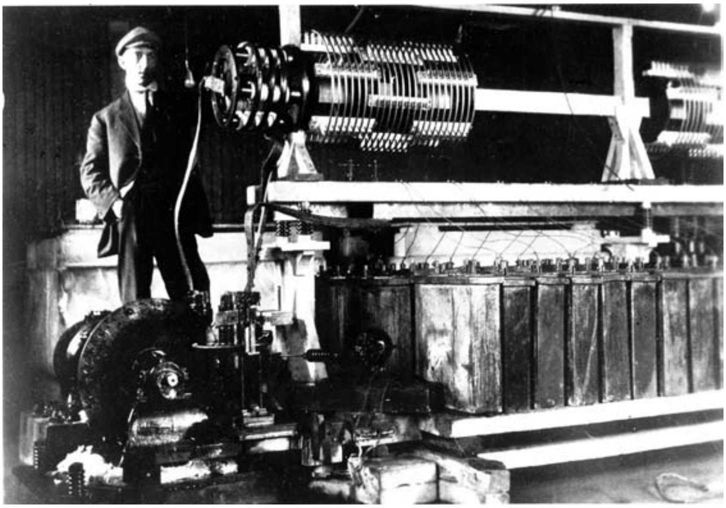

Figure 1.1-3 Joel Earl Hudson standing by Marconi's spark gap transmitter in 1907. (*Photo courtesy of Cape Cod National Seashore.*)

Regulations of the Federal Communications Commission, and have hundreds of footnotes. Since frequent changes are made in the rules and regulations, the latest issue always should be consulted [7, p. 1.8; 8].

As can be seen from Table 1.2-3, radio amateurs today enjoy many frequency allocations. This is due to the history of their pioneering efforts, partic-

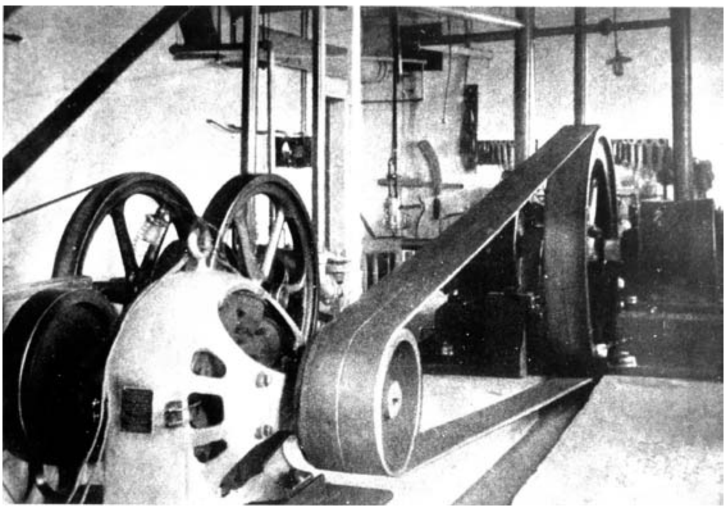

Figure 1.1-4 Prime power for Marconi's South Wellfleet transmitter. (*Photo courtesy of Cape Cod National Seashore.*)

Figure 1.1-5 Marconi's first wireless station in South Wellfleet, Cape Cod, Massachusetts. Local residents predicted that antennas would blow down in first good storm. They did, and he rebuilt them. (*Photo courtesy of Cape Cod National Seashore*.)

ularly at the higher frequencies. We owe much of the rapid development of short-wave radio to the experimental enterprise of amateur radio operators. George Southworth [5, p. 83] pointed out that, in about 1930:

> It is interesting that while the telephone people [researchers at the Bell Telephone Laboratories] were conducting intensive research on the lower frequencies ... much was happening in the outside world at higher frequencies.... It is said that the advantages of short waves were first discovered by an amateur who had built for himself a short-wave receiver and upon listening had found that he could hear the harmonics of distant broadcasting stations ... at distances far beyond those at which the fundamentals could be heard. Amateurs later built for themselves short-wave transmitters and soon thereafter carried on two-way communication.

Today, the electromagnetic spectrum is like a superhighway. There are only so many lanes and only so much traffic that it can sustain if everyone is to enjoy rapid and efficient transport.

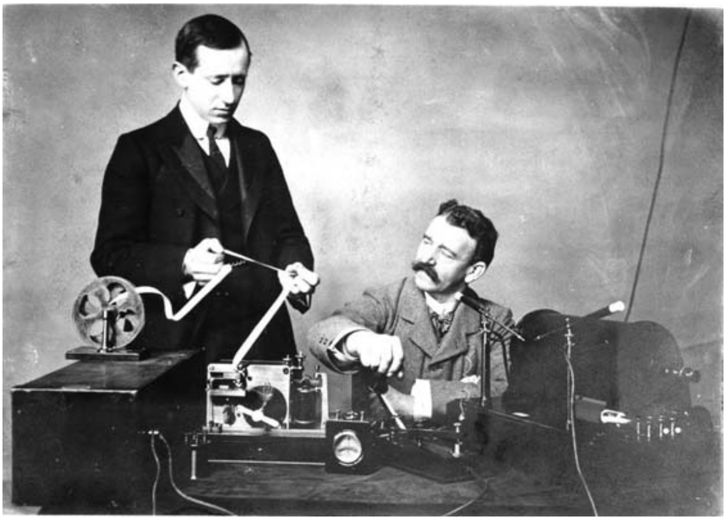

Figure 1.1-6 Guglielmo Marconi (left) received the Nobel Prize for his wireless communication work. He is shown in a 1901 photo with assistant George Kemp shortly after a successful wireless transmission test. (*Photo courtesy of Marconi, Ltd.*, UK.)

The simultaneous functioning of the intricate grid of radiation allocations, only a part of which are shown in Table 1.2-3, depend upon each user occupying his or her precise frequency, modulation format, bandwidth, and effective radiated power and, furthermore, not intruding on other frequency bands by generating spurious signals with his or her equipment. This is the task and challenge of today's high frequency engineering.

TABLE 1.2-1 General Frequency Band Designations

f	λ	Band	Description
30–300 Hz	10^4–10^3 km	ELF	Extremely low frequency
300–3000 Hz	10^3–10^2 km	VF	Voice frequency
3–30 kHz	100–10 km	VLF	Very low frequency
30–300 kHz	10–1 km	LF	Low frequency
0.3–3 MHz	1–0.1 km	MF	Medium frequency
3–30 MHz	100–10 m	HF	High frequency
30–300 MHz	10–1 m	VHF	Very high frequency
300–3000 MHz	100–10 cm	UHF	Ultra-high frequency
3–30 GHz	10–1 cm	SHF	Superhigh frequency
30–300 GHz	10–1 mm	EHF	Extremely high frequency (millimeter waves)

Source: From Reference [7, Section 1].

TABLE 1.2-2 Microwave Letter Bands

f (GHz)	Letter Band Designation
1–2	L band
2–4	S band
4–8	C band
8–12.4	X band
12.4–18	Ku band
18–26.5	K band
26.5–40	Ka band

Source: From Reference [9, p. 123].

1.3 CONVENTIONS USED IN THIS TEXT

This section lists the notational conventions used throughout this text.

Sections

Sections use a decimal number. To the left of the decimal is the chapter number and to the right is the section number. Thus, 7.10 refers to the tenth section in Chapter 7.

Equations

Equations have a number sequence that restarts in each section. Therefore, a reference to (7.15-4) is directed to the fourth equation in Section 7.15.

Figures

Figure and table numbering also restarts in each section. Therefore, a reference to Figure 7.24-2 relates to the second figure in Section 7.24.

Exercises

The exercises at the end of each chapter are numbered according to the section to which they most closely relate. For example, the exercise numbered E3.5-1 is the first exercise relating to the material in Section 3.5. Material contained in prior sections also may be needed to complete the exercise.

Symbols

The principal symbols used in this text and the quantities that they represent are listed in Appendix A. For example, c refers to the velocity of electromagnetic propagation in free space, while v refers to the velocity of propagation in

TABLE 1.2-3 Selected U.S. Radio Frequency Allocations

Frequencies in kHz	Allocated Purposes
490–510	Distress (telegraph)
510–535	Government
535–1605	AM radio
1605–1750	Land/mobile public safety
1800–2000	Amateur radio
Frequencies in MHz	**Allocated Purposes**
26.96–27.23, 462.525–467.475	Citizen band radios
30.56–32, 33–34, 35–38, 39–40, 40.02–40.98, 41.015–46.6, 47–49.6, 72–73, 74.6–74.8, 75.2–76, 150.05–156.2475, 157.1875–161.575, 162.0125–173.4	Private mobil radio (taxis, trucks, buses, railroads)
220–222, 421–430, 451–454, 456–459, 460–512	
746–824, 851–869, 896–901, 935–940	
74.8–75.2, 108–137, 328.6–335.4, 960–1215, 1427–1525, 220–2290, 2310–2320, 2345–2390	Aviation (communication and radar)
162.0125–173.2	Vehicle recovery (LoJack)
50–54, 144–148, 216–220, 222–225, 420–450, 902–928, 1240–1300, 2300–2305, 2390–2450	Amateur radio
72–73, 75.2–76, 218–219	Radio control (personal)
54–72, 76–88, 174–216, 470–608	Television broadcasting VHF and UHF
88–99, 100–108	FM radio broadcasting
824–849	Cellular telephones
1850–1990	Personal communications
1910–1930, 2390–2400	Personal comm. (unlicensed)
1215–1240, 1350–1400, 1559–1610	Global Positioning Systems (GPS)
Frequencies in GHz	**Allocated Purposes**
0.216–0.220, 0.235–0.267, 0.4061–0.45, 0.902–0.928, 0.960–1.215, 1.215–2.229, 2.320–2.345, 2.360–2.390, 2.7–3.1, 3.1–3.7, 5.0–5.47, 5.6–5.925, 8.5–10, 10.0–10.45, 10.5–10.55, 13.25–13.75, 14–14.2, 15.4–16.6, 17.2–17.7, 24.05–24.45, 33.4–36, 45–46.9, 59–64, 66–71, 76–77, 92–100	Radar, all types
2.390–2.400	LANs (unlicensed)
2.40–2.4835	Microwave ovens
45.5–46.9, 76–77, 95–100, 134–142	Vehicle, anticollision, navigation
10.5–10.55, 24.05–24.25	Police speed radar
0.902–0.928, 2.4–2.5, 5.85–5.925	Radio frequency identification (RFID)
3.7–4.2, 11.7–12.2, 14.2–14.5, 17.7–18.8, 27.5–29.1, 29.25–30, 40.5–41.5, 49.2–50.2	Geostationary satellites with fixed earth receivers

TABLE 1.2-3 *(Continued)*

Frequencies in GHz	Allocated Purposes
1.610–1626.5, 2.4835–2.5, 5.091–5.25, 6.7–7.075, 15.43–15.63	Nongeostationary satellites, mobile receivers (big LEO, global phones)
0.04066–0.0407, 902–928, 2450–2500, 5.725–5.875, 24–24.25, 59–59.9, 60–64, 71.5–72, 103.5–104, 116.5–117, 122–123, 126.5–127, 152.5–153, 244–246	Unlicensed industrial, scientific, and medical communication devices
3.3–3.5, 5.65–5.925, 10–10.5, 24–24.25, 47–47.2	Amateur radio
6.425–6.525, 12.7–13.25, 19.26–19.7, 31–31.3	Cable television relay
27.5–29.5	Local multipoint TV distribution
12.2–12.7, 24.75–25.05, 25.05–25.25	Direct broadcast TV (from satellites)
0.928–0.929, 0.932–0.935, 0.941–0.960, 1.850–1.990, 2.11–2.20, 2.450–2.690, 3.7–4.2, 5.925–6.875, 10.55–10.68, 10.7–13.25, 14.2–14.4, 17.7–19.7, 21.2–23.6, 27.55–29.5, 31–31.3, 38.6–40	Fixed microwave (public and private)

a medium for which the relative dielectric and permeability constants may be greater than unity.

Prefixes

Except where noted otherwise, this text uses the International System of Units (SI). Standard prefixes are listed in Table 1.3-1.

Fonts

The font types used throughout this text to connote variable types are listed in Table 1.3-2. Combinations of these representational styles are used to convey the dual nature of some variables. For example, in Maxwell's equation

$$\nabla \cdot \vec{\mathrm{D}} = \rho$$

$\vec{\mathrm{D}}$ is written in regular type because the equation applies to all time waveforms, not just sinusoidal variations, and $\vec{\mathrm{D}}$ is also a vector quantity. On the other hand, the Helmholtz equation is written

$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

TABLE 1.3-1 Standard Prefixes

Prefix	Abbreviation	Factor
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
hecto	h	10^{2}
deka	da	10
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto		10^{-18}

using italic type for the variable $\vec{E}$ because *this equation only applies for sinusoidal time variations*, and therefore the components of the vector $\vec{E}$ are *phasor quantities*.

Throughout this text, except where otherwise noted, the magnitudes of sinusoidal waveforms (V, I, E, D, H, B) are peak values. To obtain root-mean-square (rms) values, divide these values by $\sqrt{2}$.

1.4 VECTORS AND COORDINATES

General vector representations are three dimensional. They can be described by any three-dimensional, *orthogonal coordinate system* in which each coordinate direction is at right angles to the other two. Unless otherwise specified, *rectan-*

TABLE 1.3-2 Fonts Used in This Text to Identify Variable Types

Variable Type	Font	Examples
DC or general time-varying function (not sinusoidal)	Regular type	$\mathrm{V, I, H, E, B, D}$
Explicit general time variation	Regular type, lowercase	$\mathrm{v(t), i(t)}$
Explicit sinusoidal time variation	Italic type, lowercase	$v(t), i(t)$
Phasors, impedance, admittance, general functions, and variables, unit vectors	Italic type	V, I, H, E, B, D, Z, Y $f(x), g(y), x, y, z, \vec{x}, \vec{y}, \vec{z}$
Vectors	$\overrightarrow{\text{Arrow above}}$	$\vec{\mathrm{E}}, \vec{\mathrm{H}}, \vec{\mathrm{B}}, \vec{\mathrm{D}}, \vec{E}, \vec{H}, \vec{B}, \vec{D}$
Normalized parameters	Lowercase	$z = Z/Z_0,\ y = Y/Y_0$

gular (Cartesian) coordinates are implied. Certain circular and spherical symmetries of a case can make its analysis and solution more convenient if the geometry is described in *cylindrical coordinates* or *spherical coordinates*.

In this text all coordinate systems are *right-handed orthogonal coordinate systems*. That is,

> In a right-hand orthogonal coordinate system, rotating a vector in the direction of any coordinate into the direction of the next named coordinate causes a rotational sense that would advance a right-hand screw in the positive direction of the third respective coordinate.

We define that *unit vectors are vectors having unity amplitude and directions in the directions of the increasing value of the respective variables that they represent*.

In rectangular coordinates (Fig. 1.4-1) the order is (x, y, z) and an arbitrary point is written as $P(x, y, z)$. The unit vectors in these respective directions are $\vec{x}$, $\vec{y}$, and $\vec{z}$. Thus, a three-dimensional vector field $\vec{\mathrm{E}}$ can be written

$$\vec{\mathrm{E}} = \vec{\mathrm{E}}_x + \vec{\mathrm{E}}_y + \vec{\mathrm{E}}_z \tag{1.4-1a}$$

or

$$\vec{\mathrm{E}} = \mathrm{E}_x\vec{x} + \mathrm{E}_y\vec{y} + \mathrm{E}_z\vec{z} \tag{1.4-1b}$$

or

$$\vec{\mathrm{E}} = \vec{x}\mathrm{E}_x + \vec{y}\mathrm{E}_y + \vec{z}\mathrm{E}_z \tag{1.4-1c}$$

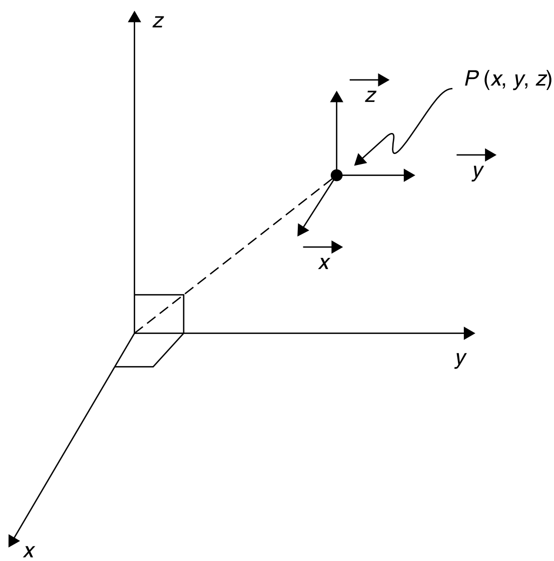

Figure 1.4-1 Rectangular (Cartesian) right-hand coordinate system.

Generally, the format of (1.4-1c) is used in this text. In the language of vector mathematics, rotating a unit vector $\vec{x}$ in the direction of another unit vector $\vec{y}$ is called *crossing* $\vec{x}$ into $\vec{y}$, and this is written as $\vec{x} \times \vec{y}$. This is a specific example of the *vector cross product*. The vector cross product can be applied to any two vectors having any magnitudes and relative orientations; but, in general, we must take into account the product of their magnitudes and the angle between them, as will be shown more specifically for the vector cross product in Chapter 7. For present purposes, since $\vec{x}$, $\vec{y}$, and $\vec{z}$ form a right-hand orthogonal set of unit vectors, we can express the right-handedness of their coordinate system by requiring that the following cross product relations apply:

$$\vec{x} \times \vec{y} = \vec{z} \tag{1.4-2a}$$

$$\vec{y} \times \vec{z} = \vec{x} \tag{1.4-2b}$$

$$\vec{z} \times \vec{x} = \vec{y} \tag{1.4-2c}$$

Notice that *the vector cross product yields a new vector that is orthogonal to the plane of the crossed vectors and in a direction that would be taken by the advance of a right-hand screw when the first vector is crossed into the second.*

Also notice that *for a right-hand coordinate system any coordinate unit vector can be crossed into the next named coordinate vector to yield the direction of positive increase of the remaining coordinate, beginning with any coordinate*. For example (x, y, z), (y, z, x), or (z, x, y) all satisfy the right-hand advancing rule, as specified by (1.4-2a) to (1.4-2c).

The cylindrical coordinate system is shown in Figure 1.4-2. The order of coordinate listing is (r, φ, z) and the unit vectors are $\vec{r}, \vec{\varphi}$, and $\vec{z}$, which satisfy

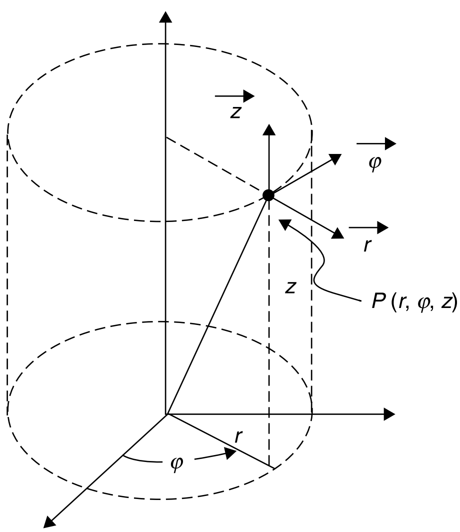

Figure 1.4-2 Cylindrical right-hand coordinate system.

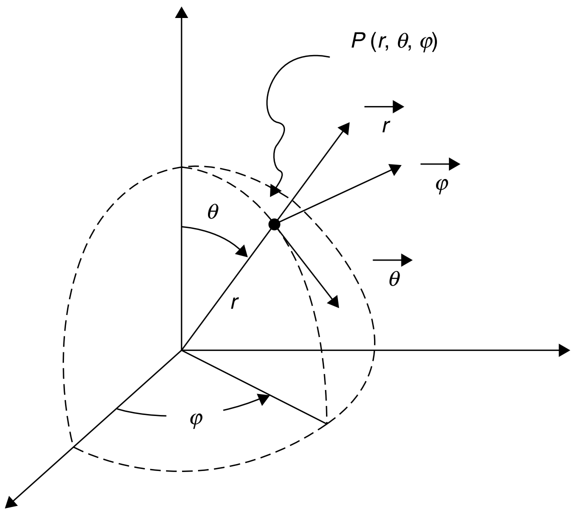

Figure 1.4-3 Spherical right-hand coordinate system.

the same sequential cross-product rules as do rectangular coordinates, namely $\vec{r} \times \vec{\varphi} = \vec{z}$, $\vec{\varphi} \times \vec{z} = \vec{r}$, and $\vec{z} \times \vec{r} = \vec{\varphi}$.

The spherical coordinate system is shown in Figure 1.4-3. The order of coordinate listing is (r, θ, φ) and the unit vectors are $\vec{r}, \vec{\theta}$, and $\vec{\varphi}$, which satisfy the sequential cross-product rules $\vec{r} \times \vec{\theta} = \vec{\varphi}$, $\vec{\theta} \times \vec{\varphi} = \vec{r}$, and $\vec{\varphi} \times \vec{r} = \vec{\theta}$. *Note that this r is not the same as the r used in cylindrical coordinates.*

1.5 GENERAL CONSTANTS AND USEFUL CONVERSIONS

There are several values of physical constants, conversion factors, and identities useful to the practice of microwave engineering. For ready reference, a selection of them is printed on the inside covers of this text.

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CHAPTER 2

Review of AC Analysis and Network Simulation

Alternating current (AC) circuit analysis is the basis for the high frequency techniques that are covered in subsequent chapters and the subject of this text. It is assumed that all readers already have been introduced to AC analysis. However, it has been the author's teaching experience that a review is usually appreciated because it provides an opportunity to put into perspective the fundamentals needed for the fluent application of AC analysis, a skill essential to the high frequency engineer.

The AC analysis makes use of complex numbers to calculate and keep track of the relative magnitudes and phases of voltages and currents. For this reason, a summary of complex mathematics is included in Appendix B. One who comfortably reads this chapter and Appendix B can follow the remainder of this book. The properties of complex equations, such as the bilinear transformation that defines the Smith chart, were developed by mathematicians probably without any practical application considerations [1].

2.1 BASIC CIRCUIT ELEMENTS

The basic building blocks of electric circuits are the resistor R, the inductor L, and the capacitor C (Fig. 2.1-1). At high frequencies these elements do not behave as pure R, L, and C components but have additional resistances and reactances called *parasitics*. More about parasitics later in the chapter.

The Resistor

The resistor passes a current I equal to the applied voltage divided by its resistance. This can be considered a definition of the resistor. Mathematically, this is written

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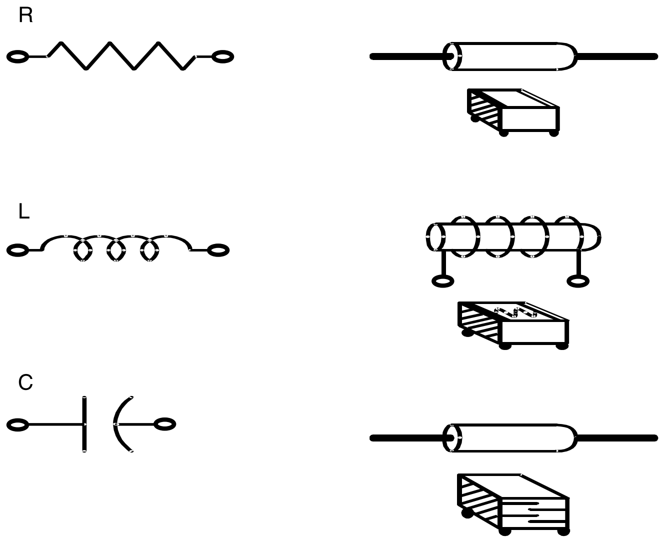

Figure 2.1-1 Practical resistor, inductor, and capacitor, the basic passive, lumped elements.

$$\mathrm{I} \equiv \frac{\mathrm{V}}{\mathrm{R}} \tag{2.1-1}$$

Notice that regular type (not italic) is used because (2.1-1) applies for direct currents (DC) as well as every time-varying waveform. For AC signals the current through and voltage across an ideal resistor are in phase (Fig. 2.1-2).

Regardless of the time variation of voltage and current, the instantaneous power dissipated in a resistor is

$$\mathrm{P(t) = v(t)i(t)} \tag{2.1-2}$$

where v(t) is the instantaneous voltage across the element and i(t) is the current through it at the same instant. With sinusoidal excitation the *average power* dissipated, P_{AVG} in any two-terminal element is

$$P_{\mathrm{AVG}} = \tfrac{1}{2} VI \cos\theta \tag{2.1-3}$$

where V and I are the peak values of voltage and current and θ is the phase angle between V and I. Frequently, *root-mean-squared (rms)* values are used to describe the voltage and current magnitudes. The rms value provides the same average power as a DC voltage or current of the same amplitude. For sinusoidal variations, the rms value is related to the peak value by

$$V_{\mathrm{rms}} = \frac{V_{\mathrm{peak}}}{\sqrt{2}} \quad \text{and} \quad I_{\mathrm{rms}} = \frac{I_{\mathrm{peak}}}{\sqrt{2}} \tag{2.1-4a, b}$$

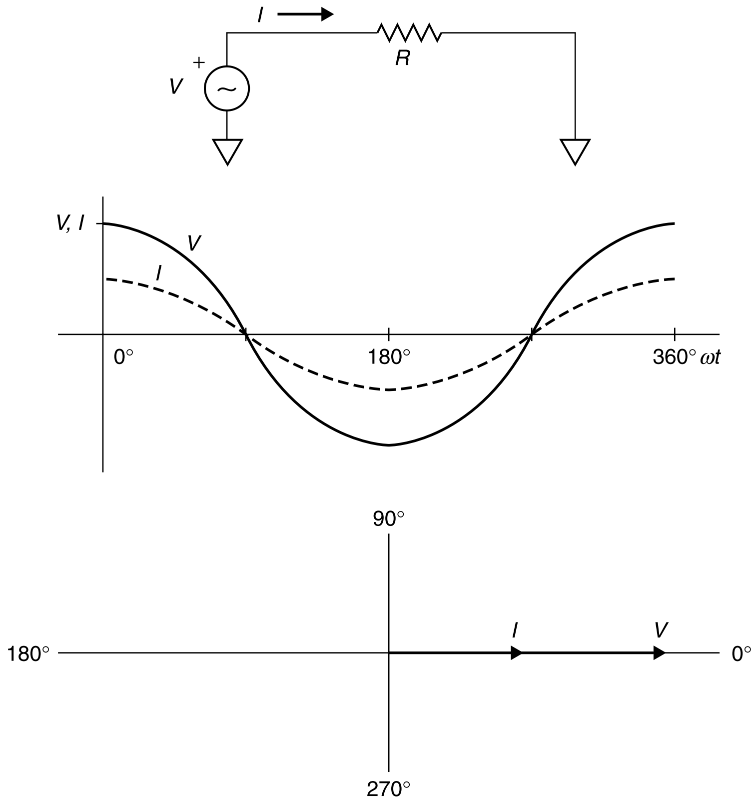

Figure 2.1-2 Ideal resistor R has AC voltage V and current I in phase.

and

$$P_{\mathrm{AVG}} = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos\theta \tag{2.1-5}$$

Note that, according to our convention, the variables are in italic type to indicate that the time waveform is sinusoidal. For an ideal resistor, $\theta = 0°$ and the instantaneous power dissipated in the resistor is equal to the product v(t)i(t). *Throughout this text, except where otherwise specified, peak values are used for voltage, current, and field amplitudes.*

Ohm's Law

Ohm's law applies to all voltage waveforms across a resistor, and states that *current through a resistor is directly proportional to the applied voltage and inversely proportional to its resistance:*

$$\mathrm{i} = \frac{\mathrm{v}}{\mathrm{R}} \tag{2.1-6}$$

where v is in volts, R in ohms, and i in amperes.

Ohm's law is a *linear relationship* (current is proportional only to the first power of voltage) and is valid for all voltage and current levels that do not change the value of resistance. It does not apply, for example, at very high voltages that cause an arc over of the resistor and/or high currents that cause the resistor to change its value due to overheating.

The Inductor

In contrast to the resistor, the ideal inductor L, cannot dissipate power. The general relationship between voltage across and current through it is

$$\mathrm{v(t)} = \mathrm{L}\frac{d\mathrm{i(t)}}{d\mathrm{t}} \tag{2.1-7}$$

The instantaneous current through the inductor is obtained by integrating

$$\mathrm{i(t)} = \int_0^{\mathrm{v}} \mathrm{v(t)}\, d\mathrm{t} \tag{2.1-8}$$

The amount of energy, U_L, stored in an inductor is equal to the time integral of the power applied to it, $v(\mathrm{t}) \times i(\mathrm{t})$, to establish a current i in it from an initial condition at which $i(\mathrm{t} = 0) = 0$:

$$\mathrm{U}_L = \int_0^{\mathrm{t}} \mathrm{v(t)} \cdot \mathrm{i(t)}\, d\mathrm{t} = \mathrm{L}\int_0^{\mathrm{t}} \frac{d\mathrm{(i)}}{d\mathrm{t}} \cdot \mathrm{i(t)}\, d\mathrm{t} \tag{2.1-9}$$

which, on integrating, gives the instantaneous energy stored as

$$U_L = \tfrac{1}{2}\mathrm{L}[i(t)]^2 \tag{2.1-10}$$

Current through the inductor *for all waveforms* is proportional to the integral of the applied voltage. With a sinusoidally applied voltage, $v(t) = V_0 \cos \omega t$, the current I is obtained by integration, noting that the constant of integration corresponds to a DC term that can be neglected for an AC solution (Fig. 2.1-3). Thus,

$$I = \frac{1}{L}\int\!\!\Big(V_0 \cos \omega t\, dt = \frac{V_0 \sin \omega t}{\omega L} \tag{2.1-11}$$

From Ohm's law current divided by voltage has the dimensions of ohms, therefore ωL must have the dimensions of "ohms." This will be used shortly in the definition of complex impedance. There is no power dissipated in an inductor. For AC excitation, the phase angle, θ, between voltage and current is -90° and the power dissipated, P_{Diss}, is given by

$$P_{\mathrm{Diss}} = \tfrac{1}{2}|V|\,|I| \cos \theta = 0 \tag{2.1-12}$$

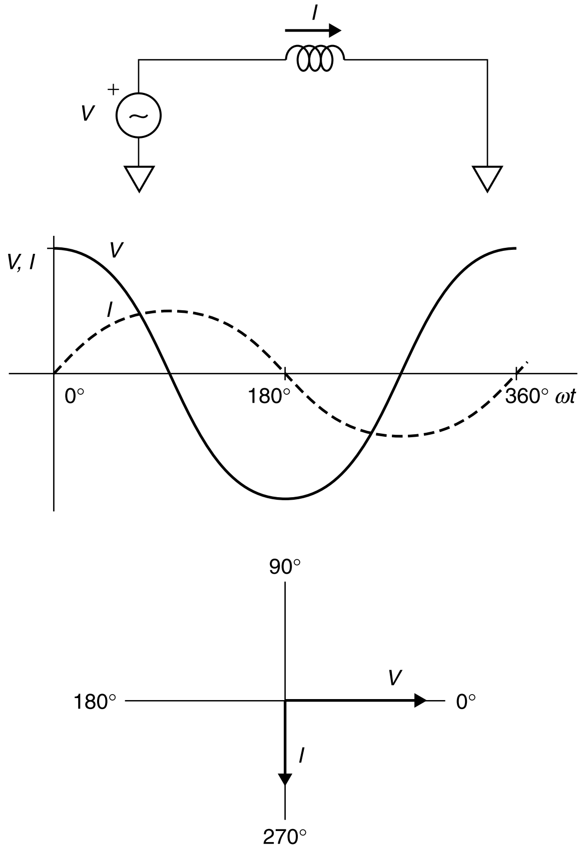

Figure 2.1-3 Sinusoidal current I through inductor lags voltage V by 90°.

The Capacitor

Like the inductor, the capacitor, C, cannot dissipate power. Rather it stores charge and, in so doing, stores energy. By definition, the capacitance C is defined as the ratio of instantaneous charge q to the instantaneous voltage v(t) at which the charge is stored:

$$C \equiv \frac{\mathrm{q}}{\mathrm{v(t)}} \tag{2.1-13}$$

With a current into the capacitor the stored charge increases. The time rate of change of q is equal to this current. Multiplying both sides of (2.1-13) by v and then differentiating with respect to t give

$$\frac{\partial \mathrm{q}}{\partial \mathrm{t}} = \mathrm{i(t)} = C\frac{\partial \mathrm{v(t)}}{\partial \mathrm{t}} \tag{2.1-14}$$

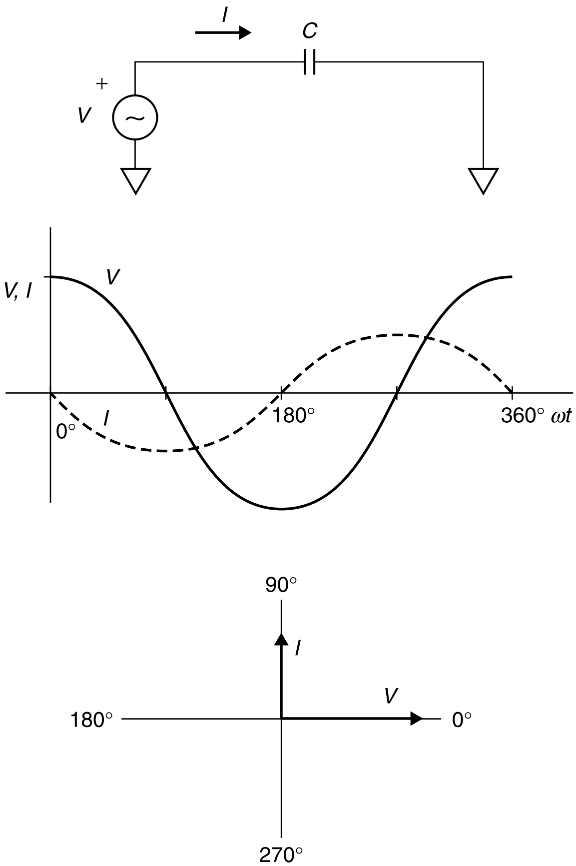

Figure 2.1-4 Ideal capacitor C is also a dissipationless component. For sinusoidal excitation, current *I* leads applied voltage *V* by 90°.

Integrating with respect to *t* gives

$$v(t) = \frac{1}{C}\int i\,dt \qquad (2.1\text{-}15)$$

When a direct current is passed into a capacitor, the voltage across the capacitor's terminals "integrates" the direct current flow from the time when the capacitor had zero volts (Fig. 2.1-4). The capacitor does not dissipate power, but rather stores energy. The energy storage can be considered the presence of charge in a potential field or the establishment of an electric field between the capacitor plates. For an initially uncharged capacitor, $v(0) = 0$, the integral with respect to time of the instantaneous power delivered to the capacitor, $v(t) \times i(t)$, is the stored energy U_C in the capacitor when it is charged to a voltage V:

$$U_C = \int_0^v \mathrm{v(t)i(t)}\,dt = C\int_0^v \mathrm{v(t)}\frac{d\mathrm{v}}{d\mathrm{t}}\,d\mathrm{t} \tag{2.1-16}$$

which, upon integrating, gives the instantaneous stored energy as

$$U_C = \tfrac{1}{2}C[\mathrm{v(t)}]^2 \tag{2.1-17}$$

This result does not depend upon the voltage or current waveforms used to store the charge. When a sinusoidal voltage is applied, the current waveform is also sinusoidal and advanced by 90°:

$$v(t) = V_0 \cos \omega t \tag{2.1-18}$$

$$i(t) = \mathrm{C}\frac{\partial v(t)}{\partial t} = -\mathrm{V_0\omega C \sin \omega t} \tag{2.1-19}$$

From (2.1-19) it follows that $1/\omega C$ has the dimensions of ohms, as did ωL. These facts prompt the definition of *complex impedance*, to be discussed shortly.

2.2 KIRCHHOFF'S LAWS

1. *Kirchhoff's voltage law: The sum of the voltage drops about a closed circuit path is zero* (Fig. 2.2-1):

$$\mathrm{v_0(t) = v_R(t) + v_L(t) + v_C(t)} \tag{2.2-1}$$

2. *Kirchhoff's current law: The sum of the currents into a circuit node is zero* (Fig. 2.2-2):

$$\mathrm{i_0(t) = i_R(t) + i_L(t) + i_C(t)} \tag{2.2-2}$$

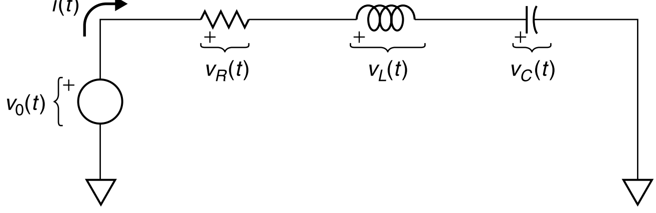

Figure 2.2-1 Kirchhoff's voltage law applied to *RLC* element loop for general time-varying applied voltage $\mathrm{v_0(t)}$.

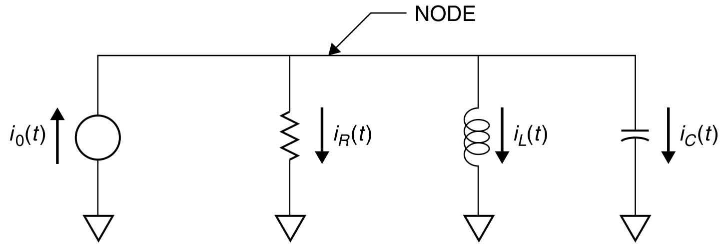

Figure 2.2-2 Kirchhoff's current law applied to node for general time-varying current $i_0(t)$.

Kirchhoff's laws apply instantaneously for all waveforms. For the series circuit of Figure 2.2-1, the applied voltage is equal to the voltage drops across the three element types. The current is continuous in the loop. Using the voltage–current relations of (2.1-6), (2.1-7), and (2.1-15) gives

$$v_0(t) = Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt \tag{2.2-3}$$

2.3 ALTERNATING CURRENT (AC) ANALYSIS

In electrical engineering it would be nice to have the complete solution in time for v(t) and i(t) whenever a circuit is analyzed in order that both the transient and steady-state behavior would be available. However, we usually find that the steady-state behavior of circuits with sinusoidal excitation is adequate, particularly since it can be obtained with much greater computational economy, as will be seen shortly. If a sinusoidal voltage or current excitation is applied to a network consisting of linear R, L, and C elements the resulting currents and voltages usually approach steady-state sinusoidal waveforms within a few RF cycles. Circuits having very high Q, to be discussed in the next chapter, require longer times, so some judgment is necessary regarding the transient effects in AC networks.

Nevertheless, ignoring transient effects and accepting a steady-state solution for an AC network is usually sufficient. Referring to (2.2-3), we notice that integral and differential expressions occur in the network equation due to the presence of L and C elements. However, the steady-state voltage and current solutions of the network are comprised solely of sinusoidal functions at a common frequency because *all integrals and derivatives of sinusoidal functions are also sinusoidal functions at the same frequency (but displaced in phase by* $\pm 90°$*)*.

For example, if we apply a voltage

$$v(t) = V_0 \cos \omega t \tag{2.3-1}$$

to the network in Figure 2.2-1, the resulting current *eventually will approach the steady-state waveform*

$$\mathrm{i(t)} = \mathrm{I_0} \cos(\omega \mathrm{t} - \varphi) \tag{2.3-2}$$

The steady-state solution that we seek is to solve for I_0 and φ in terms of V_0. Substituting this assumed solution into (2.2-3) and performing the indicated differentiation and integration gives

$$\mathrm{V_0} \cos \omega \mathrm{t} = \mathrm{I_0}\left[\mathrm{R} \cos(\omega \mathrm{t} - \varphi) - \omega \mathrm{L} \sin(\omega \mathrm{t} - \varphi) + \frac{1}{\omega \mathrm{C}} \sin(\omega \mathrm{t} - \varphi)\right] \tag{2.3-3}$$

This equation applies for all time t after sufficient time has passed to allow transients to die out (since we ignored the constant of integration associated with the third term). In particular, consider the time for which $\omega t = \pi/2 = 90^\circ$. Then, recognizing that $\cos(90^\circ - \varphi) = \sin \varphi$ and $\sin(90^\circ - \varphi) = \cos \varphi$, (2.3-3) becomes

$$0 = I_0\left\{R \sin \varphi - \left[\omega L - \frac{1}{\omega C}\right] \cos \varphi\right\} \tag{2.3-4a}$$

and solving for φ,

$$\tan \varphi = \left[\frac{\omega L - 1/\omega C}{R}\right] \tag{2.3-4b}$$

$$\varphi = \tan^{-1}\left[\frac{\omega L - 1/\omega C}{R}\right] \tag{2.3-4c}$$

If we substitute $t = 0$ into (2.3-3) and recognize that the value for φ in (2.3-4c) allows determination of $\cos \varphi$ and $\sin \varphi$, the result is

$$V_0 = I_0\left\{R \cos \varphi + \left[\omega L - \frac{1}{\omega C}\right] \sin \varphi\right\}$$

$$I_0 = \frac{V_0}{R \cos \varphi + [\omega L - 1/\omega C] \sin \varphi} \tag{2.3-5}$$

$$= \frac{V_0}{\sqrt{R^2 + [\omega L - 1/\omega C]^2}}$$

The expression in the denominator has the value of the hypotenuse of a right triangle, as shown graphically in Figure 2.3-1.

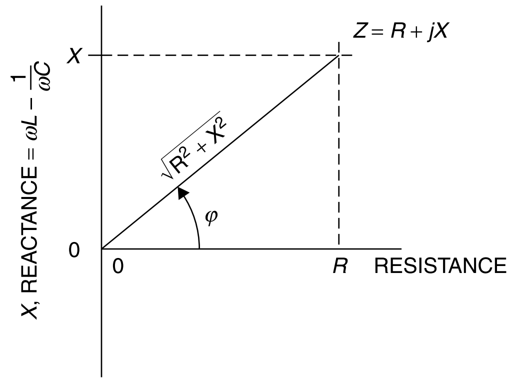

Figure 2.3-1 Orthogonal relationship between resistance R and reactance X in AC circuit. Note that if X is positive, φ is positive, which means that I lags V, consistent with (2.3-2).

The final expression of (2.3-5) appears to be in the form of Ohm's law with resistance replaced by a quantity that includes the "impeding" effects of L and C on current flow. Also, (2.3-4c) shows how L and C affect the phase relation between v and i. Both of these effects can be accounted for by defining a *complex impedance Z*.

The complex impedance Z of a series RLC combination has a real part equal to the resistance R and an imaginary part equal to j times the net reactance, $X = (\omega L - 1/\omega C)$. Thus,

$$Z = R + jX = R + j\left(\omega L - \frac{1}{\omega C}\right) = |Z|\angle\varphi \tag{2.3-6a}$$

where

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{2.3-6b}$$

$$\varphi = \tan^{-1}\frac{\omega L - 1/\omega C}{R} \tag{2.3-6c}$$

and $\omega = 2\pi f$ where f is the operating frequency in hertz.

The foregoing solution to (2.3-6) required considerable mathematical manipulation, particularly in view of the fact that only the magnitude I_0 and phase φ were unknown. We already knew that the form of $i(t)$ would be a sinusoid with time variation ωt. Clearly, an extension of Ohm's law to render the same results directly would offer a profound improvement in the efficiency of *RLC* circuit analysis.

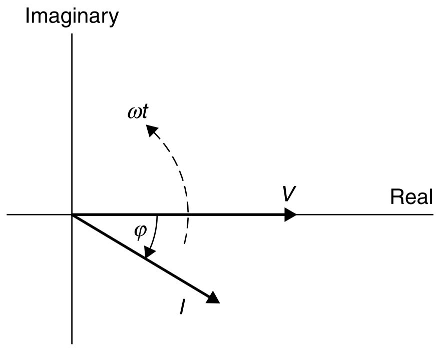

Figure 2.3-2 Voltage and current phasors for solution to network in Figure 2.2-1 where $v(t) = V_0 \cos \omega t$.

Ohm's Law in Complex Form

Ohm's law for AC circuits can be written in complex form as

$$I = \frac{V}{Z} \tag{2.3-7}$$

where V and I are complex quantities called *phasors* and Z is the AC impedance defined in (2.3-6a).

As can be seen in Figure 2.3-2, the voltage and current are now represented by complex quantities, having real and imaginary parts. They are vectors in the complex plane. However, since electric and magnetic fields are three-dimensional vectors in space, to avoid confusing voltage and current quantities with fields in space, *the Institute of Electrical and Electronic Engineers (IEEE) recommends using the term* phasor *for the complex representations of V and I, as well as for the complex representations of sinusoidally varying field magnitudes.* This convention will be assumed throughout this text, except where specifically noted otherwise. A review of complex mathematics is given in Appendix B.

2.4 VOLTAGE AND CURRENT PHASORS

The application of complex numbers to *RLC* circuit analysis is to represent the sinusoidal voltage $v(t)$ by a phasor voltage V and the resulting current $i(t)$ by a phasor current I. These phasor quantities are complex numbers, having real and imaginary parts. *Complex voltage and current phasors are mathematical artifices that do not exist in reality.* They are vectors in the complex plane that are useful in analyzing AC circuits. Phasors do not rotate; they are fixed position vectors whose purpose is to indicate the magnitude and phase of the sinusoidal waveforms that they represent.

However, if they were rotated counterclockwise in the complex plane at the rate of ω radians per second (while maintaining their angular separation, φ)

their projections on the real axis would be proportional to the instantaneous time-varying voltage and current waveforms that they represent. The horizontal axis projection represents the instantaneous wave amplitudes because we chose as a reference for this analysis $v(t) = \mathrm{V}_0 \cos \omega \mathrm{t}$, which has its maximum value at $\mathrm{t} = 0$.

For the network of Figure 2.2-1 the phasors are shown diagrammatically in Figure 2.3-2. In this drawing it is presumed that the inductive reactance, ωL, has a greater magnitude than that of the capacitive reactance, $1/\omega C$. From (2.3-6c) this means that the angle φ is positive, reflecting the fact that I lags V, as shown in Figure 2.3-2.

At a given frequency, in the time domain two values are needed to specify each sinusoidal variable, its peak magnitude and value at $t = 0$. The same information is contained in complex phasors within their real and imaginary parts. Given that $\mathrm{v(t)} = \mathrm{V}_0 \cos \omega \mathrm{t}$, to convert the complex phasors V and I to their respective time-domain variables, we interpret their projections onto the real axis as the instantaneous time value. Thus,

$$v(t) = \mathrm{Re}[Ve^{j\omega t}] \tag{2.4-1}$$

$$i(t) = \mathrm{Re}[Ie^{j\omega t}] \tag{2.4-2}$$

where V and I are the respective phasor values of voltage and current. For example, if $V = 20\angle 30°$, then $V = 20\angle 30° \equiv 20e^{j30°}$ and

$$v(t) = \mathrm{Re}\{20e^{j30°}e^{j\omega t}\} = \mathrm{Re}\{20e^{j(\omega t + 30°)}\} = 20\cos(\omega t + 30°) \tag{2.4-3}$$

The peak value of the phasor is the same as the peak value of the sinusoidal waveform that it represents. Similarly, had rms values been used, the rms magnitude of the phasor would be the same as the rms value of the sinusoid it represents.

Conventionally, the reactances of L and C elements are positive real values with the dimensions of ohms as

$$X_L = \omega L \quad \text{and} \quad X_C = \frac{1}{\omega C} \qquad \text{(in ohms)} \tag{2.4-4a,b}$$

To obtain AC impedances

$$Z_L = j\omega L \tag{2.4-5}$$

since it involves d/dt, which produces j, and

$$Z_C = -j\frac{1}{\omega C} \tag{2.4-6}$$

since it involves $\int dt$, which produces $-j$. Accordingly, the impedance of the series *RLC* circuit in Figure 2.2-1 is written

$$Z = R + j(X_L - X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{2.4-7}$$

Kirchhoff's voltage and current laws also apply to the phasor forms of V and I.

With these definitions and the rules for complex number manipulation, circuits of arbitrary complexity can be analyzed in the phasor domain to solve for the relationships between voltage and currents at any given frequency in a network of linear elements. Then (2.4-1) and (2.4-2) can be used if the instantaneous time functions are required.

2.5 IMPEDANCE

Estimating Reactance

The *reactance of an inductor, L, is* ωL, *and the reactance of a capacitor, C, is* $1/\omega C$. *Reactances have the dimensions of ohms.*

A practicing RF and microwave engineer should be able to estimate the reactances of inductors and capacitors quickly. This need arises, for example, in order to assess their effects on a circuit, the realizability of a proposed tuning method or for a variety of other design and analysis purposes.

Memorization of two reactance values and a simple scaling method permits these estimates to be made mentally, yielding a close approximation, even without a calculator. The reactance of an inductor, L, is given by

$$X_L = \omega L \quad \text{where} \quad \omega = 2\pi f \tag{2.5-1}$$

At 1 $\mathrm{GH_z}$ the reactance magnitude of a 1-nH inductor is 6.28 Ω. Therefore, the reactance of any other inductor at any other frequency is given by

$$X_L = 6.28 fL \qquad (f \text{ in gigahertz}, L \text{ in nanohenries}) \tag{2.5-2}$$

By remembering the 6.28 Ω/(nH-GHz) scale factor, other inductive reactance values are quickly estimated. For example, a 3-nH inductor at 500 MHz has a reactance

$$X_L = (6.28)(0.5)(3) = 9.42\ \Omega \tag{2.5-3}$$

Similarly, the reactance of a capacitor, C, is given by

$$X_C = \frac{1}{\omega C} = \frac{159}{fC} \qquad (f \text{ in gigahertz}, C \text{ in picofarads}) \tag{2.5-4}$$

Thus, remembering that 1 pF yields 159 Ω at 1 GHz allows other capacitive reactance values to be estimated. For example, a 2-picofarad (pF) capacitor has a reactance at 3 GHz of

$$X_C = \frac{159\ \Omega}{(2)(3)} = 26.5\ \Omega \tag{2.5-5}$$

Of course, one must also remember that *inductive reactance is directly proportional to f and L* while *capacitive reactance is inversely proportional to f and C*. In this way, memorizing the 1-GHz reactances of a 1-nH inductor and 1-pF capacitor allows simple scaling of other values.

$$X_L = 6.28fL\ (\Omega) \tag{2.5-6}$$

$$X_C = 159/fC\ (\Omega) \tag{2.5-7}$$

where f is in GHz, L in nH, and C in pF. Note that the *reactances* require a preceding j factor to become *impedances*. Thus, ωL is the reactance of an inductor, L, while $j\omega L$ is its impedance. Similarly, $1/\omega C$ is the reactance of a capacitor, while $-j/\omega C$ is its impedance.

Addition of Series Impedances

Practical circuits may have very complex interconnections. To analyze them, it is necessary to be able to combine multiple impedance and admittance values to find equivalent, overall values.

The total impedance of elements in series is obtained by adding their real and imaginary parts, respectively.

Referring to Figure 2.5-1, if

$$Z_1 = a + jb \quad \text{and} \quad Z_2 = c + jd \tag{2.5-8}$$

then,

$$Z_T = Z_1 + Z_2 = (a + c) + j(b + d) \tag{2.5-9}$$

For example, if

$$Z_1 = 5 + j9\ \Omega \quad \text{and} \quad Z_2 = 3 - j15\ \Omega$$

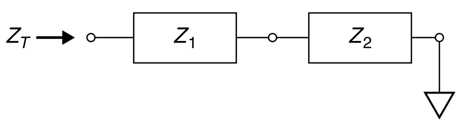

Figure 2.5-1 Series addition of impedances, $Z_T = Z_1 + Z_2$.

then

$$Z_1 + Z_2 = (5 + 3)\ \Omega + j(9 - 15)\ \Omega = 8 - j6\ \Omega$$

2.6 ADMITTANCE

Admittance Definition

The admittance Y is the complex reciprocal of impedance Z. Its unit is expressed either as the *siemen* or the *mho* (which is "ohm" spelled backward) and its symbol is the Greek capital omega written upside down (℧):

$$Y = 1/Z = G + jB \tag{2.6-1}$$

where G is the *conductance* and B is the *susceptance* of Y.

The *susceptance of an inductor* is

$$B_L = \frac{1}{X_L} = \frac{1}{\omega L} \tag{2.6-2}$$

and *the admittance of an inductor* is

$$-jB_L = \frac{1}{jX_L} = \frac{1}{j\omega L} = \frac{-j}{\omega L} \tag{2.6-3}$$

Similarly, the *susceptance of a capacitor* is

$$B_C = \frac{1}{X_C} = \frac{1}{1/\omega C} = \omega C \tag{2.6-4}$$

and the *admittance of a capacitor* is

$$jB_C = \frac{1}{-jX_C} = \frac{1}{-j/\omega C} = j\omega C \tag{2.6-5}$$

Addition of Parallel Admittances

The total admittance of elements in parallel is obtained by adding the real and imaginary parts of their admittances, respectively.

Frequently, one is given the impedances of elements that are in parallel. To determine the total admittance, find the equivalent admittances of each element and add them together. The reader is cautioned that one cannot form the admittance simply by adding together the reciprocals of the resistance and reactance parts of an impedance. Rather, it is necessary to find the complex

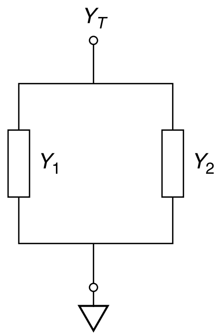

Figure 2.6-1 Parallel addition of admittances, $Y_T = Y_1 + Y_2$.

reciprocal of the impedance, which is the equivalent admittance. Thus, if the element values are expressed initially in impedance, first convert to admittance, then perform the addition of real and imaginary parts (Fig. 2.6-1).

$$Z_T = \frac{1}{Y_T} = \frac{1}{Y_1 + Y_2} \tag{2.6-6}$$

Ohm's law for AC circuits can also be written in terms of admittance,

$$I = VY \tag{2.6-7}$$

where, as before, V and I are phasor quantities and Y is admittance.

For example, suppose we wish to find the total equivalent impedance, Z_T, of a pair of elements in parallel given their individual impedances, Z_1 and Z_2 (Fig. 2.6-2).

Assume that $Z_1 = 5 + j8\ \Omega$ and $Z_2 = 3 - j5\ \Omega$. The first step is to convert these to polar form:

$$Z_1 = 9.43\angle 58^\circ\ \Omega \quad \text{and} \quad Z_2 = 5.83\angle -59.0^\circ\ \Omega$$

Then find their equivalent admitances (the complex reciprocals):

$$Y_1 = 1/Z_1 = 0.106\angle -58^\circ\ \mho \quad \text{and} \quad Y_2 = 1/Z_2 = 0.172\angle 59^\circ\ \mho$$

Next convert to rectangular form.

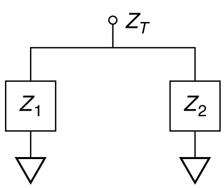

Figure 2.6-2 Parallel admittance combination example.

$$Y_1 = 0.056 - j0.090 \ \mho \qquad Y_2 = 0.089 + j0.147 \ \mho$$

Add the real and imaginary parts, respectively:

$$Y_T = Y_1 + Y_2 = 0.145 + j0.057 \ \mho$$

To find Z_T, convert Y_T to polar form:

$$\begin{aligned} Z_T = 1/Y_T &= 1/(0.156\angle 21.46^\circ) \ \Omega \\ &= 6.41\angle{-21.46^\circ} \ \Omega \\ &= 5.97 - j2.35 \ \Omega \end{aligned}$$

The Product over the Sum

Generally, to combine impedances in parallel one must first convert them to admittances, add the admittances together to form the total admittance, then convert this admittance to an impedance value, as was just demonstrated. However, when only two parallel elements are to be combined (or two admittances in series) a short cut results by forming the *product over the sum. The total impedance of two parallel impedances equals the product of the individual impedances divided by their sum.*

The validity of this rule can be shown by performing the combination in general terms and observing the result:

$$\begin{aligned} Y_T &= Y_1 + Y_2 \\ \frac{1}{Z_T} &= \frac{1}{Z_1} + \frac{1}{Z_2} \\ Z_1 Z_2 &= Z_T(Z_1 + Z_2) \\ Z_T &= \frac{Z_1 Z_2}{Z_1 + Z_2} \end{aligned} \tag{2.6-8}$$

Using the values in the previous example, $Z_1 = 5 + j8 \ \Omega$ and $Z_2 = 3 - j5 \ \Omega$,

$$\begin{aligned} Z_T &= \frac{(9.43\angle 58^\circ)(5.83\angle{-59^\circ})}{8 + j3} \ \Omega \\ &= \frac{55\angle{-1^\circ}}{8.54\angle 20.56^\circ} = 6.44\angle{-21.56^\circ} \ \Omega \\ &= 5.99 - j2.37 \ \Omega \end{aligned}$$

The same procedure applies for finding the total admittance of two admittances in series. Specifically, form their product and divide by their sum.

2.7 LLFPB NETWORKS

Up until now, the networks that we have described are composed of *LLFPB elements* to which AC analysis can be applied directly. The acronym for these networks derives from the following list of their properties. Namely, they

L: are small compared to a wavelength (lumped).
L: respond linearly to excitations (linear).
F: are of finite value (finite).
P: do not generate power (passive).
B: have the same behavior for currents in either direction (bilateral).

Thus, they are called *lumped, linear, passive, finite, bilateral (LLFPB) elements*. We shall describe additional means to evaluate networks that do not satisfy all of these criteria. For example, circuits whose dimensions are large compared to a wavelength do not satisfy the lumped criterion. Circuits containing transistors are neither passive nor bilateral.

2.8 DECIBELS, dBW, AND dBm

Logarithms (Logs)

In wireless engineering, radio signals may be transmitted over miles of distance with substantial diminution of their signal strengths. Then they may be amplified by factors of thousands or millions of times so that their information can be heard in an audio speaker, viewed on a television screen, or employed in a digital processor. To handle the large signal ratios involved, it is easier to represent them as powers of 10 (logarithms). This measure employs the *decibel scale*. It is insightful to review some mathematical background when applying the *decibel* measures that will be described shortly.

The logarithm of Y to the base X is the power L to which X must be raised to give Y. Thus,

$$Y = X^L \tag{2.8-1}$$

For example, if we choose base 10, then $X = 10$ and

$$Y = 10^L$$

Table 2.8-1 lists sample values of base 10 logarithms.

TABLE 2.8-1 Selected Values of *Y* and *L* for Base 10 Logarithms

Y	1000	100	20	10	2	1	1/2	1/10	1/20	0.01	0.001
Y	10^3	10^2	$10^{1.3}$	10^1	$10^{0.3}$	10^0	$10^{-0.3}$	10^{-1}	$10^{-1.3}$	10^{-2}	10^{-3}
L	3	2	1.3	1	0.3	0	−0.3	−1	−1.3	−2	−3

Multiplying by Adding Logs

To multiply numbers having the same base, write down the base and add their logs. For example,

$$10 \times 1000 = (10^1)(10^3) = 10^{1+3} = 10^4 = 10{,}000$$

and

$$100 \times 2 \times 5 = (10^2)(10^{0.3})(10^{0.7}) = 10^3 = 1000$$

Dividing by Subtracting Logs

To divide two numbers having the same base, write down the base and subtract the exponent of the denominator from the exponent of the numerator. For example, $10^3/10^2 = 10^{3-2} = 10^1 = 10$.

Zero Powers

Any number divided by itself, except zero, must equal unity. Regardless of the base, use of logarithms must produce this same result. As a consequence it follows that: *The zero power of any number, except zero, is 1.* For example,

$$10/10 = 10^{1-1} = 10^0 = 1$$
$$17/17 = 17^{1-1} = 17^0 = 1$$
$$459/459 = 459^{1-1} = 459^0 = 1$$

Bel Scale

The convenience of multiplying by adding logarithms as well as representing very large numbers by their logarithms first prompted the use of the *Bel scale* (named after its originator), which was simply the logarithm to the base 10 of the number. This proper name remains embedded in the present *decibel* nomenclature.

$$Y \text{ (in bels)} = \log_{10}(Y) \tag{2.8-2}$$

Accordingly,

$$100 \text{ (in bels)} = 2 \text{ bel}$$
$$10 \text{ (in bels)} = 1 \text{ bel}$$
$$2 \text{ (in bels)} = 0.3 \text{ bel}$$
$$1 \text{ (in bels)} = 0 \text{ bel}$$

$$\tfrac{1}{2} \text{ (in bels)} = -0.3 \text{ bel}$$
$$\tfrac{1}{10} \text{ (in bels)} = -1 \text{ bel}$$

Decibel Scale

The Bel scale was quickly recognized as a useful innovation, but its steps were inconveniently large, an increase of 1 bel being a factor of 10. This objection was accommodated by a transition to the *decibel (dB) scale*.

Electrical engineers took a further step. Inherently, bels and decibels are simply means of expressing ratios between similar quantities. However, in electrical engineering, *decibels are defined to be the ratio of two values of electrical power:*

$$P/P_0 \text{ (in decibels)} = 10 \log(P/P_0) \tag{2.8-3}$$

where, henceforth in this text, "log" shall represent the logarithm to the base 10 and "ln" shall represent the logarithm to the base e. Because power is proportional to the square of voltage (whether rms or peak), *the ratio of two voltages or currents referenced to the same impedance level* is defined as

$$V/V_0 \text{ (in decibels)} = 20 \log(V/V_0) \tag{2.8-4}$$

Obviously, the decibel value is just 10 times the Bel scale. Thus, a factor of 10 in power is 10 dB, a factor of 2 is 3 dB (more precisely 3.01 dB), and the value of unity remains 0 dB.

In using decibels, one need only memorize the logarithms of numbers between 1 and 10. In fact, for most purposes, knowing the values corresponding to 0.5, 1, 2, 3, and 10 dB is usually sufficient for most estimating purposes (Table 2.8-2).

Often one can deduce the decibel value of a number from the decibel values known for other quantities. For example, since $2 \times 5 = 10$, and since the factor 2 corresponds to 3 dB, and the factor 10 is 10 dB, it follows that the factor 5 corresponds to 10 dB − 3 dB = 7 dB. Similarly, if one remembers that the factor 1.2 corresponds to about 0.8 dB, then the factor $1.2 \times 1.2 = 1.2^2 = 1.44$ corresponds to about 0.8 dB + 0.8 dB = 1.6 dB.

Decibels—Relative Measures

For the system example in Figure 2.8-1, using numeric multiplication we calculate

$$P_{\mathrm{OUT}}/P_{\mathrm{IN}} = 0.8 \times 20 \times 0.5 = 8$$

With decibel multiplication (adding logs), the same result is obtained as

TABLE 2.8-2 Selected Logarithms and Decibel Values

P/P_0	V/V_0	$\log(P/P_0)$	dB[a]
0.01	0.1	−2	−20
0.1	0.316	−1	−10
0.5	0.707	−0.3	−3
1	1	0	0
1.05	1.025	0.021	0.21
1.1	1.05	0.041	0.41
1.12	1.06	0.05	0.5
1.2	1.10	0.08	0.8
1.26	1.12	0.10	1
1.58	1.26	0.2	2
2	1.414	0.3	3
2.51	1.58	0.4	4
3.16	1.78	0.5	5
4	2	0.6	6
5	2.24	0.7	7
6.3	2.5	0.8	8
8	2.82	0.9	9
10	3.16	1	10

[a] dB = $10 \log(P/P_0)$ and $20 \log(V/V_0)$.

follows:

$$P_{\text{OUT}}/P_{\text{IN}} = -1 \text{ dB} + 13 \text{ dB} - 3 \text{ dB} = 9 \text{ dB}$$

$$P_{\text{OUT}}/P_{\text{IN}} = 9 \text{ dB}$$

$$P_{\text{OUT}}/P_{\text{IN}} = \text{antilog}(0.9) = 8$$

Due to rounding off the values of the logarithms, we may obtain a slight difference in the answers by the numeric and decibel calculation methods. However, sufficient accuracy always can be obtained by using more decimal places for the log values, but for most planning purposes carrying values to within 0.1 dB provides enough accuracy for estimating.

Decibels are dimensionless because each is proportional to the logarithm of the ratio of two numbers. When decibels are used with dimensioned quantities, such as watts or milliwatts, *both quantities in the ratio must have the same units.*

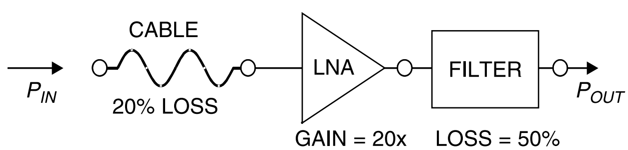

Figure 2.8-1 Subsystem design calculation using numeric and decibel methods.

Thus, for the gain of a power amplifier we might write:

$$\begin{aligned} \text{Gain} &= P_{\text{OUT}}\ (\text{watts})/P_{\text{IN}}\ (\text{watts}) \\ &= 8\ \text{W}/1\ \text{W} \\ &= 8 \\ \text{Gain (dB)} &= 10 \log(8) = 9\ \text{dB} \end{aligned}$$

Absolute Power Levels—dBm and dBW

Because the decibel scale is so useful, it has become common practice to use it to represent *absolute power levels*. This is done by the simple step of referencing to a standard power level, either 1 W (dBW) or 1 mW (dBm) (Table 2.8-3).

These absolute power references are defined as:

$$\text{dBW} = 10 \log[P\ (\text{watts})/1\ \text{W}] \qquad (\text{referenced to 1 W}) \qquad (2.8\text{-}5)$$

$$\text{dBm} = 10 \log[P\ (\text{watts})/(0.001\ \text{W})] \qquad (\text{referenced to 1 mW}) \qquad (2.8\text{-}6)$$

$$\text{dBm} = 10 \log[P\ (\text{mW})/(1\ \text{mW})]$$

Thus, 10 W (10,000 mW) can be expressed as +10 dBW or +40 dBm. One-tenth of a watt can be expressed as −10 dBW or +20 dBm.

TABLE 2.8-3 Power Levels in dBW and dBm

dBm	dBW	Power	Power (W)
−120	−150	1 fW	10^{-15}
−90	−120	1 pW	10^{-12}
−60	−90	1 nW	10^{-9}
−30	−60	1 μW	10^{-6}
−3	−33	0.5 mW	0.5×10^{-3}
0	−30	1 mW	10^{-3}
+10	−20	10 mW	10^{-2}
+20	−10	100 mW	10^{-1}
+30	0	1 W	1
+33	+3	2 W	2
+37	+7	5 W	5
+40	+10	10 W	10
+50	+20	100 W	10^{2}
+60	+30	1 kW	10^{3}
+70	+40	10 kW	10^{4}
+80	+50	100 kW	10^{5}
+90	+60	1 MW	10^{6}
+100	+70	10 MW	10^{7}
+110	+80	100 MW	10^{8}
+120	+90	1 GW	10^{9}

Customarily the sign is stated explicitly in dBW and dBm specifications to minimize the chance of misinterpretation. For example, one would say that 2 W is "plus 3 dBW."

Decibel Power Scales

Notice that this enormous power range can be represented in decibel notation simply as 240 dB, but to represent it numerically the ratio of the highest to the lowest power in the table would be expressed as the ratio 1,000,000,000,000,000,000,000,000 to 1!

2.9 POWER TRANSFER

Calculating Power Transfer

We saw in (2.1-2) that the instantaneous flow of power is given by

$$P_{\text{instantaneous}} = vi \tag{2.9-1}$$

where v and i are the instantaneous voltage and current. However, this power flow may represent the power that is dissipated, often considered the *real power flow*, and power that flows to store energy in an inductor or capacitor, which we call the *imaginary power*. Real power flow is dissipated or, if v and i apply to antenna terminals, is radiated into space. The imaginary power flow in an AC circuit flows back and forth as the inductors and capacitors cycle from peak-to-zero energy storage conditions. When applied to an antenna, imaginary power flow goes into energy storage in the near fields of the antenna.

Usually we are more interested in real power flow in a system design. In the AC case, the real power is given by (2.1-3). This is rewritten below as the peak and average power flow that will be understood to mean dissipated or radiated power in the remainder of this text. Thus

$$P_{\text{Peak}} = VI \cos\theta \tag{2.9-2}$$

$$P_{\text{Av}} = \tfrac{1}{2} VI \cos\theta \tag{2.9-3}$$

The expressions can be used throughout this text, since voltage, current, and field values usually are specified in terms of their AC zero-to-peak values as, for example, an AC voltage is written $v(t) = V_0 \sin \omega t$.

It is useful to be able to express the real power values directly in terms of the complex phasors related to them without having to determine θ explicitly. To do this consider voltage and current phasors V and I shown in Figure 2.9-1.

To find the peak real power transfer, we wish to calculate

$$P_{\text{Peak}} = |V|\,|I| \cos\theta \tag{2.9-4}$$

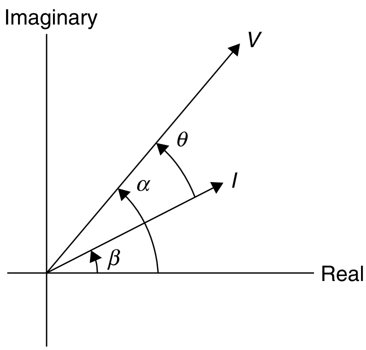

Figure 2.9-1 Voltage and current phasors used for power calculation.

where $\theta = \alpha - \beta$. By taking the complex product of V and the complex conjugate of I, I^*, the subtraction of the angles $\alpha - \beta$ produces the difference θ. Then taking the real part of the complex product yields the necessary multiplication by $\cos\theta$. Thus,

$$P_{\text{Peak}} = \text{Re}(VI^*) = |V|\,|I|\cos\theta \tag{2.9-5}$$

$$P_{\text{Av}} = \tfrac{1}{2}\,\text{Re}(VI^*) = \tfrac{1}{2}|V|\,|I|\cos\theta \tag{2.9-6}$$

These expressions, and their extensions to $\vec{E}$ and $\vec{H}$ fields, will prove very useful in deriving expressions for propagating power in waveguides and free space.

Maximum Power Transfer

The schematic diagram in Figure 2.9-2 represents an AC voltage source with internal impedance Z_G connected to a load impedance Z_L. In general, both Z_G and Z_L can be complex (each can have nonzero real and imaginary parts).

To determine the relationship between generator and load impedances for maximum power transfer, we first note that the maximum power will be delivered to the load when $jX_G = -jX_L$, since any net reactive impedance in the

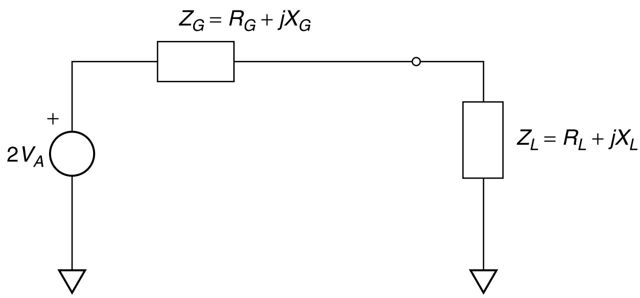

Figure 2.9-2 Maximum power transfer from generator to load.

loop would reduce the magnitude of I and with it the power delivered to the load.

The peak power (since, in our convention, V_A is a peak voltage amplitude), P_L, delivered to the load is then

$$P_L = I^2 R_L = \frac{(2V_A)^2}{(R_G + R_L)^2} R_L \tag{2.9-7}$$

and therefore

$$\frac{\partial P_L}{\partial R_L} = 4V_A^2 \frac{(R_G + R_L)^2 - R_L(2R_G + 2R_L)}{(R_G + R_L)^4} \tag{2.9-8}$$

Now set the numerator equal to zero to establish the condition for the maximum value of P_L (the zero slope of P_L versus R_L):

$$\begin{gathered} 0 = R_G^2 + 2R_G R_L + R_L^2 - 2R_G R_L - 2R_L^2 \\ R_G^2 = R_L^2 \qquad R_G = R_L \end{gathered} \tag{2.9-9}$$

Thus, the maximum power transfer occurs when the load impedance is set equal to the complex conjugate of the generator impedance. That is,

$$Z_L = Z_G^* \tag{2.9-10}$$

The maximum peak power available to be delivered to the load is

$$P_A = V_A^2 / R_G \tag{2.9-11}$$

Usually, at microwave frequencies a generator with real impedance, Z_0, is assumed. Then the maximum available peak power becomes

$$P_A = V_A^2 / Z_0 \tag{2.9-12}$$

and this will be delivered to the Z_0 load. The symbol V_A is called the *available voltage* from the generator, not because it is the largest voltage available to the load (which would occur if Z_0 is replaced with an open circuit) but because *V_A is the voltage available to the load under the condition of maximum power transfer*.

2.10 SPECIFYING LOSS

Insertion Loss

We have seen in the preceding section that maximum power transfer occurs between generator and load when their impedances are the complex conjugates of one another. In practice, one rarely knows the equivalent impedance of a

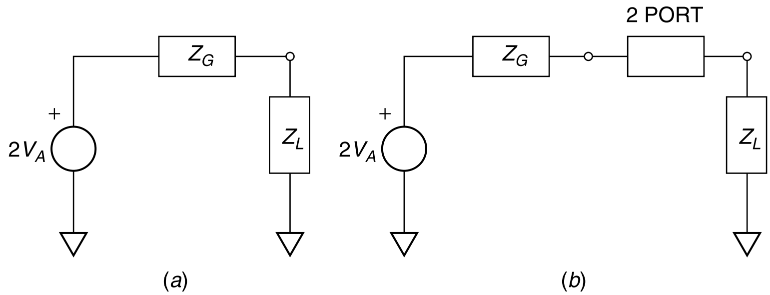

Figure 2.10-1 Insertion loss measurement.

microwave test source. Yet one of the most common measurements is that of *insertion loss*, consisting of the following procedure.

A generator with source impedance Z_G is connected to a load Z_L (Fig. 2.10-1*a*), and the power delivered to the load is found to be P_{L1}. Next, a two-port network is interposed between generator and load (Fig. 2.10-1*b*) and the power delivered to the load found to be P_{L2}. The insertion loss (IL) is defined as

$$\mathrm{IL} = \frac{P_{L2}}{P_{L1}}$$

$$\mathrm{IL} = 10 \log \frac{P_{L2}}{P_{L1}} \quad \text{(in decibels)} \tag{2.10-1}$$

The reader can see the problem with this procedure. The value of IL is strongly dependent upon the values of Z_G and Z_L. Without knowing these values the effect of inserting a two-port network cannot be predicted accurately.

If the two port is a passive network, one might expect that the insertion loss could range from 0 dB (no loss) to some finite loss (a positive decibel value). But this is not necessarily so. For example, if the load is 25 Ω and the generator impedance is 50 Ω, installing a two port that is a low-loss transformer might increase the power delivered to the load, resulting in a negative decibel loss value, or power gain, and this would be obtained with a passive two-port network.

Transducer Loss

The prospect of "gain" and the lack of defined source and load impedances of the insertion loss method led to the specification of an alternate measurement called *transducer loss* (TL). It is defined as

$$\mathrm{TL} = \frac{P_A}{P_L}$$

$$\mathrm{TL} = 10 \log \frac{P_A}{P_L} \quad \text{(in decibels)} \tag{2.10-2}$$

where P_A is the available power from the generator. Since the maximum power that can be delivered to the load, with or without a passive two port, is P_A, the transducer loss can never be less than unity (always a positive value when expressed in decibels).

When the generator and load impedances are complex conjugates of each other, the insertion loss and the transducer loss are equal. Commonly, the generator and load are matched to the Z_0 of the test cables interconnecting them, satisfying the complex conjugate requirement.

Within the industry, it is customary to use the insertion loss measurement. This is because of the ease of making a substitution measurement, first attaching generator and load using a "thru" connection to obtain P_{L1} and then substituting the two port to obtain P_{L2}. Fortunately, when performed with a well-padded generator (a matched, high attenuation pad at the output of the generator cable) and a well-matched power detecting load, the effect of this common practice, in effect, is actually to perform a proper transducer loss measurement, since generator and load essentially are matched to a common Z_0. Similarly, when performing "insertion loss" measurements with a network analyzer, the source and load are usually resistive and equal to the Z_0 of the test system, ensuring that a transducer loss measurement is also made. In practice, however, these techniques are commonly referred to as "insertion loss measurements."

Loss Due to a Series Impedance

The transducer loss caused by a series impedance Z placed between matched generator and load is a commonly encountered condition, often in switching or attenuating circuits (Fig. 2.10-2).

The transducer loss may be termed *isolation* when Z is made large enough to block most of the power from reaching the load, as in the "off" state of a switch. The transducer loss (or isolation), as a function of Z and Z_0 is calculated as follows. Before Z is installed, under the matched generator and load conditions, the voltage at the load is the available or *line voltage* V_A and the current through the load, called *line current*, is $I_A = V_A/Z_0$. Then, with the series Z in place, the current through the load is reduced. The power delivered

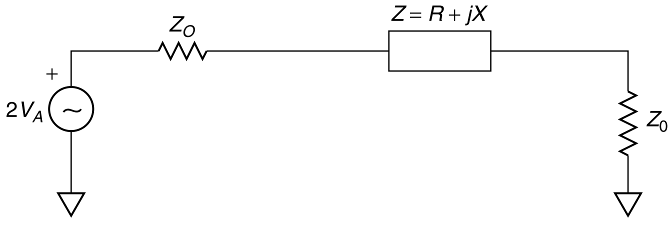

Figure 2.10-2 Equivalent circuit of impedance installed between matched generator and load.

to the load is proportional to the square of the load current. Thus,

$$\mathrm{TL} = \frac{P_A}{P_L} = \frac{V_A^2/Z_0}{I^2 Z_0} = \frac{V_A^2/Z_0}{[(2V_A)^2/|2Z_0+Z|^2]Z_0} = \frac{|2Z_0+Z|^2}{(2Z_0)^2}$$
$$= \left|1 + \frac{Z}{2Z_0}\right|^2 \tag{2.10-3}$$

Since Z/Z_0 is the *normalized impedance*, $z = r + jx$, the transducer loss (or isolation) can be written

$$\mathrm{TL} = \left|1 + \frac{z}{2}\right|^2 = 1 + r + \frac{r^2}{4} + \frac{|x|^2}{4} \tag{2.10-4}$$

For example, if 5 Ω resistance is placed between a 50-Ω source and a 50-Ω load, the transducer loss is

$$\mathrm{TL} = 1 + \tfrac{5}{50} + \left(\tfrac{1}{4}\right)\left(\tfrac{5}{50}\right)^2 = 1.10 = 0.42 \text{ dB}$$

This might represent the "on state" (forward bias) of a series PIN diode installed in the transmission line to provide switching. When the diode's bias state is switched (to reverse bias), it might be represented as 0.2 pF capacitance in the "off state." The transducer isolation at, say, 500 MHz, for which the diode's reactance is 1590 Ω, is

$$\mathrm{TL} = 1 + \left(\tfrac{1}{4}\right)\left(\tfrac{1590}{50}\right)^2 = 254 = 24 \text{ dB}$$

Loss Due to a Shunt Admittance

Another common transducer loss example is that of a shunt-connected admittance between matched generator and load (Fig. 2.10-3). An analysis similar to that used for the series Z gives

$$\mathrm{TL} = \left|1 + \frac{y}{2}\right|^2 = 1 + g + \frac{g^2}{4} + \frac{|b|^2}{4} \tag{2.10-5}$$

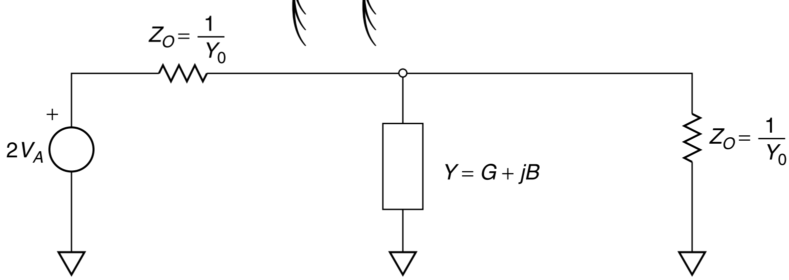

Figure 2.10-3 Equivalent circuit used with admittance transducer loss formula.

in which the normalized admittance is

$$y = \frac{Y}{Y_0} = \frac{G + jB}{Y_0} = g + jb \tag{2.10-6}$$

Loss in Terms of Scattering Parameters

When using a circuit simulator, transducer loss is obtained by calculating the square of the magnitude of the scattering parameter, S_{21}, since for passive networks $|S_{21}| \leq 1$ the transducer loss (or isolation) as a function of S_{21} is

$$\begin{aligned} \mathrm{TL} &= |S_{21}|^{-2} \\ &= -20 \log|S_{21}| \quad \text{(in decibels)} \end{aligned} \tag{2.10-7}$$

where the negative signs has been added so that the loss ratio is equal to or greater than unity and a positive number when expressed in decibels. The S parameters are described in Chapter 5.

2.11 REAL RLC MODELS

Resistor with Parasitics

Real resistors have series inductance L and shunt capacitance C. These unintended elements are called *parasitic elements* or simply *parasitics*. For a standard carbon resistor these elements (typically L = 5 nH and C = 1 pF) have negligible effects at 10 MHz, but at 1 GHz their presence causes the resistance value to be changed profoundly. Figure 2.11-1 shows their effects on a 50-Ω resistor with frequency. At resonance, a little over 1.5 GHz, the resistance has doubled in value.

Inductor with Parasitics

A real inductor L has some series resistance R and some shunt capacitance C. Consider a 20-nH inductor with $Q = 60$ at 500 MHz (R = 1 Ω). Also assume a shunt capacitance of 0.5 pF. The behavior with frequency is as shown in Figure 2.11-2. The inductor goes through parallel resonance, f_R, at about 1.6 GHz. It approximates a linear inductor only up to about 600 MHz (about 40% of f_R). Above f_R its net reactance actually is capacitive!

Capacitor with Parasitics

Similarly, capacitors also have parasitic inductance, capacitance, and resistance. For the capacitor, a four-element model is required to provide an accu-

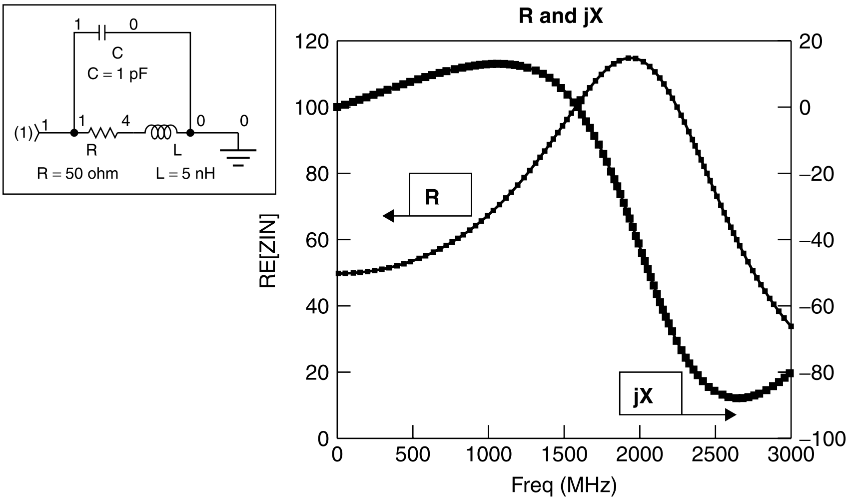

Figure 2.11-1 Frequency behavior of 50-Ω resistor with 5-nH-series-inductance and 1-pF-parallel-capacitance parasitics.

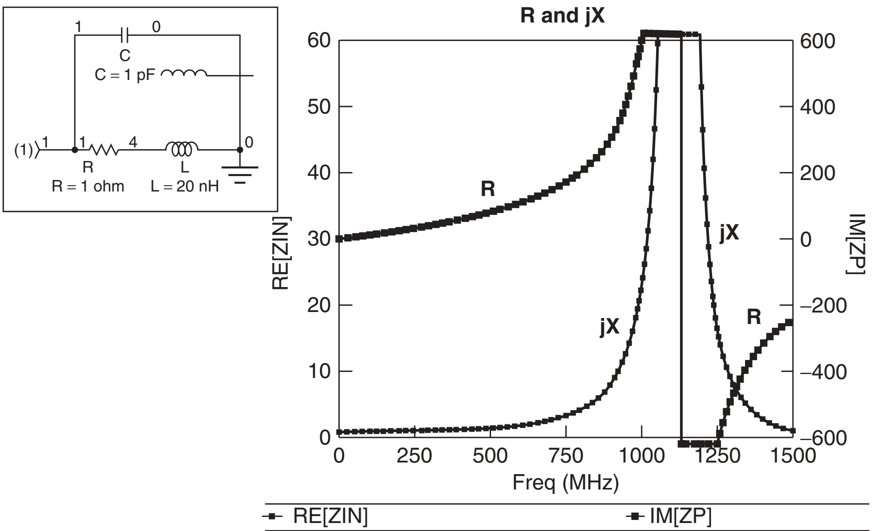

Figure 2.11-2 Frequency behavior of 20-nH inductor having parasitic shunt capacitance of 1 pF.

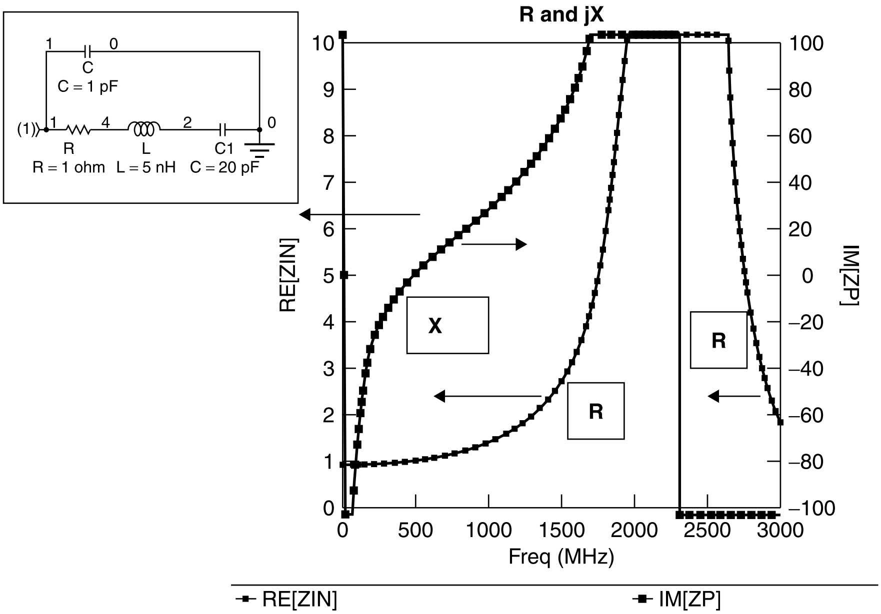

Figure 2.11-3 Reactance and impedance magnitude of 20-pF capacitor with 5-nH-series inductance and 1-pF parallel capacitance parasitics.

rate simulation over a wide frequency range (Fig. 2.11-3). Notice that the circuit goes through series resonance ($jX = 0$) at about 500 MHz and through parallel resonance ($jX = \infty$) at about 2.3 GHz.

2.12 DESIGNING *LC* ELEMENTS

Lumped Coils

The inductance of a single-layer, air core coil (Fig. 2.12-1), which can be wound on a cardboard or wood cylindrical mandrel, is given by Wheeler [2]:

$$L = n^2\left[\frac{r^2}{9r + 10l}\right] \quad (\mu\text{H}) \tag{2.12-1}$$

where n is the number of turns, $r = D/2$ is the radius of the coil, and l is its length, both in inches. In this formula the wire thickness from which the coil is formed is assumed to be small compared to the turns' spacing.

For example, suppose that it is desired to make an RF choke for an integrated circuit module. There is enough space for a 150-mil-long coil having 10 turns of 36-gauge wire (5 mil diameter) about a 25-mil-diameter form. What

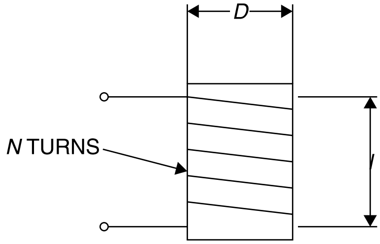

Figure 2.12-1 An "air core" coil wound on a nonmetallic form.

is L?

$$L = (10)^2 \left[\frac{0.0125^2}{9(0.0125) + 10(0.150)} \right] = 0.0097\ \mu\text{H} = 9.7\ \text{nH}$$

Highest Q generally results when the coil diameter $D = 2r$ is approximately equal to its length. Then (2.12-1) can be written

$$L = \frac{n^2 D}{58} \quad (\mu\text{H}) \tag{2.12-2}$$

Solving for n,

$$n = \sqrt{\frac{58L}{D}} \tag{2.12-3}$$

For example, suppose we wish to make a high-Q RF choke for use in a radio transmitter having 200 nH (0.20 μH) of inductance and wound as a $\frac{3}{8}$-in.-diameter coil on a nonmagnetic form. How many turns are required if we choose $l = D$?

$$n = \sqrt{\frac{(58)(0.2)}{0.375}} = 5.56\ \text{turns}$$

High μ Inductor Cores—the Hysteresis Curve

If a coil or transformer is wound on a high-permeability material core ($\mu_R \gg 1$), such as ferrite material, the magnetic spins of the atoms of the material can be aligned with the applied H field, yielding a much higher flux density, B. This can produce a much higher inductance for a given coil size.

However, the effect is complicated by the factors illustrated in Figure 2.12-2. Starting with an initially unmagnetized core, we experience the behavior that originates at the $\vec{B}$–$\vec{H}$ axes' origin. The $\vec{B}$ increases very rapidly, having a slope of $\mu_{EFF} = \mu_0 \mu_R$. Then, as all of the core's atoms align with the applied $\vec{H}$ field, saturation occurs at $\vec{B}_{SAT}$ and thereafter $\vec{B}$ increases with applied $\vec{H}$ at μ_0 slope.

When the $\vec{H}$ field is removed, many of the core's atoms remain aligned, leaving $\vec{B}_{REM}$. As $\vec{H}$ is cycled positively and negatively, a *hysteresis curve* (a

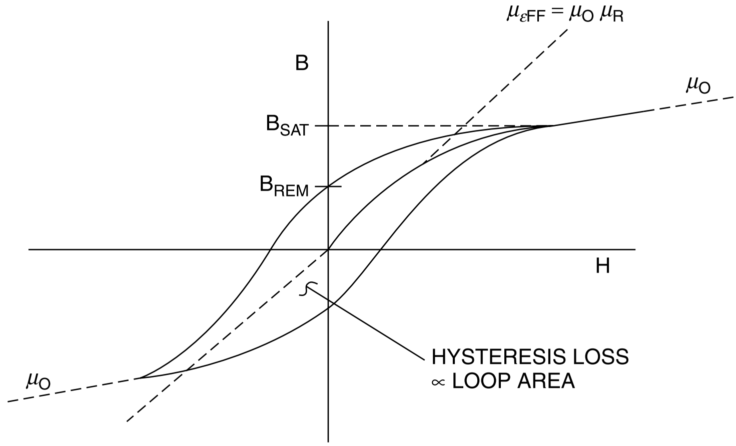

Figure 2.12-2 Use of magnetic core material can greatly increase flux density for given magnetic field ($\vec{B} = \mu_0\mu_R\vec{H}$).

curve having memory) is traced. The area within the curve is proportional to the AC losses when the coil is sinusoidally excited. Suppliers of inductors having high-permeability cores specify a maximum current for the coil's use. This is not an indication of the coil's power-handling capability, but rather the maximum current for which the coil provides its small signal inductance (operates below B_{SAT}). The hysteresis curve is also a function of temperature, and the coil's specification includes an indication of the highest temperature at which the coil's inductance specification is met. Above a critical temperature T_C the spins of the material's atoms do not align with the applied H field, and the effective permeability is only μ_0.

Estimating Wire Inductance

There are numerous formulas given for the inductance of an isolated straight wire. These are approximations, having an implicit assumption about the nature of the return current path (Fig. 2.12-3). *An inductance determination requires a closed path* just as the current in the coil requires a closed path. The inductance of a truly isolated straight wire is as undefined as the sound of one hand clapping! An exact relation (7.25-17) exists for the high frequency inductance per unit length of a coaxial cable. The high frequency approximation assumes that the $\vec{E}$ and $\vec{H}$ fields are contained between the conductors and do not penetrate the conductors appreciably.

$$L = \frac{\mu_0\mu_R}{2\pi} \ln \frac{b}{a} \text{ H/m} \tag{2.12-4}$$

$$L = 5.08 \ln \frac{b}{a} \text{ nH/in.} \tag{2.12-5}$$

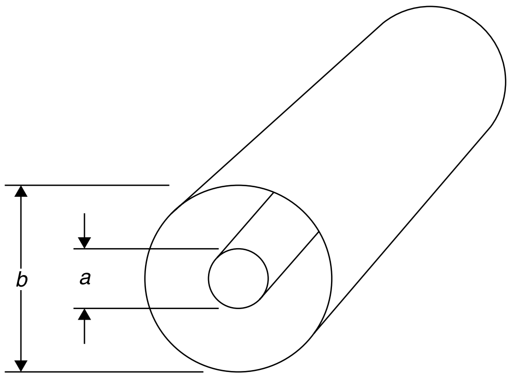

Figure 2.12-3 Inductance of isolated wire can be approached by that of coaxial line whose outer conductor is made very large.

where $\mu_R = 1$ for all nonmagnetic insulators. If we assume b is very large compared to a, then the calculated value for L approximates that of a straight wire with far removed return path.

For example, if $b/a = 7$

$$L \approx 10 \text{ nH/in.} \tag{2.12-6}$$

This is a reasonable approximation for the inductance of thin wires elevated above ground planes in microstrip circuits.

Note, however, that if we assume a value of $b = 50a$, the value for $\ln(b/a)$ doubles, and the inductance becomes 20 nH/in., also a value in the vicinity of those measured for real circuits in which the thin wire is well above the ground plane. The point to observe from the formula for the inductance of coaxial line is that the *inductance continues to increase as the outer conductor diameter is increased*, that is, as the return path for the inductive current is further removed from the center conductor "wire." It is a common but incorrect practice *to speak of the inductance of an "isolated" straight wire*. The better procedure is to approximate the inductance of a wire by the known inductance per unit length of an appropriately similar transmission line.

Parallel Plate Capacitors

The capacitance C of a parallel plate capacitor (Fig. 2.12-4) having plate area A separation d, and relative dielectric constant ε_R is given by

$$C = \varepsilon_0 \varepsilon_R \frac{A}{d} \tag{2.12-7}$$

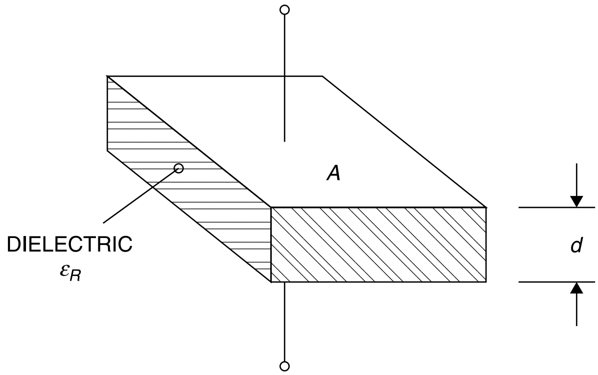

Figure 2.12-4 Parallel plate capacitor.

where

$$\varepsilon_0 = \frac{10^{-9}}{36\pi} \text{ F/m} \tag{2.12-8}$$

is the permittivity of free space. These expressions neglect the E field fringing at the edges of the capacitor. With fringing the capacitance is larger depending on the plate area-to-separation aspect ratio.

For engineering work, a more convenient format to estimate the parallel plate capacitance is [3, Appendix A]

$$\begin{aligned} C &= \varepsilon_0 \varepsilon_R \frac{A}{d} \\ &= \varepsilon_R (0.0885) \frac{A}{d} \text{ pF/cm} \\ &= \varepsilon_R (0.225) \frac{A}{d} \text{ pF/in.} \end{aligned} \tag{2.12-9a,b,c}$$

For example, suppose that we wish to construct an RF blocking capacitor. We have space for a 0.5-in.-diameter capacitor and will use 1-mil Mylar tape having a relative dielectric constant of 3 [3, Appendix B] as the plate separation dielectric. What will be the value of C?

$$C = (3)(0.225)\frac{(3.14)(0.25^2)}{0.001} = 132 \text{ pF}$$

This capacitor would have a reactance of only 1.2 Ω at 1 GHz, a reasonably small impedance in a 50 Ω system.

2.13 SKIN EFFECT

An important consideration in the design of circuits is that the effective resistance of conductors increases considerably at high frequencies. This is due to *skin effect*, which is a result of the crowding of current densities near conductor surfaces. This is analyzed in Section 7.18.

At any point, current flows in accordance with the incremental (or "point") form of Ohm's law, namely

$$\vec{J} = \sigma\vec{E} \tag{2.13-1}$$

where $\vec{J}$ (A/m^2) is the current density, $\vec{E}$ (V/m) is the electric field at the point, and σ (℧/m) is the conductivity of the material. This equation applies for any time variations. At high frequencies (sinusoidal excitations) Maxwell's equations play a special role in analyzing the effects in a conductor. The sinusoidally varying electric field $\vec{E}$ produces a sinusoidally varying current density $\vec{J}$, which in turn produces a sinusoidally varying $\vec{H}$ field, which in turn produces an opposing electric field. More about Maxwell's equations in Chapter 7. The net result is that the $\vec{E}$ field in the conductor falls off rapidly with depth from the surface, and with it so does the current density [4, p. 234; 5, p. 43] (Fig. 2.13-1).

The current density in the conductor is

$$\vec{J} = \vec{J}_0 e^{-z/\delta_S} \tag{2.13-2}$$

where the *skin depth* is

$$\delta_S = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{2.13-3}$$

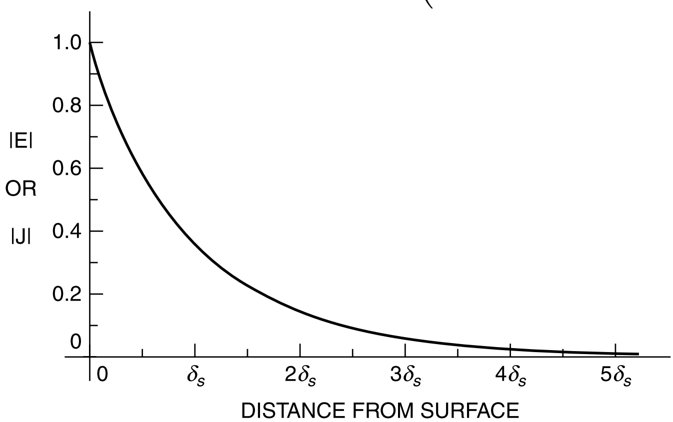

Figure 2.13-1 Fall-off of electric field $\vec{E}$ and current density $\vec{J}$ with distance from surface in skin depth units.

and f = frequency (Hz), $\mu = 4\pi \times 10^{-7}$ (H/m) for nonmagnetic materials, and $\sigma = 6.17 \times 10^{7}$ $\mho$/m for silver.

At a frequency of 1 GHz, the skin depth in silver is calculated as

$$\begin{aligned}\delta_S &= \frac{1}{\sqrt{(3.14)(10^9\ \mathrm{Hz})(4)(3.14)(10^{-7}\ \mathrm{H/m})(6.17)(10^7\ \mho/\mathrm{m})}} \\ &= 2.03 \times 10^{-6}\ \mathrm{m}\end{aligned}$$

Thus, a silver conductor has a skin depth at 1 GHz of only 2 μm or about 0.1 mil! Since copper has about 90% the conductivity of silver, its skin depth is only 5% greater than that of silver, or about 2.1 μm at 1 GHz.

The term *skin depth* should not be misinterpreted. *Current does flow below the skin depth*, but its density at this depth is reduced to 1/e (about 37%) of its surface value. At 10 skin depths (about 1 mil in silver or copper) the current density would be reduced to 1/22,026 of its surface value!

For calculation purposes, it turns out that *when the skin depth is small, the AC resistance of a conductor has the same value that it would have if all the current were carried in a material depth equal to the skin depth* [5, p. 45].

For round wire the AC resistance, R_{AC}, is determined by the cross section of a hollow tube having a wire diameter D and a thickness equal to the skin depth, δ_S (Fig. 2.13-2). This can be compared with the direct current (DC) resistance R_{DC} for which the entire cross section of the wire is effective:

$$R_{\mathrm{AC}} = \frac{l}{\pi D \delta_S \sigma} \tag{2.13-4}$$

$$R_{\mathrm{DC}} = \frac{l}{\pi (D^2/4)\sigma} \tag{2.13-5}$$

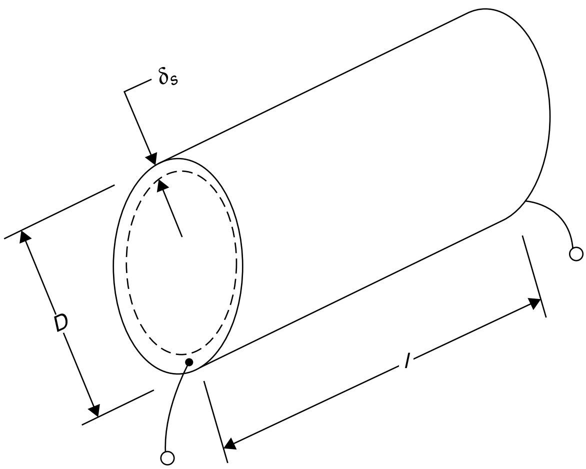

Figure 2.13-2 Skin depth region for solid round wire.

The ratio of the AC-to-DC resistance is

$$\frac{R_{\mathrm{AC}}}{R_{\mathrm{DC}}} = \frac{D}{4\delta_S} \tag{2.13-6}$$

Thus, a 30-gauge (10-mil-diameter) hookup wire used in a circuit at 1 GHz has

$$\frac{R_{\mathrm{AC}}}{R_{\mathrm{DC}}} = \frac{D}{4\delta_S} = \frac{(0.010 \text{ in.} \times 0.0254 \text{ m/in.})}{4(2 \times 10^{-6} \text{ m})} = 32$$

The DC resistance of this wire is only 0.1 Ω/ft, but its resistance at 1 GHz is 3.2 Ω/ft, a factor of 32 greater.

2.14 NETWORK SIMULATION

Using complex circuit analysis, it is possible to analyze any network to determine its performance as, for example, how much of an applied signal passes through the network to a specified load and what input impedance the network presents to a specified source. However, the application of complex mathematics is difficult for all but the simplest of networks, much more so if the analysis is to be conducted over numerous frequencies. Over the years, methods of simplifying the analysis were devised. These included filter theory (Chapter 9) and the Smith chart (Chapter 5). However, use of such techniques is generally limited to networks having specific circuit element values and relationships. For example, filter theory defines precisely the relationship of the circuit values to one another. The Smith chart is useful when the transmission line has uniform characteristic impedance.

Beginning in the 1960s computer programs became available for which a *variable topology* network could be specified and the network's performance calculated over a band of frequencies. These programs have been refined to a high degree, usually operate on a personal computer, and are generally termed *network simulators*. A sample network simulator screen is shown in Figure 2.14-1 using the Genesys software. The initial screen shown allows specification of a network. One selects components, such as resistors, inductors, capacitors, transmission lines, and so forth, from the toolbar above the workspace and *drags* them to the desired location in the schematic. Then *double clicking* on the component gives a submenu in which the component name and value can be specified. The sample network in Figure 2.14-1 was created in this manner. The program provides input and outputs whose impedances can be specified. These default to 50 Ω if no other values are specified.

Once the circuit is specified, selection of the *outputs* (a left menu option in Fig. 2.14-1) provides options for different kinds of outputs, such as rectangular or polar plots, a Smith chart format, an antenna plot, a three-dimensional plot, and so forth. It also allows specification of the functional output such as S_{11} or

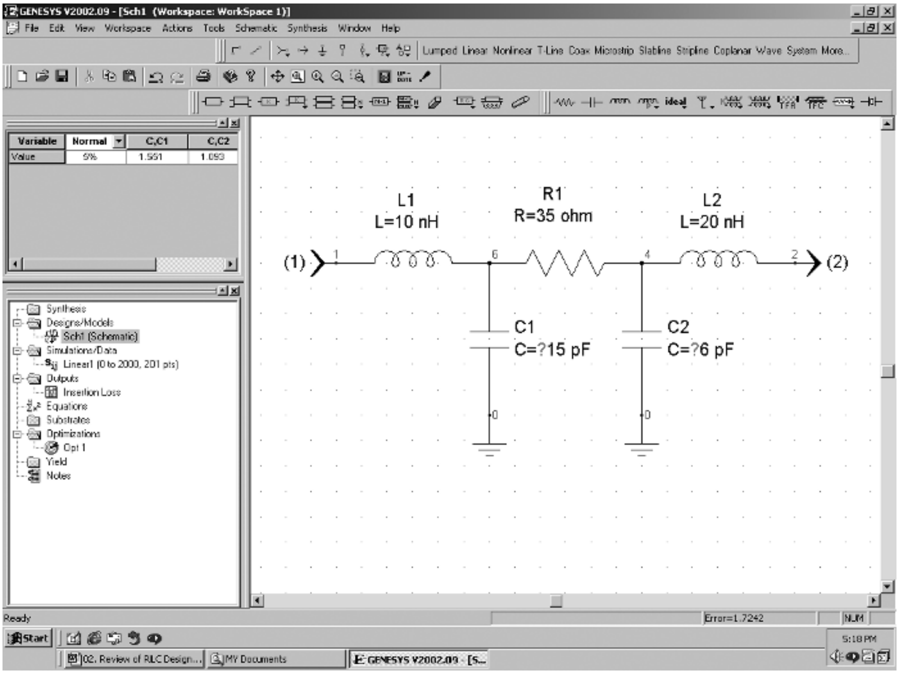

Figure 2.14-1 Inputting circuit to network simulator.

voltage standing wave ratio (VSWR) (to show reflection properties), the magnitude of S_{21} (for insertion loss) and phase of S_{21} to show transmission phase, and other functions. S parameters are described in Section 6.5. The insertion loss $|S_{21}|$ of the circuit in Figure 2.14-1 over the bandwidth 0 to 2 GHz is shown in Figure 2.14-2.

Even this simple graph would take hours or days to calculate manually. In this particular execution the insertion loss was calculated at 201 different frequencies, and each dot in the plot represents a separate calculation. However, the generation of the graph appeared almost instantaneously, since the personal computer on which it was performed can perform millions of calculations per second.

A *tuning* feature of modern network simulators causes the performance to be recalculated when the value of a given component is varied. This can be done manually or automatically. The automatic mode is termed *optimization*. For example, when the program varies the values of C1 and C2 and is directed to achieve a value of $|S_{21}| = 0$ at 1 GHz, the result is that shown in Figure 2.14-3. In this program, the variable nature of these components is denoted by the leading question mark before their values in Figure 2.14-1.

To achieve this optimization, the software varies each component whose value is allowed to be changed in a random manner and calculates the resulting

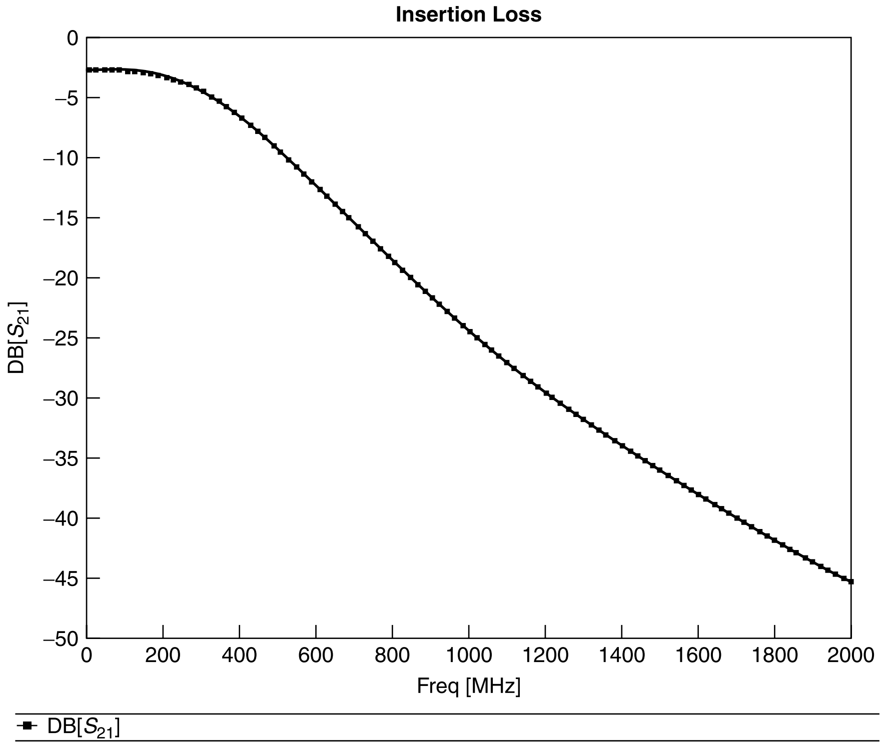

Figure 2.14-2 Insertion loss versus frequency for circuit shown in Figure 2.14-1.

performance, in this case the magnitude of S_{21}. If the change results in performance closer to the desired value, the change is retained and another adjustable component's value is changed. This process is repeated, usually millions of times, until the desired result is obtained. In this example, the optimization was terminated after a few seconds because zero insertion loss cannot be obtained due to the finite value of R1, whose value was not allowed to vary. Usually, such an optimization procedure would be wholly impractical to perform by manual calculation methods.

Simulation software has become an indispensable engineering tool. However, the termination just described exemplifies the need for human interaction with the computer. The computer does not appreciate that zero loss is unobtainable in a network having finite resistance, and the optimization routine would have continued indefinitely if left unattended.

Another application of varying element values is to perform a *yield analysis* of a circuit by which circuit element values are allowed to vary in accordance with a given distribution, such as a Gaussian distribution. The program *builds* a specified number of circuits to determine what the yield of circuits would be to

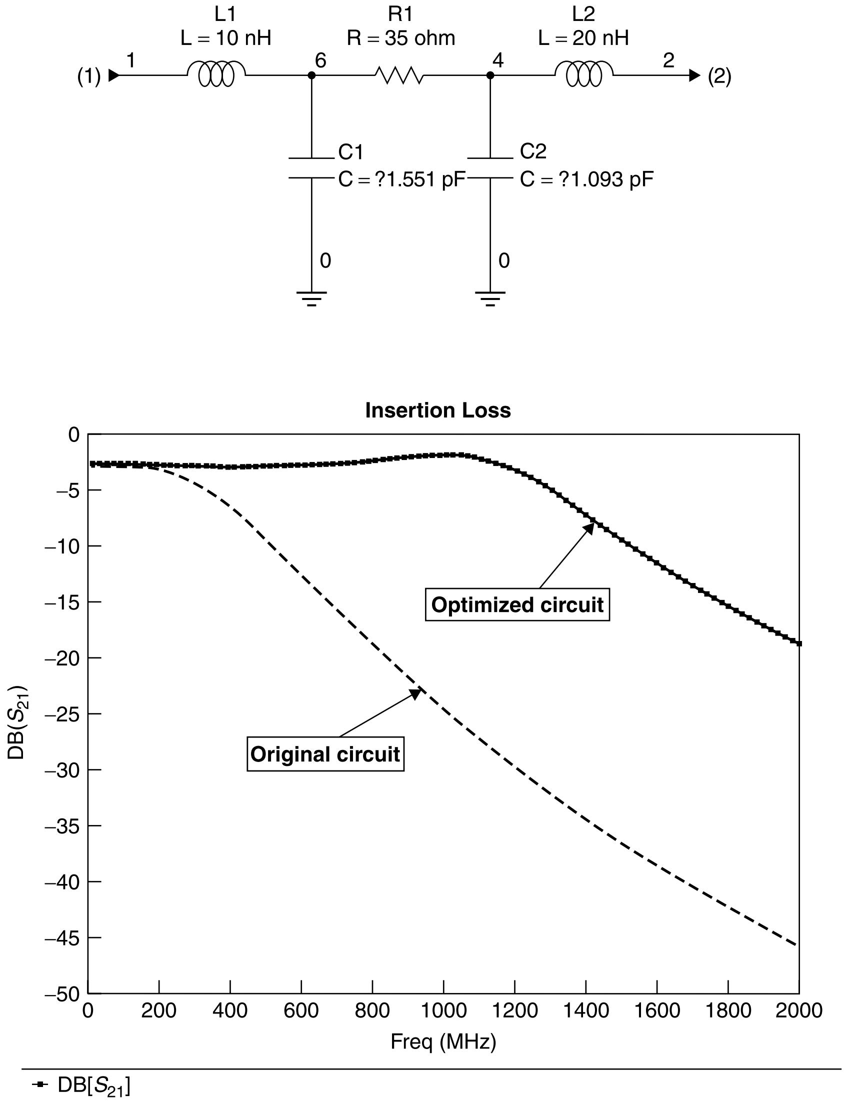

Figure 2.14-3 Result of optimizing values of C1 and C2 to minimize insertion loss at 1 GHz.

some required specification when the component parts have values with that statistical distribution. This analysis is demonstrated in Section 9.19.

For the most rigorous evaluation of networks *electromagnetic simulation* (called *EM simulation*) is performed, by which the electric and magnetic fields surrounding a circuit are computed and from them the circuit performance. This is demonstrated in Section 7.34.

The use of network simulation is especially important in transistor amplifier design. Most simulation programs contain a library of available transistors. The program can input a transistor's *S parameter table* and analyze the performance of a circuit containing the transistor (see Chapter 10 for examples). This permits stability and gain analyses at all frequencies over which the transistor has gain, a process that is essential to modern amplifier design.

These features of network simulation software have become essential to the conduct of efficient RF and microwave design. Each software package has its own format and features, but most simulation software available today performs at least the functions described. Throughout this text, network simulation is used to demonstrate the principles discussed, and network simulation exercises are provided to permit the reader to gain proficiency in these techniques.

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EXERCISES

E2.4-1 Four alternating currents at the same frequency have their reference directions into a common node. They are represented by:

$$i_1(t) = 10\cos(\omega t + 36.9^\circ)\ \mathrm{A}$$

$$I_2 = 14.1\angle 315^\circ\ \mathrm{A}$$

$$I_3 = 20\angle 126.9^\circ\ \mathrm{A}$$

$$I_4 = 13.4\angle 243.4^\circ\ \mathrm{A}$$

What is the total current (amplitude and phase) into the node?

E2.5-1 Realizing that you understand AC circuits, your friend asks for your help. He wishes to install a central air conditioner in his home. The unit requires $V = 220$ *V AC rms at 60 Hz* with a steady-state running current of $I = 50$ *A rms*. The maximum running current his wiring can provide is *40 A rms*. You note that the machine develops 10 horsepower (1 hp = 746 W) and that it is nearly 100% efficient.

a. What is the power factor of the machine (*power factor* = cos θ, where θ is the phase angle between V and I)?

b. Assuming the machine's motor presents an inductive load, what impedance Z does this machine present to the power line?

c. What is the capacitance C of a capacitor connected in parallel with this machine that would make the power factor unity? What steady-state rms current must it sustain?

d. What total steady-state, rms current would the machine draw with this capacitor installed in parallel?

E2.9-1 A 500-Ω load is to be connected to a 50-Ω transmission line to match load terminate it at 500 MHz. However, the hookup wire has 5 nH of series inductance. How can this be tuned such that the net reactance in series with the 50-Ω load is zero at 500 MHz?

E2.10-1 Show that the insertion loss (or isolation) of a normalized admittance, $y = g + jb$, mounted in shunt with a matched generator (Figure 2.10-3) and load of $Z_0 = 1$ is given by

$$\mathrm{IL} = \left|1 + \frac{y}{2}\right|^2 = 1 + g + \frac{g^2}{4} + \frac{|b|^2}{4}$$

E2.12-1 It is proposed to realize the tuning capacitor of Exercise 2.9-1 as a pair of circular parallel plates separated by 1 mil of double-sided tape having a dielectric constant of 3. What should the diameter D of the plates be made to realize the required capacitance?

E2.12-2 A circuit designer requires a high Q small volume coil of $L = 50$ nH to be formed on a wood mandrel of 0.125-in. diameter using #30 gauge copper wire having a diameter of 0.010 in. (10 mil).

a. How many turns of wire are required?

b. What is the length of the wire required (not counting the ends used for hook-up)?

E2.13-1 Derive an expression for the Q of the coil in E2.12-2 in terms of its dimensions, wire conductivity, and frequency, taking skin effect into account. Then, using this expression, solve for the Q at 10, 100, and 1000 MHz.

CHAPTER 3

LC Resonance and Matching Networks

3.1 *LC* RESONANCE

When the reactance magnitudes of L and C are equal, the pair resonates. At resonance the net reactance of a series-connected *LC* circuit is zero (a short circuit), and the net susceptance of a parallel-connected *LC* circuit is zero and the net reactance is infinite (an open circuit).

The resonant frequency is obtained by equating the magnitudes of the reactances of the *L* and *C* elements. Since their impedances are of opposite signs, the net reactance is zero for a series circuit or infinity for a parallel circuit, resulting in *resonance* at ω_0:

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{3.1-1}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{3.1-2}$$

$$\omega = 2\pi f \tag{3.1-3}$$

$$f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{3.1-4}$$

The resonance solution can be obtained graphically. Given an 80-nH inductor, the reactance magnitude is 5 Ω at 10 MHz. Draw a line through this value having an upward slope and passing through successive 5's on each decade of log–log graph paper (Fig. 3.1-1). Next, suppose the capacitor to resonate when it has 0.32 pF, or a 5000-Ω-reactance magnitude at 100 MHz. Similarly, draw a line through this point having a downward slope, and passing through successive 5's. The intersection of the two lines defines the resonant frequency, seen to be 1 GHz for this example.

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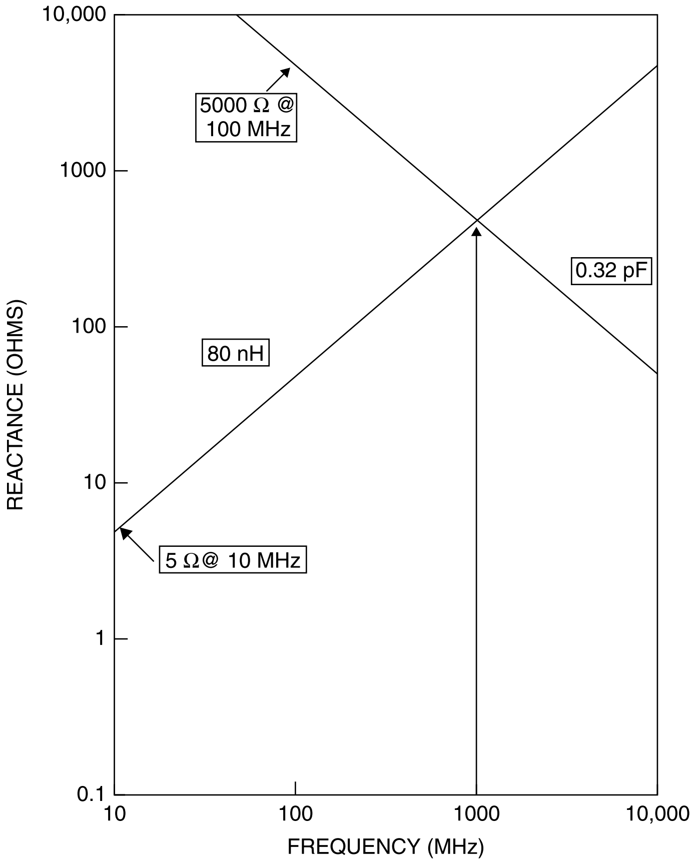

Figure 3.1-1 *L* and *C* reactance chart for determining resonance.

3.2 SERIES CIRCUIT QUALITY FACTORS

Q of Inductors and Capacitors

The *quality factor* Q of an inductor or capacitor is the ratio of its reactance magnitude to its resistance (Fig. 3.2-1). For an inductor,

$$Q = \frac{X_L}{R_L} = \frac{\omega L}{R_L} \tag{3.2-1}$$

and for a capacitor,

$$Q = \frac{X_C}{R_C} = \frac{1}{\omega C R_C} \tag{3.2-2}$$

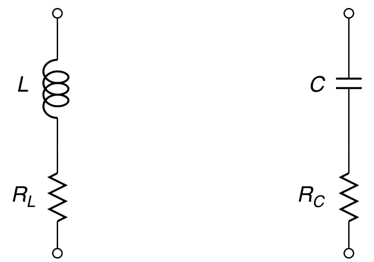

Figure 3.2-1 L and C models for Q_U definition.

When a series resonant circuit is formed from a LC pair, one can define a Q for the circuit at the resonant frequency ω_0. Called the *unloaded* Q, Q_U, it relates the energy stored in the reactive elements to the power loss in the resistive elements and, hence, is a measure of the quality of the resonator, itself. Thus,

$$Q_U = \frac{\omega_0 L}{R_L + R_C} = \frac{1}{\omega_0 C(R_L + R_C)} \tag{3.2-3}$$

Q_E, External Q

Even when a resonator is formed from an ideal, lossless LC pair, there is power dissipation when it is connected to external loads (Fig. 3.2-2). This dissipation affects the sharpness of the resonant response, which depends upon the ratio of the amount of energy stored in the capacitor or inductor relative to that dissipated in the external load or loads.

Under these conditions, *the external quality factor Q_E is the ratio of the magnitude of the reactance of either the L or C at resonance to the total external resistance:*

$$Q_E = \frac{\omega_0 L}{Z_0 + Z_0} = \frac{1}{\omega_0 C(Z_0 + Z_0)} \tag{3.2-4}$$

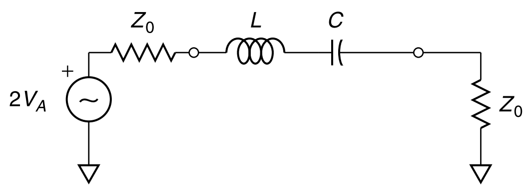

Figure 3.2-2 Series equivalent circuit of LC resonator with dissipation only in the external loads (Q_E model).

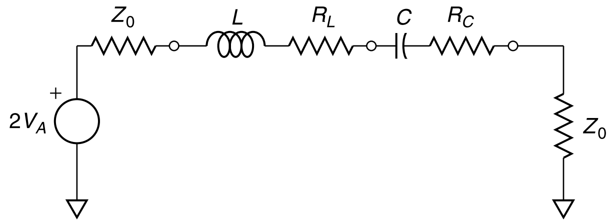

Figure 3.2-3 Q_L includes the effect of resonator resistances and external load.

Q_L, Loaded Q

The loaded quality factor Q_L is the ratio of the magnitude of the reactance of either reactor at resonance to the total circuit resistance (Fig. 3.2-3). In the case of a *doubly match-terminated resonator*, that is, one having a source resistance Z_0 and a load impedance Z_0, the total external resistance is $2Z_0$:

$$Q_L = \frac{\omega_0 L}{Z_0 + Z_0 + R_L + R_C} = \frac{1}{\omega_0 C_0 (Z_0 + Z_0 + R_L + R_C)} \tag{3.2-5}$$

From (3.2-3), (3.2-4), and (3.2-5), it follows that

$$\frac{1}{Q_L} = \frac{1}{Q_E} + \frac{1}{Q_U} \tag{3.2-6}$$

Thus, it is seen that the three Q factors add reciprocally. Equivalently, the product over the sum method can be used to calculate Q_L from Q_U and Q_E:

$$Q_L = \frac{Q_U Q_E}{Q_U + Q_E} \tag{3.2-7}$$

3.3 PARALLEL CIRCUIT QUALITY FACTORS

The dual of the circuit of Figure 3.2-3 is that shown in Figure 3.3-1 in which the resonator consists of shunt-connected elements. For the shunt equivalent circuits the Q's are defined as the ratio of the susceptance of *either* the L or the C to the appropriate conductances, which are, for the inductor,

$$Q = \frac{B_L}{G_L} = \frac{1}{\omega L G_L} \tag{3.3-1}$$

and for the capacitor

$$Q = \frac{B_C}{G_L} = \frac{\omega C}{G_L} \tag{3.3-2}$$

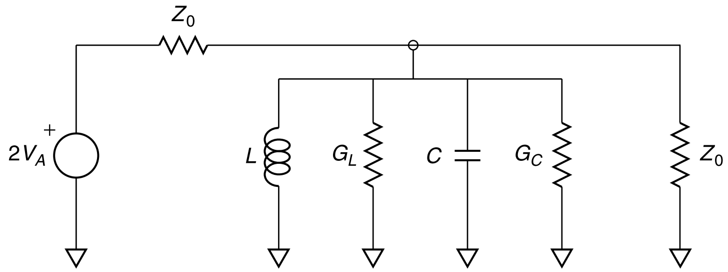

Figure 3.3-1 Equivalent circuit of parallel element resonator.

The overall Q values for the LC resonator are defined at ω_0 and given by

$$Q_U = \frac{\omega_0 C}{G_L + G_C} \quad Q_E = \frac{\omega_0 C}{2Y_0} \quad Q_L = \frac{\omega_0 C}{G_L + G_C + 2Y_0} \qquad (3.3\text{-}3\text{a, b, c})$$

The relationships among the Q values apply equally to the series and parallel resonant circuits.

3.4 COUPLED RESONATORS

Direct Coupled Resonators

When an LC resonator is placed in series or in shunt between matched generator and load, the result is *direct resonator coupling*. For ideal lossless resonator elements the insertion loss is *0 dB* at the resonance frequency, since all of the available generator power is delivered to the matched load.

For Q_L values above 10, the fractional 3-dB bandwidth of an LC resonator is well approximated as $1/Q_L$ [1, p. 315]. This is shown in Figure 3.4-1 for the case of a direct coupled series resonator in a 50-Ω system. Note that R_L and R_C are zero in this case.

A similar result is obtained when a parallel LC combination is directly coupled between generator and load, as shown in Figure 3.4-2. Note that in this case Q_L is calculated from (3.3-3c) with $G_L = G_C = 0$.

Lightly Coupled Resonators

A more general definition [1, p. 314] of Q_L for a resonator is

$$Q_L = \frac{\omega_0(\text{time-average energy stored in the system})}{\text{energy loss per second in the system}} \qquad (3.4\text{-}1)$$

where ω_0 is the resonant frequency of the resonator. This more general defini-

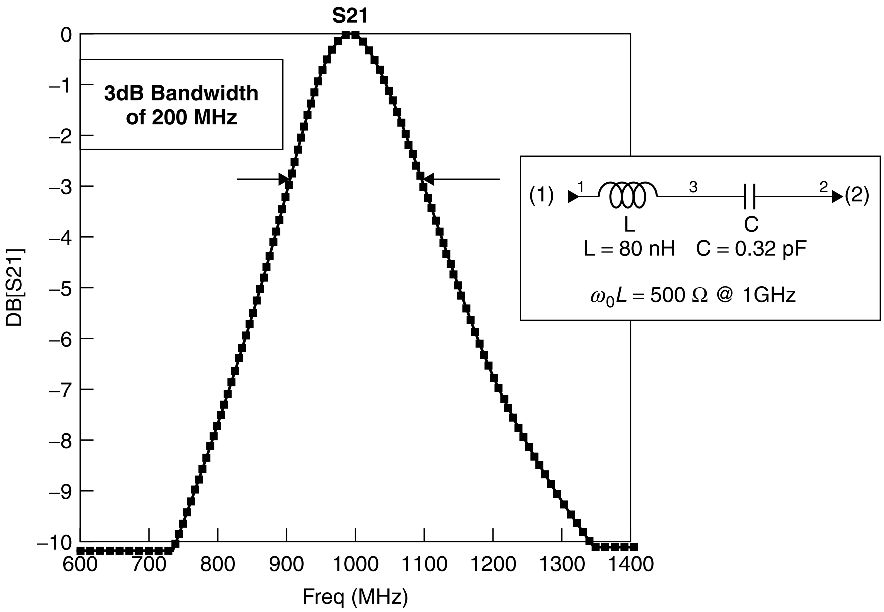

Figure 3.4-1 Resonant LC with $Q_L = 5$. Note that $Q_L = 500/(50 + 50) = 5$ and therefore the 3-dB bandwidth is $f_0/Q_L = 1000/5 = 200$ MHz.

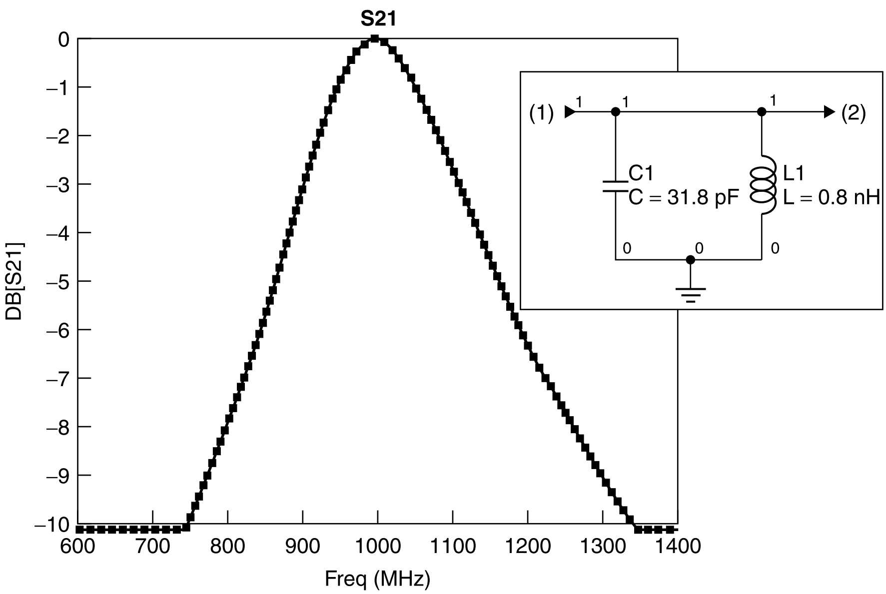

Figure 3.4-2 Shunt LC resonator mounted between 50-Ω source and load. The 3-dB bandwidth is 200 MHz, about 1000 MHz center frequency for a loaded $Q = 5$.

tion has the advantage that it can be applied to any type of resonator, including cavity and dielectric resonators. As with the lumped element model, the fractional 3-dB bandwidth remains a good approximation to the reciprocal of the loaded Q when $Q_L = 10$ or more.

The series and shunt resonators shown in the previous two examples are *directly coupled*. That is, either the current through the series resonator is the same as that through the load or the voltage across the parallel resonator is the same as that across the load. Consequently, to achieve a high Q_L the reactance magnitudes of the series elements at resonance must be relatively high compared to the load impedance. In a parallel circuit the magnitude of the shunt susceptance of each element at resonance must be correspondingly higher than the load conductance.

Practically, direct coupling constrains the amount of energy that can be stored in the LC resonator relative to that which is dissipated in the load because it is difficult to realize large ratios between resonator and load impedance magnitudes. For example, to obtain the modest Q_L of 5 at 1 GHz in the previous example with a load impedance of 50 Ω required that the reactance of the shunt inductance be 5 Ω, resulting in a very low inductance value of 0.8 nH.

If, instead, the resonator is only lightly coupled to the generator and load, the voltage across and the current through the L and C elements is theoretically unlimited. This can permit a much higher loaded Q, as the following example demonstrates. Of course, real resonators have finite losses, and therefore the Q_L is not unlimited in practice, but the following lossless example (Fig. 3.4-3) serves to illustrate the principles of the lightly coupled resonator.

Paradoxically, *the lighter the coupling the higher the stored energy in the resonator*. For comparison we first consider a direct coupled parallel resonator with resonant frequency of 1000 MHz, $L = 3.18$ nH and $C = 7.95$ pF. Directly coupled in a 50-Ω system, this results in a loaded Q of only 1.25, so low that the actual 3-dB bandwidth, as determined from circuit simulation, is about 800 MHz.

Next, the same resonator is coupled to generator and load through 1.16-pF capacitors, as shown in the schematic in Figure 3.4-3. The resulting 3-dB bandwidth is only 100 MHz. Some retuning of the resonator's capacitor from 7.95 to 5.911 pF was required because the capacitive coupling reactance increases the total capacitance of the circuit and would lower the resonant frequency if uncompensated. This configuration is called a *top-C coupled parallel resonator filter* [2, p. 155]. Notice that the resonator's inductor is 3.18 nH, four times as large as, and more practically realizable than, the 0.8 nH of the preceding 200-MHz direct coupled, shunt resonator example.

Physically, the action of the network can be visualized by considering what occurs when the generator is first turned on. This is a transient condition, not covered by our steady-state AC analysis. Due to the light coupling only a small amount of energy enters the resonator. The peak voltage across the capacitor does not build up as rapidly as it would with direct coupling because energy must enter through the high impedance of the coupling capacitor. Hence it may

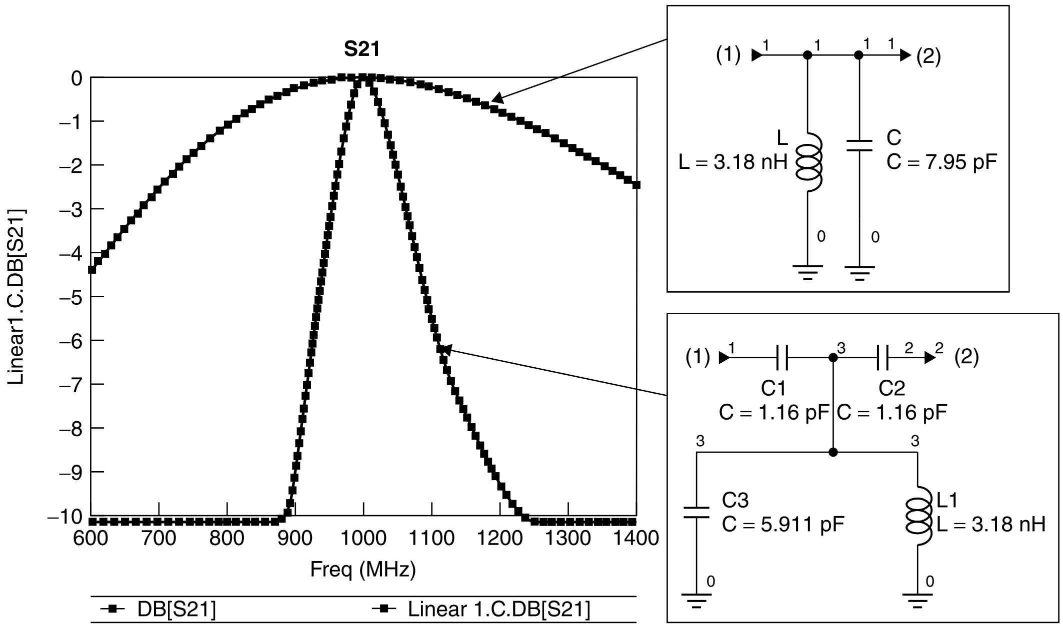

Figure 3.4-3 Performance comparison of direct and light coupling of *LC* resonator between matched generator and load.

take several RF cycles for the steady-state AC voltage to become established across the resonator.

The voltage increases, from cycle to cycle, until it is sufficient to block further energy increase from the generator. Since the network is symmetric in this case, in the steady state energy flows out of the resonator to the load at the same rate that it enters from the generator. For this ideal, lossless LC case there is no insertion loss at the resonant frequency. The higher the reactance of the coupling capacitors, the greater the resonator voltage (to block further power from entering) and the greater the stored energy in the resonant LC. This results in a higher Q_L and a narrower 3-dB bandwidth.

Why did the loaded Q of the lightly coupled resonator circuit increase by a factor of 8 relative to that of the directly coupled resonator having the same LC elements? We can infer that the energy stored in the coupled resonator also increased by a factor of about 8. This energy storage in a parallel LC resonator is proportional to the square of the voltage across it [1, pp. 314–315]. Clearly, the voltage across the top coupled resonator must have increased relative to the line voltage V_A of the direct coupled case.

This must be true because the same power reaches the load in both cases, yet for the top coupled circuit the load current must be driven through not only the 50 Ω load resistance but the series 1.16 pF capacitance as well, an additional $-j137\ \Omega$ at 1 GHz. The impedance of capacitor and load is $(50 - j137)\ \Omega$, and its magnitude is 146 Ω, requiring a voltage increase of 146/50 or 2.92. The corresponding energy storage increase in the LC resonator of the top coupled case is therefore a factor of 2.92^2 for a factor of 8.5, approximately equal to the observed 3-dB bandwidth decrease by a factor of 8.

3.5 *Q* MATCHING

A very useful method for accomplishing resistive impedance transformation at a given frequency is that of the LC network in which one element, either an inductor or a capacitor, is placed in series or parallel with the load to be transformed and the remaining element is made the opposite kind and placed, respectively, in parallel or series with the combination. Interestingly, the mathematical development of this matching technique arises from the formula for converting a series impedance to its parallel equivalent, or vice versa. To demonstrate the method and its basis, we review the *series-to-parallel equivalent circuit transformation* (Fig. 3.5-1).

Low to High Resistance

For this development, it is illustrative to derive the equivalence between Z and Y by first expressing Z in terms of its Q. As will be seen, this leads to a lumped circuit resistance transformation [3, p. 35]. In this exercise, we simply refer to the quality factor of Z as Q. We derive the result starting with a positive

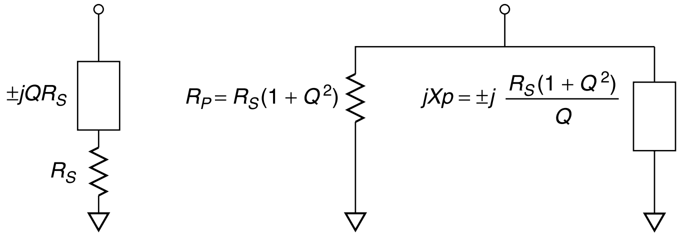

Figure 3.5-1 Series-to-parallel equivalent circuits.

imaginary value for Z, but the derivation is similar if begun with a negative imaginary value:

$$Z = R_S + jX$$

$$Z = R_S(1 + jQ) \tag{3.5-1}$$

$$Y = \frac{1}{Z} = \frac{1}{R_S(1 + jQ)} \cdot \frac{1 - jQ}{1 - jQ}$$

$$Y = \frac{1 - jQ}{R_S(1 + Q^2)} \tag{3.5-2}$$

$$Y = G - jB$$

$$G = \frac{1}{R_S(1 + Q^2)} \tag{3.5-3}$$

$$B = \frac{Q}{R_S(1 + Q^2)} \tag{3.5-4}$$

Converting the parallel admittance terms into parallel resistance R_P and parallel reactance X_P,

$$R_P = R_S(1 + Q^2) \tag{3.5-5}$$

$$X_P = \frac{R_S(1 + Q^2)}{Q} \tag{3.5-6}$$

The important point to notice is that *the parallel equivalent circuit of a series impedance has a resistance value that is higher than the series value by the factor* $(1 + Q^2)$. This suggests that resonating the reactance in the parallel equivalent circuit [4, p. 120] yields a lumped circuit, resistance transforming method. *To transform a low resistance to one of higher resistance, add sufficient series reactance to bring the parallel equivalent resistance to the value desired, and then resonate the resulting circuit by adding an appropriate parallel susceptance.*

Generally, the purpose in transforming a resistive load to a different value is to match the load to a circuit of different source impedance. This method can be called *LC matching* (because opposite reactor types must be used), *L matching* (because of the *L* orientation of the reactors), or *Q matching* [5] (because the *Q* is chosen based on the transformation ratio).

In the remainder of this text we will call the method *Q matching*. As an example, suppose we wish to transform 5 to 50 Ω at 1 GHz. This requires that we *increase* the resistance by a factor of 10. Accordingly

$$1 + Q^2 = 10$$
$$Q = 3$$

Since we wish to increase resistance, add a series reactor, say L_1 (either L or C could be used), to the 5-Ω resistor. Examination of (3.5-5) and (3.5-6) shows that *the parallel equivalent circuit has the same Q as the series circuit*. For example, if the series circuit consisted of a resistor and an inductor, its Q is given by

$$Q = \frac{\omega L}{R_S} = \frac{X_L}{R_S}$$

The Q of a parallel LC circuit is

$$Q = \frac{B_L}{G}$$

By substituting the values for B_L and G from (3.5-5) and (3.5-6), we obtain

$$Q = \frac{B_L}{G} = \frac{R_P}{X_P} = \frac{R_S(1 + Q^2)}{R_S(1 + Q^2)/Q} = Q$$

which demonstrates that the series and parallel LR circuits have the same Q. We expect this result since the series and parallel circuits are equivalent and therefore must have the same quality factor.

Given this condition, the reactance of L_2 in the parallel equivalent circuit can be written down immediately. It must have a parallel reactance of $50/Q = 16.7\ \Omega$ because in the parallel circuit the reactance magnitude is the resistance divided by the Q.

Next, add a capacitive reactance of equal magnitude to parallel resonate the circuit. The resulting net admittance is 0.02 ℧, or 50 Ω impedance at 1 GHz, the frequency at which the match is performed. Notice that *in the Q matching method, the two reactors are always of opposite type* (Fig. 3.5-2). This is because the final circuit is resonated to remove any net reactance or susceptance. In the present example, when these reactances are converted to their corresponding L and C values, the actual Q matched circuit is shown in Figure 3.5-3. Notice that the resistance peaks just above 1000 MHz. This circuit might be made to

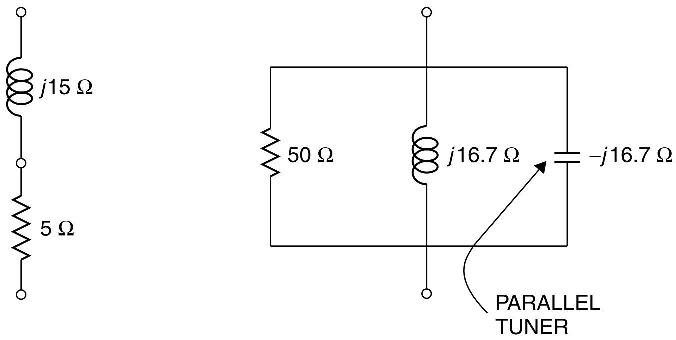

Figure 3.5-2 Q matching example, 5 to 50 Ω at 1 GHz.

have a broader bandwidth (closer to 50 Ω over a wider spread of frequencies around 1 GHz) by using the network simulator optimizer. Nevertheless, the transformation from 5 to 50 Ω was determined very simply using the Q method (Fig. 3.5-4).

Broadbanding the *Q* Matching Method

A single *LC* matching circuit has limited bandwidth. *To increase bandwidth, use multiple matching circuits, changing the resistance in smaller, equal ratio steps.*

For n-section Q matching of a resistance ratio R, make each successive section change the load impedance by the ratio $\sqrt[n]{R}$. Thus,

$$1 + Q^2 = \sqrt[n]{R} \tag{3.5-7}$$

To illustrate this multiple resistance step transformation, repeat the previous example using three sections. Note that $\sqrt[3]{10} = 2.154 = 1 + Q^2$; then $Q = 1.074$ for each matching section.

Therefore, the first section transforms the 5-Ω load to 2.154(5 Ω) = 10.77 Ω. The second section transforms 10.77 Ω to 2.154(10.77) = 23.20 Ω. The third section transforms 23.20 Ω to 2.154(23.20) = 50 Ω. When the Q matching method is performed three times to obtain the transformations listed above, the result is that shown in Figure 3.5-5.

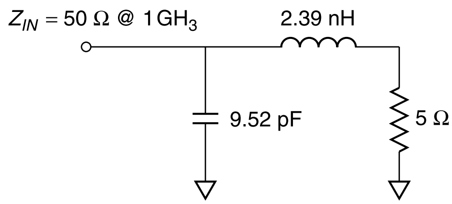

Figure 3.5-3 Final Q matched resistor network.

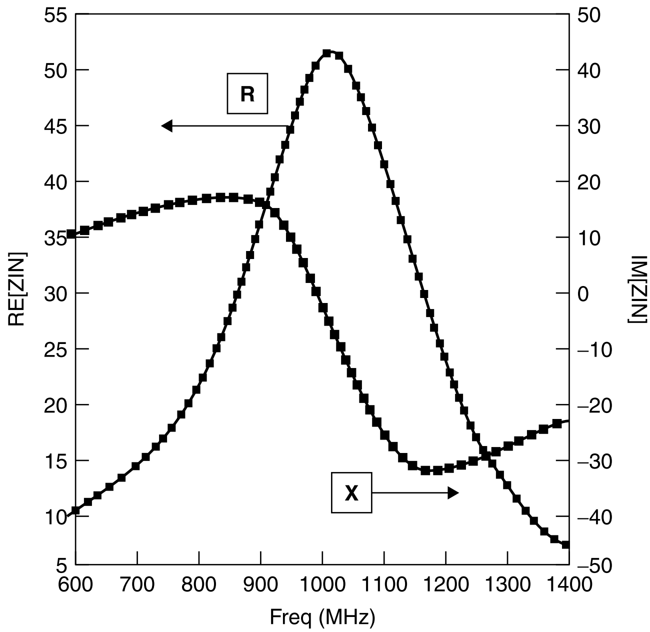

Figure 3.5-4 Frequency response of *Q* transformation network from 5 to 50 Ω.

Notice that with the three-section matching network, the bandwidth is much greater. The input resistance R is within 5 of 50 Ω from 800 to 1250 MHz, and the imaginary part, jX, is very near zero over the same range. This is a much broader bandwidth than the 970 to 1100 MHz bandwidth (Fig. 3.5-4) over which the single-stage matching yielded within 5 of 50 Ω for the real part and a much larger series reactance.

High to Low Resistance

If the resistance of a load is higher than desired, the same Q matching method can be used except that a shunt reactor is used as the first transformation element.

The same equivalent circuits and circuit element relations given in Figure 3.5-1 are employed, but in this case the transformation is from right to left, that is, from the parallel equivalent to the series equivalent circuit. We demonstrate this with a load circuit consisting of a 250-Ω resistor in series with a 0.7958-pF capacitor ($-j200$ Ω at 1 GHz). This example also demonstrates how to absorb the existing series reactance. We wish to transform this impedance to 50 Ω at 1 GHz.

The $(250 - j200)$-Ω load is shown in Figure 3.5-6a. Since the 250-Ω real part exceeds the desired 50 Ω, a parallel circuit is needed to use the *Q* transformation approach. The first step is to transform this series circuit to its parallel

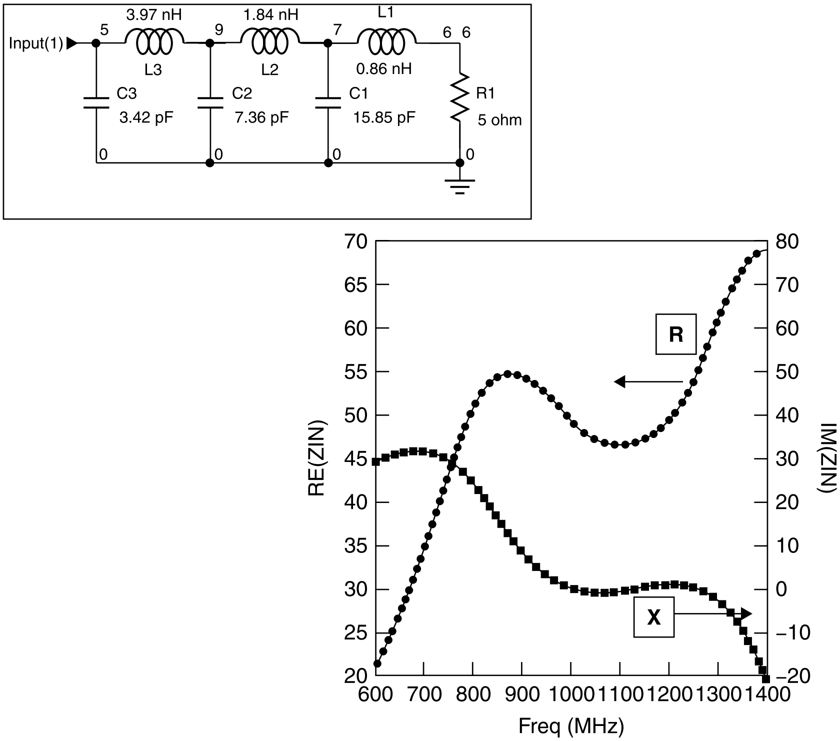

Figure 3.5-5 Three-section Q matching from 5 to 50 Ω.

equivalent. The parallel equivalent is easily obtained by noting that the Q of the load is 0.8. In the parallel equivalent circuit the shunt resistor will be larger than 250 Ω by the factor $(1 + Q^2) = 1.64$. Thus the shunt resistor is 410 Ω. The shunt equivalent circuit must have the same Q. Observing that in the parallel equivalent circuit the shunt reactance is related to the shunt resistor by the factor $(1/Q)$, we see that the shunt capacitive reactance magnitude must be $410/0.8\ \Omega = 512.5\ \Omega$. At 1 GHz this corresponds equivalently to a capacitance

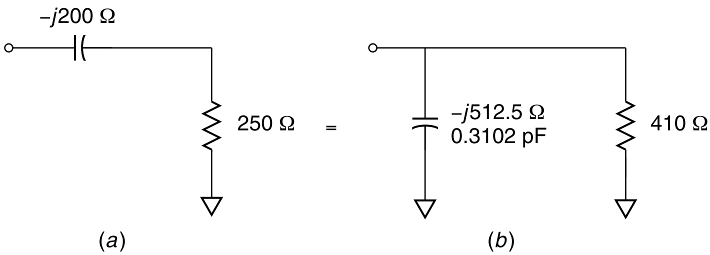

Figure 3.5-6 (*a*) The (250 − *j*200)-Ω load and (*b*) its parallel equivalent circuit.

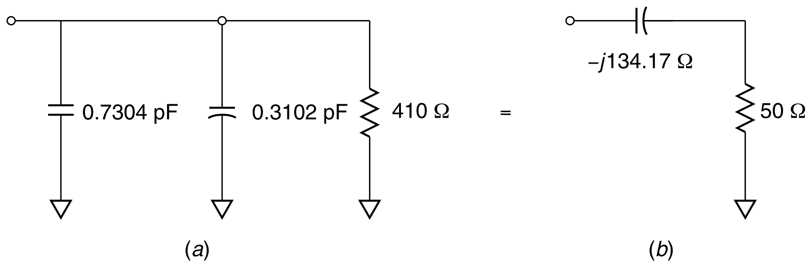

Figure 3.5-7 Equivalent circuit of load after the addition of 0.7304 pF capacitor.

of 0.3102 pF. Note that *the nature of the reactance is unchanged between the series and parallel equivalent circuits*. In this case, the reactance is capacitive.

Next, the task is to transform 410 to 50 Ω. This is a ratio of 8.2. Therefore, $1 + Q^2 = 8.2$ and $Q = 2.6833$. The large number of decimal places is carried to show that the method produces an arbitrarily accurate result if executed with sufficient precision. In order to change the Q of the parallel equivalent circuit in Figure 3.5-6b, we add shunt capacitance to produce a total shunt reactance of $410\ \Omega/2.6833 = 152.80\ \Omega$. This corresponds to a total capacitance of 1.0406 pF. Since 0.3102 pF is already available, only an additional 0.7304 pF need be added, as shown in Figure 3.5-7a. In this way the original 200 Ω reactance has been absorbed within the Q matching circuit.

The series equivalent circuit (Fig. 3.5-7b) has the required 50 Ω resistance. The magnitude of the series reactance must have the same Q of the parallel circuit, in this case $Q = 2.6833$. The resulting capacitive reactance magnitude is $50\ \Omega \times 2.6833 = 134.17\ \Omega$. The reactance type is unchanged by expressing the circuit in series format; hence, the reactance is capacitive, as shown in Figure 3.5-7b.

The final step is to resonate the $-j134.7\ \Omega$ with a series inductor (21.353 nH) of $+j134.7\ \Omega$ to obtain the desired 50 Ω real input impedance. The resulting circuit and its performance with frequency is shown in Figure 3.5-8.

This tuning could have been performed equally well, with the same Q values, by using a shunt inductor as the first tuning element. The inductive susceptance would be larger than that of the capacitor actually used, since it would be required to neutralize the extant capacitive susceptance already in place and then to provide the necessary susceptance magnitude required. But this would not change the required *Q of* 2.6833. With this procedure, the second tuning element would have been a capacitor instead of an inductor.

In some circuit applications having a series capacitor could be advantageous as a "DC block" to prevent bias currents from passing to the signal terminal. In other circuits the presence of a series inductor to conduct bias current might be useful. *The availability of two different circuit topologies for Q matching is a design advantage.*

As was demonstrated, the Q matching method is not limited to transforming a pure load resistance. The load can be complex. This method of tuning of

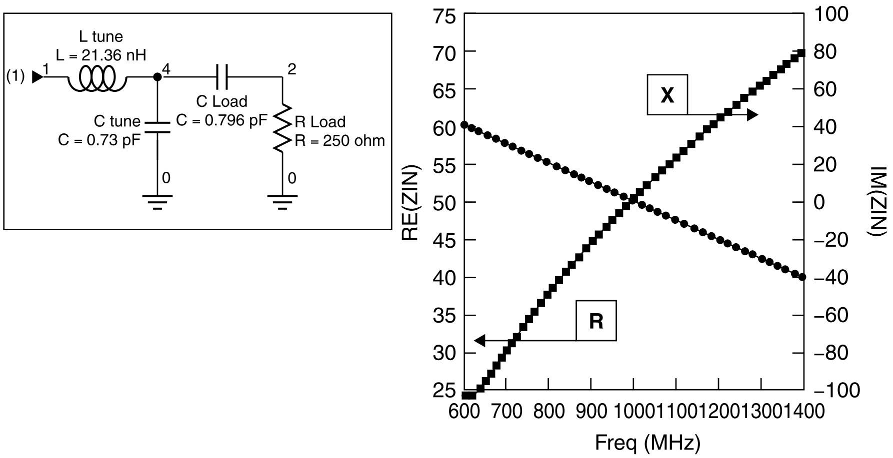

Figure 3.5-8 Circuit to tune $(250 - j200)$-Ω load to 50 Ω at 1 GHz.

using one series reactor and one shunt reactor can transform any lossy impedance to any resistance. This distinction about the load being lossy is necessary because, using lossless L and C elements, *a lossless load cannot be transformed into a lossy one, nor vice versa.*

In practice the range of transformations is limited by the fact that the tuning reactors do not have zero resistance and they have parasitic inductances and capacitances. Nevertheless, the typically realizable range and efficiency of tuning in practical circuits makes this Q matching technique a valuable design tool.

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3. Chris Bowick, *RF Circuit Design*, Butterworth-Heinemann, Woburn, MA, 1982. *This is a very readable and practical design guide to filter design and matching circuits.*
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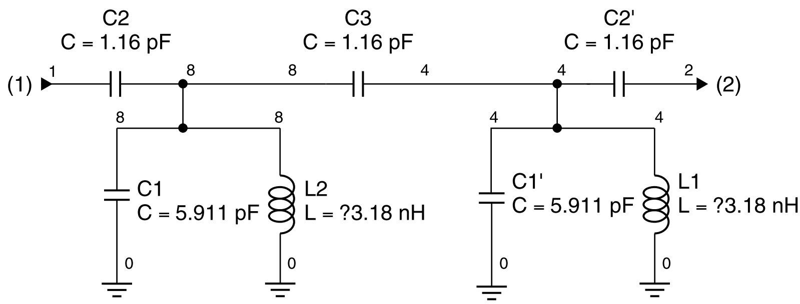
5. Pieter L. D. Abrie, *Q-Match: Lumped-Element Impedance Matching Software and User's Guide*, Artech House, Norwood, MA, 2000.

EXERCISES

Note. For exercises requiring the use of a network simulator, plot the magnitude of S_{21} in decibels for insertion loss and the magnitude of S_{11} in decibels for return loss. For a definition of S parameters, see Section 6.5 and for a return loss definition see Section 4.6.

E3.2-1 Use a network simulator to find the loaded Q of an 80-nH inductor having 5 Ω of series resistance and a 0.032-pF capacitor having 50,000 Ω of parallel resistance when the two are connected in series and used as a resonator in a doubly match-terminated 50-Ω line.

E3.2-2 Use a network simulator to design a dual resonator, top-C coupled network to provide a 3-dB bandwidth of 100 MHz and a center frequency of 1000 MHz. Start with the circuit values shown below, keep L1 and L2 fixed, and maintain symmetry with respect to the input and output ports of 50 Ω. Vary C1, C2, and C3 to obtain a 3-dB bandwidth of 100 MHz.



a. How does the frequency response compare with that of the single resonator, lightly coupled circuit of Figure 3.4-3?

b. What roles do the respective capacitors C1, C2, and C3 play in the frequency response of this network?

c. What points does this exercise demonstrate?

E3.2-3 If the inductors for the circuit in E3.2-2 are realized using a straight length of copper wire 0.025-in. diameter (22 gauge) and the conductivity of copper is 5.8×10^7 ℧/m:

a. What wire length is needed for the 3.18-nH inductors, assuming the wire is well removed from ground?

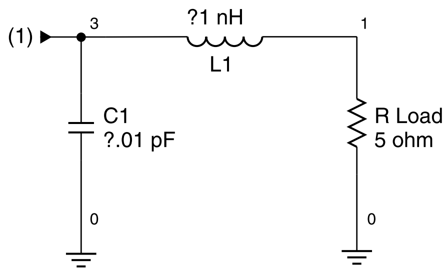
b. What is the inductor's resistance at 1000 MHz?

c. What is the inductor's Q at 1000 MHz.

d. How does this affect the resonant frequency insertion loss of the network?

e. What can you say about the practicality of this network as a radio filter?

E3.5-1 Use a network simulator to design a *LC* matching network to match a 5-Ω resistor to 50 Ω at 1 GHz. Do not use the optimizer. *Hint. Start with a series inductor L1 near the 5-Ω resistor and try to get the real part of the admittance at node 3 to be 0.02* ℧, *then parallel resonate the input with C1.*



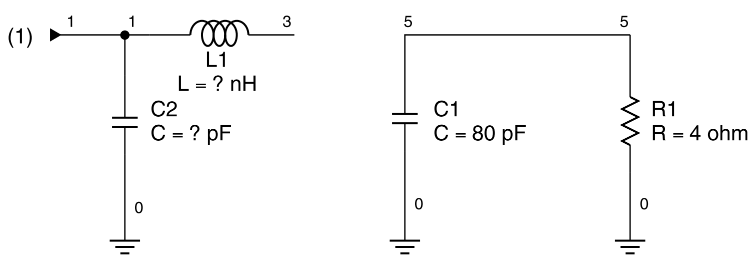
E3.5-2 **a.** Repeat E3.5-1 but this time use the *Q* matching method and transform 10 to 50 Ω.

b. Use a circuit simulator to compare the return loss versus frequency of this circuit with that used to transform 5 to 50 Ω in E3.5-1.

c. Was this manual calculation easier than the computer-aided solution used for E3.5-1?

d. How do the bandwidths of the transformations from 5 to 50 and from 10 to 50 Ω compare? Why is one narrower?

E3.5-3 A semiconductor manufacturer has hired you as a consultant. He wants to build a power transistor for 2-GHz applications and expects that it will have an input equivalent circuit consisting of 4 Ω in parallel with 80 pF. He has been told that a *LC* network can be designed to tune any impedance to 50 Ω, so he believes that his proposed transistor equivalent circuit should be all right.



a. What are the values for L1 and C2 of the tuner?

b. Examine the frequency behavior of the tuned circuit using a circuit simulator.

c. Comment on the practicality of this tuning arrangement.

E3.5-4 **a.** Use the Q matching method to design an 800-MHz LC matching circuit to match a transistor having 10 Ω input resistance in series with 0.8 nH of lead inductance to a 50-Ω source.

b. Use a circuit simulator and determine: How wide is the 20-dB-return-loss bandwidth? How wide is the 10-dB-return-loss bandwidth?

E3.5-5 **a.** For the application of E3.5-4, design a two-stage matching network.

b. Compare the performance of this circuit with that of the previous single-stage matching design. How much bandwidth improvement is obtained?

E3.5-6 Use the network simulator with the optimizer, varying all four elements of the two-stage tuner developed in E3.5-5. Set as an objective the achievement of a return loss of less than -20 dB (the magnitude of $S_{11} < -20$ dB) from 600 to 1000 MHz.

CHAPTER 4

Distributed Circuits

4.1 TRANSMISSION LINES

A single inch of wire can have 10 nH of inductance, an impedance of $+j63\ \Omega$ at 1 GHz. Interconnecting circuit parts with uncontrolled wire lengths would lead to large, unwanted reactances. It would also lead to unpredictable circuit behavior because circuit components having transmission paths larger than about one tenth of a wavelength impart not only reactive changes but resistive transformations in impedance as well.

Today, with computer assistance, it is possible to analyze circuits having nearly arbitrary geometry, and therefore any hookup wire lengths theoretically can be handled analytically. Nevertheless, it is inadvisable to use irregular transmission line geometries except when there is a specific good reason for doing so.

In the early 1930s computers were not available to analyze circuits, and it was necessary that engineers confine transmission circuit designs to simple, regular cross sections in order that performance could be predicted. These were called *uniform transmission lines*, and they permitted more predictable behavior for distributed circuits.

Even today, for *distributed circuits*, wherein *the lengths of the transmission paths are appreciable compared to the operating wavelength*, it is generally advisable to interconnect circuits and circuit parts with uniform transmission lines. Unless a transmission line is kept uniform in cross section, its effect on the circuit becomes impractically difficult to compute by hand, and it is equally difficult to gain insight into how the circuit works.

A transmission line is a set of conductors that are long compared to a wavelength (generally considered to be $>\lambda/10$*) and have a uniform cross section along their length for which a* characteristic impedance Z_0 *can be defined.*

In this chapter we first describe without proof the behavior of signals on transmission lines in order to provide a perspective of their operation. Later these transmission line circuit behaviors are derived analytically.

High Frequency Techniques: An Introduction to RF and Microwave Engineering, By Joseph F. White.
ISBN 0-471-45591-1 © 2004 John Wiley & Sons, Inc.

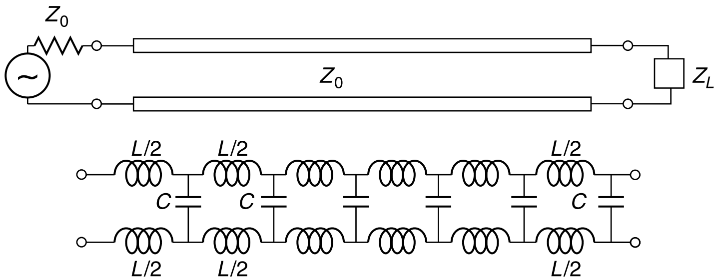


Figure 4.1-1 Balanced, two-wire, lossless transmission line with generator and load connections and the equivalent circuit.

The lumped circuit model for a lossless, balanced, uniform transmission line is shown in Figure 4.1-1. In the equivalent circuit, *L is the inductance/unit length* and *C is the capacitance/unit length* of the uniform transmission line. For the lossless case the *characteristic impedance* Z_0 of the line is given by

$$Z_0 = \sqrt{\frac{L}{C}}(\Omega) \tag{4.1-1}$$

where Z_0 *is defined as the ratio of voltage-to-current for a traveling wave in either direction*. For the lossless line Z_0 is independent of frequency, a fact that is most important for broadband matching. *When a transmission line is terminated in its characteristic impedance* (Z_0)*, no reflections occur*. That is, all power traveling toward the load is absorbed at the load, and none is reflected back toward the generator. There are other ways to accomplish total absorption of a traveling wave by the load, and we shall examine them as matching techniques.

The transmission line shown in Figure 4.1-2 is an example of a *balanced line* because both of its conductors have the same impedance to ground. For this reason stray voltage and currents induced onto either conductor of the line from interfering sources tend to be identical on both conductors, producing *common mode* voltage on the conductors and canceling insofar as the signal is concerned. The signal voltage is the difference in potential between the two conductors, and accordingly is called the *differential mode* voltage.

A balanced line example is the twisted pair phone line. The twist is added to further ensure that the two lines have identical impedances to ground, and, consequently, noise pickup is almost entirely in the common mode. This noise is not induced onto the signals carried by the conductors since the signals are impressed on the line in the differential mode.

Another example of a two-wire balanced transmission line is the *twin lead* used between an outside roof antenna and a TV set. Its geometry yields a characteristic impedance of about 300 Ω, more closely matching that of the source impedance of television antennas. Its balanced nature provides the same noise immunity to such sources as automobile ignitions, motors, and other

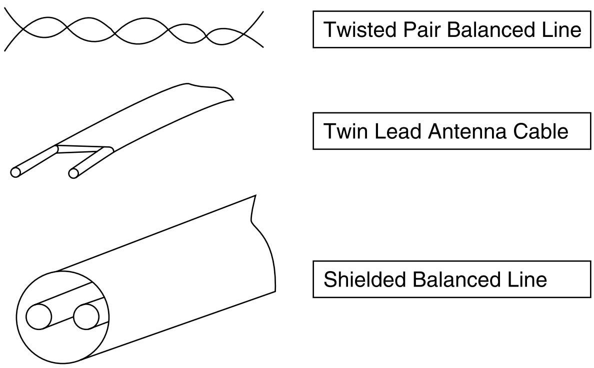


Figure 4.1-2 Examples of balanced transmission lines.

noise sources. However, it cannot provide immunity to noise that enters the system through radiation reception by the television aerial itself.

The balanced line pair can be further protected from interference by enclosing it within a shield, in which case the line is a *balanced shielded transmission line*.

For actual radio circuits *unbalanced transmission lines*, one of whose conductors is generally at ground potential, are more common (Fig. 4.1-3). The primary examples are stripline and microstrip, requiring only a "center conductor" pattern that can be photoetched for low cost and easy reproducibility.

Coaxial lines (also called *coax*) were developed to provide a flexible line with inherent shielding. They are used throughout the microwave bands, for example, up to 1 GHz for television cable systems and to 50 GHz or more in instrumentation applications.

Stripline evolved from coax in about 1950 and was considered a breakthrough due to its manufacturing ease. Its lack of mechanical flexibility was not a disadvantage since it was generally used to interconnect circuit parts installed on a common circuit board.

Microstrip evolved after stripline. It shared the reproducibility advantages of stripline but had the further advantage of providing easy access to circuit com-

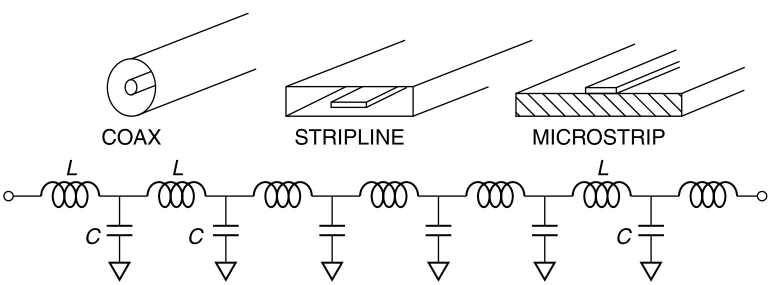


Figure 4.1-3 Unbalanced transmission line formats commonly used in wireless applications.

ponents since there was no upper ground plane to get in the way of component installation and tuning. However, microstrip's format usually results in an *inhomogeneous dielectric* medium, that is, a differing dielectric media for various portions of the transmission line's cross section. This occurs because microstrip is usually implemented as a conductor pattern on a dielectric material with air dielectric above the transmission line patterns.

As frequency increases, the capacitance between center conductor and ground causes an increasingly larger percentage of the energy to propagate within the dielectric, resulting in a greater signal delay for higher frequencies than for lower frequencies. This variation in signal delay with frequency distorts wideband signals because their various frequency components experience differing transmission delays through the inhomogeneous transmission line medium.

4.2 WAVELENGTH IN A DIELECTRIC

The *wavelength is the distance a sinusoid propagates in its period.* Wavelength is reduced in proportion to the square root of the dielectric constant ε_R of the propagating medium, since the speed of propagation is reduced in that proportion (Fig. 4.2-1). For a nonmagnetic dielectric

$$\lambda = \frac{\lambda_0}{\sqrt{\varepsilon_R}} \tag{4.2-1a}$$

$$v = \frac{c}{\sqrt{\varepsilon_R}} \tag{4.2-1b}$$

$$f\lambda = v \tag{4.2-1c}$$

$$f\lambda_0 = c \tag{4.2-1d}$$

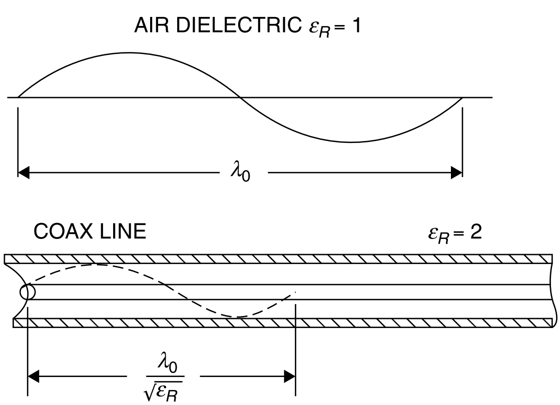


Figure 4.2-1 Effect of dielectric material is to decrease propagation speed, and with it, wavelength.

where λ_0 is the free space wavelength, c is the velocity of light in free space (2.9979×10^8 m/s), and λ is the wavelength in the dielectric medium.

Typical transmission line dielectrics range from pure Teflon with relative dielectric constant, $\varepsilon_R = 2.03$, to alumina (Al_2O_3) with $\varepsilon_R = 10$. Sapphire is the single-crystalline form of alumina and is an ideal, albeit expensive, microwave dielectric. Ceramic materials are available with relative dielectric constants up to 100 for use as primary resonators for oscillator circuits. Pure water has a dielectric constant of 81 [1] at RF and microwave frequencies.

Teflon material has very low dissipative loss and is a primary choice for microwave circuit boards. However, in pure form, it both expands and cold flows substantially with temperature and therefore is usually reinforced with embedded glass fibers for dimensional stability. The glass often is woven into a cloth fabric for reinforcing printed circuit (PC) boards. The presence of glass fibers increases the dielectric constant, and, if the fibers have a particular average orientation, as occurs with woven fabrics, the result is a dielectric constant that is a function of direction in the dielectric. Such a material is called an *anisotropic dielectric*.

To overcome the directional variation of dielectric constant the dielectric is sometimes reinforced using small, randomly oriented glass fibers (microfibers), making it uniform in all directions. A popular microfiber circuit board material is Rogers *Duroid*.

4.3 PULSES ON TRANSMISSION LINES

The concept of reflections on transmission lines can be visualized by considering the reaction of an open-circuited transmission line when a voltage pulse is sent down it.

Consider a voltage pulse applied to a lossless transmission line. The pulse propagates at the speed of light in a transmission line with air or vacuum surrounding the conductors or propagates at reduced speed if the line is embedded within a higher dielectric constant medium. For a lossless line, the pulse propagates along the line, as depicted in sketches 1 to 4 in Figure 4.3-1, its voltage undiminished with distance since there are assumed to be no dissipative or radiative losses. Associated with the voltage wave is a current wave. The ratio of voltage to current is Z_0.

In this theoretical idealization, on reaching the open-circuited end, all of the incident energy must be reflected since it can neither be radiated nor stored there. Of course, on a real open-circuited transmission line, some dissipative and radiative losses would occur, but we ignore them in this example.

Since all of the power of the pulse must be reflected at the open end of the line, the situation is equivalent to the introduction of a reverse-going pulse that is otherwise identical to the incident pulse. At the open circuit the current must be zero. To satisfy this condition, the reflected current must be directed opposite to the incident current. This requires that the associated reflected voltage

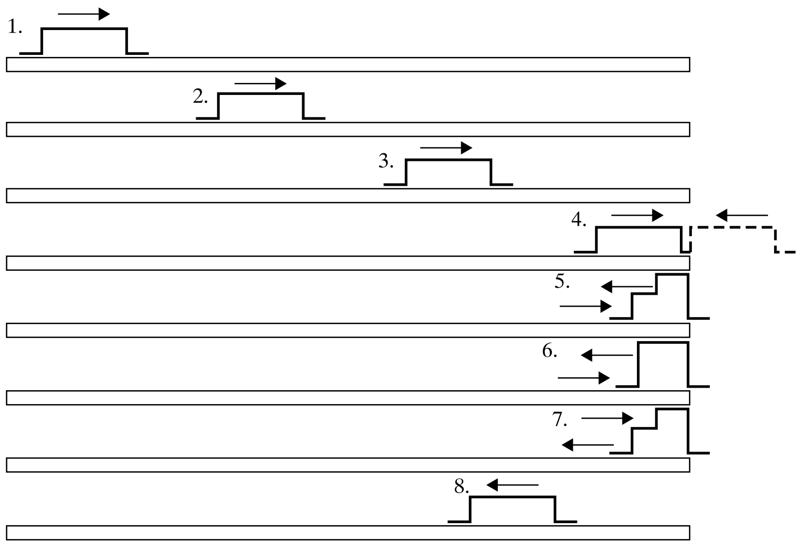


Figure 4.3-1 Pulse on lossless transmission line and its reflection from the open-circuited end.

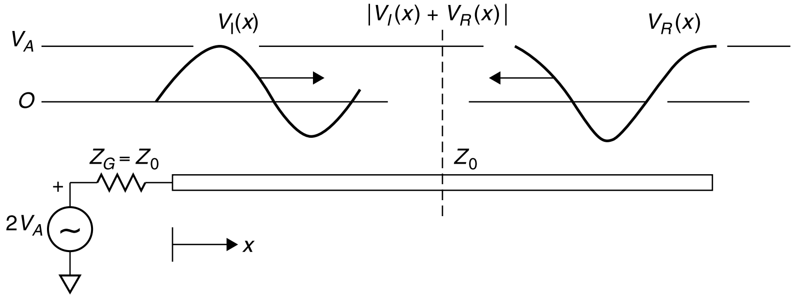
have the same polarity as the incident voltage, which results in a doubling of the voltage at the end of the line.

The total voltage on the line is equal to the sum of the incident and reverse-going pulses (sketches 4 to 7 of Fig. 4.3-1). Since the reflected voltage wave has the same amplitude as the incident wave, *reflection from an open-circuit terminated line produces a doubling of the voltage at the end of the line as the voltages of the incident and reflected waves add together*. The current at the end of the line is the sum of an incident current pulse and an equal magnitude but oppositely directed reflected current pulse, resulting in a net current of zero. We would expect this at an open-circuit termination.

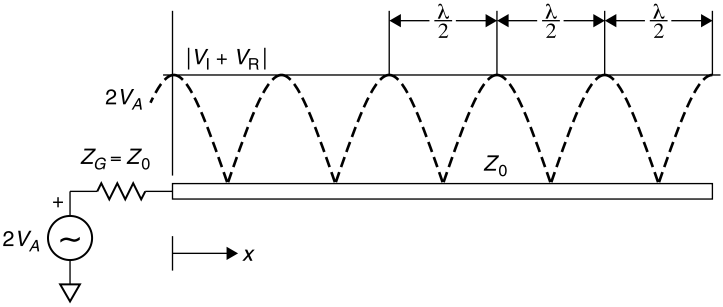
After reflection at the open-circuit end of the line, the pulse travels back toward its source. At the source the pulse could again reflect if the source impedance is not equal to the line's characteristic impedance Z_0. This pulse example is useful in demonstrating physically how the doubling of voltage occurs at the open-circuit termination of a transmission line. Similar reasoning shows that the current at a short-circuit termination of the transmission line produces a doubling of current at the line's end and a zero voltage.

4.4 INCIDENT AND REFLECTED WAVES

Just as pulses on a transmission line that is open circuited produce a doubling of the incident voltage, so also do continuous sine waves when applied to the same transmission line termination condition.



(a)



(b)

Figure 4.4-1 (*a*) Incident and reflected sine waves on open-circuit terminated line and (*b*) the peak voltage magnitude as function of position x on the line.

In this case, however, the sine wave, of voltage V_I, is continuously applied at the source end. The reflected voltage V_R, an identical sine wave propagating back toward the generator, combines with it to produce a *standing-wave pattern*, creating voltage maxima when V_I and V_R are in phase and minima when they are out of phase (Fig. 4.4-1). The peak voltage magnitude of the standing-wave pattern is shown in Figure 4.4-1*b*. The *maxima and minima are separated by a half wavelength due to the doubling of their relative velocities resulting from their opposite directions of travel.*

In general, the net voltage at any point on the line is the result of the phasor addition of V_I and V_R at that point. Note that the amplitude of the standing-wave pattern is not a sine wave. It is the envelope of the peak AC voltage as a function of distance along the line. Such voltage (and the associated current) standing waves are analogous to those of waves in water that reflect from a seawall or other reflecting obstacle. In this example of a lossless line terminated in an open circuit, the peak voltage is $2V_A$ and the minimum voltage is zero. At the voltage nulls the current is $2I_A$ where $I_A = V_A/Z_0$. This will be seen from the analysis of transmission lines to follow.

4.5 REFLECTION COEFFICIENT

In the open-circuit load, lossless line case, the magnitudes of incident and reflected voltages are equal, and at the "load" end of the line they are in phase. This produces the voltage doubling there. But for arbitrary loads, V_I and V_R are neither equal in magnitude nor do they have the same phase. We define a complex *reflection coefficient* at any point, x, on the line as

$$\Gamma(x) = \frac{V_R(x)}{V_I(x)} = \rho\angle\varphi \tag{4.5-1}$$

where V_R and V_I are reflected and incident phasor voltages, respectively.

On a lossless line the respective magnitudes of V_I and V_R do not change, thus *the magnitude of the reflection coefficient, ρ, does not change either.* Only the angle φ changes as the two waves travel by each other in opposite directions; φ depends upon the relative phases of V_R and V_I at the load as well as the electrical distance from the load.

For example, returning to the open-circuited case in Figure 4.4-1, since $\varphi_{\text{LOAD}} = 0°$, $V_{\text{LOAD-INC}} + V_{\text{LOAD-REF}} = 2$, a voltage *peak* occurs at the load. On the other hand, if there is a short circuit at the load, the total voltage at the end of the line must be zero. Then $\varphi_{\text{LOAD}} = 180°$, $V_{\text{LOAD-INC}} + V_{\text{LOAD-REF}} = 0$, and there is a voltage *null* at the load position.

In general, a line is terminated in some complex load, and the incident and reflected voltages have neither the same magnitudes nor are they precisely in or out of phase at the line's terminus. The magnitudes of the voltage peak and null values are

$$V_{\text{MAX}} = V_{\text{Incident}}(1 + \rho) \tag{4.5-2a}$$

$$V_{\text{MIN}} = V_{\text{Incident}}(1 - \rho) \tag{4.5-2b}$$

where ρ is the ratio of the magnitude of the reflected wave to that of the incident wave. One of the earliest microwave measurement techniques consists of cutting a narrow longitudinal slot in a transmission line and moving a rectifying diode along the line to determine $|V_{\text{MAX}}|$ and $|V_{\text{MIN}}|$ as well as the locations of these maximum and minimum voltages relative to the load position. This is called a *slotted line measurement*. Prior to the development of the network analyzer (about 1965) this was the only practical way to measure microwave impedance. Special microwave oscillators having a 1-kHz amplitude modulation were used in conjunction with *standing-wave ratio (SWR) meters*, consisting of tuned and calibrated 1-kHz amplifiers, to make the measurement more sensitive and precise. The ratio of the maximum and minimum voltage amplitudes was and continues to be called the *voltage standing-wave ratio*, or *VSWR*, sometimes just SWR:

$$\mathrm{VSWR} = \frac{|V_{\mathrm{MAX}}|}{|V_{\mathrm{MIN}}|} = \frac{1+\rho}{1-\rho} \tag{4.5-3a}$$

and conversely

$$\rho = \frac{\mathrm{VSWR} - 1}{\mathrm{VSWR} + 1} \tag{4.5-3b}$$

4.6 RETURN LOSS

Relative to the incident power, return loss is the fraction of power returned from the load:

$$\text{Return loss} = \rho^2 \text{ (fraction)} = 20 \log \rho \text{ (dB)} \tag{4.6-1}$$

Within the industry, VSWR and return loss are used as alternative means for specifying match, even though the slotted line is rarely if ever used for measurements. A perfect match of the load to the transmission line occurs when

$$Z_L = Z_0 \tag{4.6-2}$$

This condition produces no reflected wave, hence $\rho = 0$ and VSWR $= 1$. Then all incident power is absorbed in the load; no power is reflected, and, when expressed in decibels, the return loss is $-\infty$.

4.7 MISMATCH LOSS

Relative to the incident power, mismatch loss is the fraction of power absorbed (not returned) from the load:

$$\text{Mismatch loss} = 1 - \rho^2 \text{ (fraction)} = 10 \log(1 - \rho^2) \text{ (dB)} \tag{4.7-1}$$

In fractional form, return loss and mismatch loss sum to unity. Care should be taken to study these definitions. It is a common error to confuse them. Table 4.7-1 lists values of return loss and mismatch loss for various values of the reflection coefficient, ρ.

As an example, if the VSWR of a load is 1.5, ρ *is 0.2*, return loss is 0.04, and mismatch loss is 0.96. This means that the load absorbs 96% of the incident power and 4% is reflected. In decibels, the return loss is 14 dB and the mismatch loss is 0.18 dB. Notice that the minus sign for the losses in decibels is dropped conversationally because it is implied in the term "loss." However, *when using a network simulator, losses in decibels must be entered as negative quantities*.

TABLE 4.7-1 Relationships Among Reflection Coefficient Magnitude, ρ, VSWR, Return Loss and Mismatch Loss

ρ	VSWR $\frac{1+\rho}{1-\rho}$	Return Loss Fraction ρ^2	Return Loss (dB) 20 log ρ	Mismatch Loss Fraction $1-\rho^2$	Mismatch Loss (dB) $10\log(1-\rho^2)$
0	1.0	0	$-\infty$	1.0	0
0.01	1.02	0.0001	−40.0	0.9999	−0.0004
0.02	1.04	0.0004	−34.0	0.9996	−0.0017
0.05	1.11	0.0025	−26.0	0.9975	−0.011
0.10	1.22	0.01	−20.0	0.99	−0.044
0.15	1.35	0.0225	−16.5	0.9775	−0.099
0.2	1.5	0.04	−14.0	0.96	−0.18
0.3	1.86	0.09	−10.5	0.91	−0.41
0.33	2	0.11	−9.6	0.89	−0.50
0.4	2.3	0.16	−8.0	0.84	−0.76
0.5	3	0.25	−6.0	0.75	−1.25
0.6	4	0.36	−4.4	0.64	−1.94
0.7	5.7	0.49	−3.1	0.51	−2.92
0.707	5.8	0.50	−3.01	0.50	−3.01
0.8	9	0.64	−1.94	0.36	−4.44
0.9	19	0.81	−0.92	0.19	−7.21
1.0	∞	1.00	0.0	0.0	$-\infty$

4.8 MISMATCH ERROR

In Sections 4.6 and 4.7 return loss and mismatch loss were defined in terms of reflections at a mismatched load. However, these terms also can be applied to the input of a two-port network terminated by a matched or mismatched load. For example, a reactance connected in shunt with an otherwise match-terminated transmission line produces a reflection. For this circuit return loss and mismatch loss can be calculated.

Now, if there are two or more sources of reflection on the line (Fig. 4.8-1), the resulting reflections can combine to produce an overall reflection that depends not only upon the individual reflections but their reflective interactions as

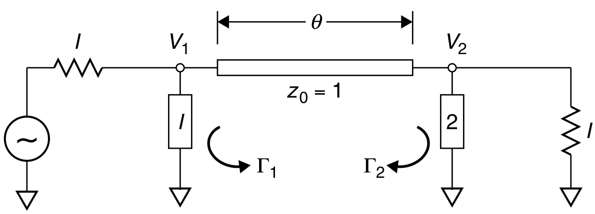


Figure 4.8-1 Interaction between two reflecting networks separated by transmission line length.

well. To evaluate this combination, suppose two circuits, having Γ_1 and Γ_2 complex reflection coefficients are interconnected by a lossless transmission line of electrical length θ. For this discussion it is assumed that generator and load are matched to each other and to the characteristic impedance of the line, in this case normalized to $Z_0 = 1$.

Due to the multiple reflections between them and their spacing, the obstacles will interact in a manner that causes their total insertion loss to vary. The loss of the combination may be higher or lower than their *simple loss*, that is, *the sum of their individual mismatch losses* that would be measured when they are separately connected between generator and load.

The amount by which the actual loss differs from the simple loss is called the *mismatch error*, usually expressed in decibels. The term *mismatch error* arises because, if we ignore their interaction and estimate their combined loss to be their *simple loss*, we would encounter an error equal to this mismatch error value. To analyze this reflection interaction it is useful to define the *transmission coefficient T*.

At the point of reflection on a transmission line, we defined an incident voltage V_I and a reflected voltage V_R. The phasor sum of these two voltages is the *total voltage* V_T. This is the voltage on the line at the point of reflection:

$$V_T = V_I + V_R \tag{4.8-1}$$

The transmission coefficient T is the ratio of V_T to V_I, just as the reflection coefficient Γ is the ratio of V_R to V_I. Then it follows that

$$T = 1 + \Gamma \tag{4.8-2}$$

where both T and Γ are complex numbers. Note that T can be greater than unity. In fact, this occurs at the end of an open-circuited transmission line at which $\Gamma = 1$ and $T = 2$.

The situation when two sources of reflection exist on a transmission line is shown in Figure 4.8-2. An incident voltage $V_1 = 1$ encounters the first obstacle and the voltage at the first obstacle is T_1. This voltage proceeds on toward the second obstacle. When T_1 reaches the second obstacle, it is further affected. In

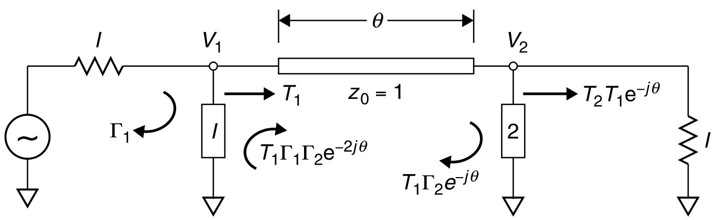


Figure 4.8-2 The reflection interaction between two circuits separated by lossless line length [1, p 407].

the absence of reflections between the obstacles, the resulting voltage at the second obstacle would be V_2 (*expected*). This voltage launches a wave traveling toward the load as

$$V_2 \text{ (expected)} = T_1 T_2 e^{-j\vartheta} \tag{4.8-3}$$

The "expected" insertion loss (IL), or simple loss, of the two obstacles, neglecting their reflective interactions, would be the sum of their individual mismatch losses or

$$\text{IL (expected)} = 20 \log|T_1|^{-1} + 20 \log|T_2|^{-1} \tag{4.8-4}$$

where the negative exponents are used in order that IL is a positive number when expressed in decibels.

However, when T_1 reaches the second obstacle, a part of its energy is reflected by it. This reflected wave travels back toward the first obstacle at which a third reflection occurs. The process continues in a manner similar to that of the multiple reflections occurring between facing, partially reflecting mirrors, creating an infinite series of reflections and re-reflections.

The initial waves and the multiple reflections are as described in Figure 4.8-2. Each time the reflections make a round trip between obstacles, they gain an additional phase change of $e^{-2j\theta}$ as well as the factor $\Gamma_1\Gamma_2$. Each such round trip path adds another term to the expression for the V_2 wave traveling toward the load. The result is an infinite series of terms:

$$\frac{V_2}{V_1} = T_1 T_2 e^{-j\theta}[1 + \Gamma_1\Gamma_2 e^{-2j\theta} + (\Gamma_1\Gamma_2 e^{-2j\theta})^2 + \cdots] \tag{4.8-5}$$

The magnitude of the quantity outside the brackets, $|T_1 T_2|$, is the simple loss ratio. The quantity inside the square brackets, $1 + \Gamma_1\Gamma_2 e^{-2j\theta} + (\Gamma_1\Gamma_2 e^{-2j\theta})^2 + \cdots$, is the effect of the reflective interaction between the two obstacles on the transmission line. Fortunately, this infinite series has a closed form [2, pp. 2–3], namely

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1 \tag{4.8-6}$$

Because $|\Gamma_1\Gamma_2 e^{-2j\theta}| < 1$ for passive obstacles, (4.8-5) reduces to

$$\frac{V_2}{V_1} = T_1 T_2 e^{-j\theta}[1 + \Gamma_1\Gamma_2 e^{-2j\theta}]^{-1} \tag{4.8-7}$$

Since insertion loss is $\text{IL} = 20 \log|V_1/V_2|$, the actual insertion loss for the circuit in Figure 4.8-2 is

$$\text{IL (actual)} = 20 \log|T_1|^{-1} + 20 \log|T_2|^{-1} + |\text{ME}| \text{ (dB)} \tag{4.8-8}$$

where ME is called the *mismatch error* and given by

$$\mathrm{ME} = 20 \log_{10}|1 - \Gamma_1\Gamma_2 e^{-j2\theta}| \text{ (dB)} \tag{4.8-9}$$

The mismatch error can either *increase* or *decrease* the base insertion loss of the two obstacles. The extreme values for ME are given, respectively, by using the $+$ and $-$ options. That is,

$$\mathrm{ME} \text{ (extreme values)} = 20 \log[1 \pm \rho_1\rho_2] \tag{4.8-10}$$

where ρ_1 and ρ_2 are the magnitudes of the reflection coefficients.

The larger the product of the reflection coefficients, the larger the mismatch error, and accordingly the larger the spread between maximum and minimum mismatch loss values. Customarily, *it is usually the extreme mismatch error values that are called the mismatch errors* because they are the error limits that can be experienced by ignoring the interaction of two reflections spaced along a transmission line. Notice that the *mismatch error occurs between each pair of obstacles*. If there are three obstacles spaced along a transmission line, there can be two separate mismatch errors.

A mismatch error can occur between a practical generator (or measurement system) and the reflections of a device under test if the generator is not perfectly matched to the transmission line. Mismatch errors also occur between connectors when neither has a unity VSWR and/or when one or more imperfectly matched connectors interact with an imperfectly matched device to which they are attached. In summary *a mismatch error can arise whenever two causes of reflection are spaced along a transmission system.*

Mismatch errors are especially large with cascaded reactive filters because these circuits accomplish their filtering action by reflecting power in the stopband. For example, suppose that two reactive filters are connected in cascade, each having 20 dB of isolation (when installed between matched source and load) at a frequency we wish to block. We would expect, neglecting reflective interaction, to obtain 40 dB of isolation in the stopband, the sum of the separate isolation values of the two filters. However, to provide 20 dB of isolation reactively, each filter must have $\rho = 0.995$, or $\rho^2 = 0.99$. For the pair of filters the mismatch error limits are

$$\mathrm{ME}_+ = 20 \log(1 + 0.99) = +6 \text{ dB} \tag{4.8-11a}$$

$$\mathrm{ME}_- = 20 \log(1 - 0.99) = -40 \text{ dB} \tag{4.8-11b}$$

In the first case, ME_+ has the same sign as the base losses. Thus, under ideal spacing of the filters, we can obtain 46 dB of stopband isolation. This is 6 dB greater isolation than expected from the simple addition of the individual isolation values. However, in the second case, ME_-, which occurs with the least optimal spacing, *placing two lossless, reactive 20-dB filters in cascade may yield*

0 dB of total attenuation. Nor is this unfavorable spacing unlikely since the stopband may represent a very wide frequency bandwidth over which the least favorable electrical spacing is likely to occur at various stopband frequencies.

Of course, real filters, even if designed as reactive networks, have finite loss, and the observation of 0 dB loss for a pair of them is a limiting, but unrealizable condition. Nevertheless, the calculation shown above suggests that a substantial reduction in isolation from that expected when ignoring VSWR interaction is readily possible, dependent upon their electrical spacing at each stopband frequency.

Interestingly, there is another way of treating an interacting pair of mismatches using their VSWR values [3, Example 4.5, 8]. The overall VSWR due to a pair of dissipationless mismatches having individual VSWR values SWR_1 and SWR_2 (where $SWR_1 > SWR_2$) that are referenced to and separated by a length of lossless Z_0 line can range from SWR_1/SWR_2 to $(SWR_1)(SWR_2)$.

Applying this to the previous example, for a reactive filter providing 20 dB of isolation, $1/(1-\rho^2) = 100$, therefore $\rho = 0.994987$ and the corresponding $SWR_1 = SWR_2 = 397.997487$. Since the VSWRs are equal, $SWR_1/SWR_2 = 1$ and their minimum insertion loss is 0 dB. The maximum VSWR $= (397.99)^2 = 158{,}402$. The corresponding $\rho = 0.999987$. The return loss is $1/(1-\rho^2) = 39{,}601 = 45.98$ dB, which probably agrees with the prior estimates of 0 and 46 dB within the accuracy of the significant figures carried in the calculations.

4.9 THE TELEGRAPHER EQUATIONS

To establish a mathematical basis for the analysis of transmission lines, we shall use a uniform line equivalent circuit that includes line losses, as shown in Figure 4.9-1. Note that, while the series elements are shown only in the upper conductor, this equivalent circuit is applicable to both balanced and unbalanced transmission lines. Also, this analysis applies to any waveforms that propagate on the transmission line, not just sinusoidal excitations.

In this equivalent circuit R, L, G, and C are, respectively, the series resistance, series inductance, shunt conductance, and shunt capacitance per unit

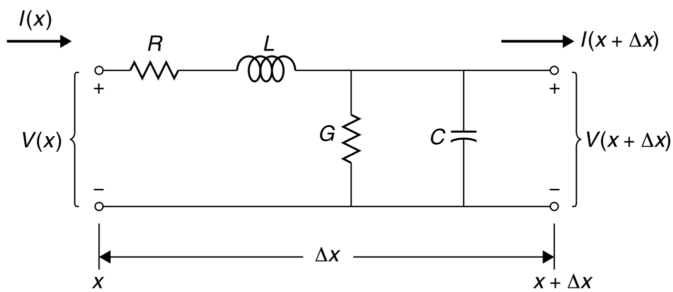


Figure 4.9-1 Equivalent circuit, including losses, for infinitesimally short length of uniform transmission line.

length of the transmission line. The small section of transmission line can be analyzed by applying Kirchhoff's laws and taking the llimit as this circuit length tends to zero to obtain the differential equations for the circuit [4, p. 23]. *Kirchhoff's voltage law* requires that the sum of the voltages about a closed loop is zero. Thus

$$v(x,t) - R\,\Delta x\, i(x,t) - L\,\Delta x \frac{\partial i(x,t)}{\partial t} - v(x+\Delta x, t) = 0 \qquad (4.9\text{-}1)$$

Similarly, applying Kirchhoff's current law, which requires that the sum of the currents into a circuit node be zero, gives

$$i(x,t) - G\,\Delta x\, v(x+\Delta x,t) - C\,\Delta x \frac{\partial v(x+\Delta x,t)}{\partial t} - i(x+\Delta x, t) = 0 \quad (4.9\text{-}2)$$

For the above two equations, divide all terms by Δx and let $\Delta x \to 0$. Note that as $\Delta x \to 0$

$$v(x+\Delta x,t) \to v(x,t)$$
$$i(x+\Delta x,t) \to i(x,t)$$

Also,

$$\frac{v(x+\Delta x,t) - v(x,t)}{\Delta x} \to \frac{\partial v(x,t)}{\partial t}$$
$$\frac{i(x+\Delta x,t) - i(x,t)}{\Delta x} \to \frac{\partial i(x,t)}{\partial t}$$

The resulting differential equations in (4.9-3) are the time-domain form of the transmission line equations and are called the *telegrapher equations*, presumably because their initial use was in the analysis and design of long telegraph lines:

$$\frac{\partial v(x,t)}{\partial x} = -Ri(x,t) - L\frac{\partial i(x,t)}{\partial t} \qquad (4.9\text{-}3a)$$
$$\frac{\partial i(x,t)}{\partial x} = -Gv(x,t) - C\frac{\partial v(x,t)}{\partial t} \qquad (4.9\text{-}3b)$$

4.10 TRANSMISSION LINE WAVE EQUATIONS

These telegrapher equations apply for any time-varying $v(x,t)$ and $i(x,t)$ on the transmission line. In the steady-state sinusoidally excited case, they reduce to

$$\frac{dV(x)}{dx} = -(R + j\omega L)I(x) \tag{4.10-1a}$$

$$\frac{dI(x)}{dx} = -(G + j\omega C)V(x) \tag{4.10-1b}$$

in which R, L, G, and C are per unit length and $V(x)$ and $I(x)$ are the phasor forms of $v(x,t)$ and $i(x,t)$, whose time variation is implicit. These two equations can be solved simultaneously to yield two equations, one a function of only V and the other a function of only I. To do so, for example, differentiate the second equation with respect to x to get

$$\frac{d^2I(x)}{dx^2} = -(G + j\omega C)\frac{dV(x)}{dx}$$

or

$$\frac{dV(x)}{dx} = -\frac{1}{G + j\omega C}\frac{d^2I(x)}{dx^2} \tag{4.10-2}$$

Then substitute this value for $dV(x)/dx$ into (4.10-1a).

$$\frac{d^2I(x)}{dx^2} - (R + j\omega L)(G + j\omega C)I(x) = 0$$

or

$$\frac{d^2I(x)}{dx^2} - \gamma^2 I(x) = 0 \tag{4.10-3a}$$

Using the same procedure, (4.10-1b) yields

$$\frac{d^2V(x)}{dx^2} - \gamma^2 V(x) = 0 \tag{4.10-3b}$$

in which we define

$$\gamma \equiv \sqrt{(R + j\omega L)(G + j\omega C)} \tag{4.10-4}$$

Solving differential equations involves removing the differentials (integrating) by guessing or recognizing the solution from prior differentiation experience. The only functions that are unchanged after double differentiation are the exponentials, $e^{\pm\gamma x}$. (A close possibility would have been $\sin \gamma x$, however, twice differentiating $\sin \gamma x$ would give *minus* $\sin \gamma x$.) A second-order differential equation such as this should have two independent solutions. In this case the two solutions correspond to the two values for the $\pm$ sign, they are

$$\frac{d^2 e^{\gamma x}}{dx^2} = \gamma^2 e^{\gamma x} \quad \text{and} \quad \frac{d^2 e^{-\gamma x}}{dx^2} = \gamma^2 e^{-\gamma x}$$

Accordingly, the sum of these two solutions represents the complete solution to either (4.10-3a) or (4.10-3b).

$$V(x) = V_I e^{-\gamma x} + V_R e^{+\gamma x} \tag{4.10-5a}$$

$$I(x) = I_I e^{-\gamma x} - I_R e^{+\gamma x} \tag{4.10-5b}$$

where γ is the propagation constant for the voltage and current waves. Use of the negative sign before I_R in (4.10-5b) results in the same value for the reflection coefficient for I as for V. Physically the two terms in each equation correspond to waves propagating, respectively, in the $+x$ and $-x$ directions. These two equations are called the *wave equations* for a uniform transmission line.

4.11 WAVE PROPAGATION

The terms with negative exponents (such as $V_I e^{-\gamma x}$) correspond to a wave propagating in the $+x$ direction, from left to right in Figure 4.9-1. Since the generator is attached at the left and the load at the right in Figure 4.9-1, this corresponds to an *incident wave*, hence the subscript I. Similarly, the positive exponent terms (such as $V_R e^{+\gamma x}$) correspond to a wave traveling in the $-x$ direction and represent a *reflected wave* in our convention. These use the subscript R.

The total voltage at any x location on the line is the phasor sum of the incident and reflected voltages, $V(x) = V_I(x) + V_R(x)$ and $I(x) = I_I(x) - I_R(x)$. We can cast the relationship between $V(x)$ and $I(x)$ into the format of impedance by resorting to their interrelation expressed in either (4.10-1a) or (4.10-1b). For example, invoking (4.10-1a) and substituting into it the value for $V(x)$ given by (4.10-5a) gives

$$\begin{aligned} I(x) &= -\frac{1}{R + j\omega L}\frac{dV(x)}{dx} = -\frac{1}{R + j\omega L}\frac{d(V_I e^{-\gamma x} + V_R e^{+\gamma x})}{dx} \\ I(x) &= \frac{\gamma}{R + j\omega L}(V_I e^{-\gamma x} - V_R e^{+\gamma x}) \end{aligned} \tag{4.11-1}$$

From this we see that the current, $I(x)$ has terms that are of the same form as those of the voltage $V(x)$, but there is a negative sign between them. This is because the current convention for reflected waves reverses while that for voltages does not.

The factor $(R + j\omega L)/\gamma$ has the dimensions of impedance and is called the *characteristic impedance* Z_0 of the transmission line. It can be rewritten as

$$Z_0 = \frac{R + j\omega L}{\gamma} = \frac{R + j\omega L}{\sqrt{(R + j\omega L)(G + j\omega C)}}$$
$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \tag{4.11-2}$$

With this definition, the equation for $I(x)$ has exactly the form of Ohm's law, noting that the minus sign between terms accounts for the reversal of current direction for a reflected wave. In other words, Z_0 *is the ratio of the respective voltage to current of waves traveling in either direction:*

$$I(x) = I_I(x) - I_R(x) = \frac{V_I}{Z_0} e^{-\gamma x} - \frac{V_R}{Z_0} e^{+\gamma x} \tag{4.11-3}$$

Notice that for a lossy line, for which R and G are nonzero, the characteristic impedance is a complex quantity that varies with frequency. However, *for the lossless line, the characteristic impedance is a constant, independent of frequency* and given by

$$Z_0 = \sqrt{\frac{L}{C}} \tag{4.11-4}$$

For high frequency, low-loss transmission lines, (4.11-4) is a good approximation, since $\omega L \gg R$ and $\omega C \gg G$. For a transmission line with loss, the propagation constant γ is a complex quantity. Conventionally this is written with real and imaginary parts as

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \tag{4.11-5}$$

where α and β are, respectively, the attenuation and phase constants of the transmission line with units of reciprocal length. The real term, α, is responsible for loss and is expressed in *nepers/unit length*; and the imaginary quantity, β, is responsible for phase change and is expressed in *radians/unit length*.

For a transmission line having loss, both the incident and reflected waves diminish in amplitude as they travel along the line due to dissipative and/or radiative losses. For an incident wave the amplitude diminishes with x according to the factor $e^{-\alpha x}$. For example, if α were 0.1 nepers/meter, then a wave traveling 1 m would have its amplitude decreased by the factor $e^{-0.1} = 0.9$. That is, the voltage of the traveling wave would be reduced by 10%. Since power is proportional to the square of the voltage, the power loss would be about 19% or 0.8686 dB for this 1-m length of line.

Notice that this example describes the conversion between loss in nepers and decibels, namely Loss (dB) = 8.686 loss (nepers), conversely Loss (nepers) = 0.115 loss (dB). These conversions are listed in the inside cover summaries for convenient reference.

As noted, the imaginary portion of the propagation constant β produces a phase change in the wave as a function of distance. It has the dimensions of *radians/unit length* or, optionally, *degrees/unit length*. An any instant of time, with time frozen so to speak, a snapshot of the voltage amplitude along the line of the incident wave (assuming there is no reflected wave for this example) would be a sinusoid. The distance between adjacent wave crests, or between adjacent troughs, or between any other similar pair of corresponding points of the wave sinusoid is the wavelength λ. Since the sinusoid completes a cycle in 2π radians, it follows that

$$\beta = \frac{2\pi}{\lambda} \text{ radians/unit length} \tag{4.11-6a}$$

$$\beta = \frac{360^\circ}{\lambda} \text{ degrees/unit length} \tag{4.11-6b}$$

For the lossless line,

$$\alpha = 0 \tag{4.11-7a}$$

$$\beta = \omega\sqrt{LC} \tag{4.11-7b}$$

Usually one does not know explicitly the distributed L and C values for a transmission line. Conventionally, the transmission line Z_0 is provided, along with the effective relative dielectric constant ε_R of the line. Therefore, it is more convenient to determine electrical length by first determining the wavelength on the line, finding the fraction of a wavelength of the circuit in question, and multiplying either by 2π for length in radians or by 360° to obtain electrical length in degrees. This is demonstrated by the following example.

Electrical length can be expressed in either radians or degrees. For example, consider a 6 in. length ($x = 6$ in.) of coaxial cable having a polyethylene insulator between its inner and outer conductors, for which the relative dielectric constant is 2.26. What is the cable's electrical length θ at 1 GHz? Since the wavelength in free space for a 1-GHz sinusoid is 11.8 in.,

$$\theta = \beta x = \frac{x}{\lambda} 360^\circ = \frac{x}{\lambda_0/\sqrt{e_R}} 360^\circ = \frac{6 \text{ in.}}{11.8 \text{ in.}/\sqrt{2.26}} 360^\circ = 275^\circ \tag{4.11-8}$$

Since 2π radians corresponds to 360°, the electrical length in radians is

$$\theta = 275^\circ \frac{6.28}{360^\circ} = 4.80 \text{ rad} \tag{4.11-9}$$

Notice that it was not necessary to determine β explicitly. However, since the wavelength in the cable at 1 GHz is 7.85 in., then

$$\beta = \frac{360^\circ}{7.85\text{ in.}} = 45.9^\circ/\text{in. at 1 GHz} \tag{4.11-10}$$

Note from (4.11-7b) that β is frequency dependent. In a nondispersive transmission media (for which electrical length is proportional to frequency), β is directly proportional to frequency. Thus, at 1.2 GHz the same cable would have a propagation constant of

$$\beta = \frac{1.2\text{ GHz}}{1\text{ GHz}} 45.9^\circ/\text{in.} = 55.1^\circ/\text{in. at 1.2 GHz} \tag{4.11-11}$$

and the same 6-in. cable would have an electrical length of 330° or 5.76 rad at 1.2 GHz.

4.12 PHASE AND GROUP VELOCITIES

The incident wave, traveling from the generator toward the load, is, from (4.10-5a)

$$V(x)_{\text{Incident}} = V_I e^{-\gamma x} \tag{4.12-1}$$

in which V_I is a phasor with implicit sinusoidal time variation. Written explicitly

$$v(x)_I = \text{Re}[V_I e^{-\gamma x} e^{j\omega t}] = Ce^{-\alpha x}\cos(\omega t - \beta x + \varphi) \tag{4.12-2}$$

where C is a constant amplitude factor and φ is the initial phase of the wave at $t = x = 0$. This is interpreted as a sinusoidal wave propagating in the $+x$ direction with an exponential attenuation due to the factor α. The phase constant β is given in (4.11-6), namely, $\beta = 2\pi/\lambda$. A similar equation can be written for the reflected wave that travels back toward the generator in the $-x$ direction. The voltage $v(x,t)_I$ has a constant phase for

$$\beta x = \omega t \quad \text{or} \quad x = \frac{\omega t}{\beta} \tag{4.12-3}$$

Therefore, a constant phase point on the traveling wave moves at a *phase velocity* given by dx/dt or

$$v_P = \frac{\omega}{\beta} \tag{4.12-4}$$

Thus, the voltage on the transmission line for either an incident or reflected wave varies as [4, pp. 45–46]

$$e^{j(\omega t-\beta x)} = e^{j\omega(t-x/v_P)} \tag{4.12-5}$$

Any periodic function of time can be represented as a sum of sinusoidal waves using a Fourier analysis (see Section 7.27). If v_P is the same for all of them, then their addition will faithfully reproduce along the line the temporal wave shape at the input, only delayed by the time of propagation x/v_P. This is the case for a lossless transmission line having $\gamma = j\beta = \omega\sqrt{LC}$ for which

$$v_P = \frac{1}{\sqrt{LC}} \tag{4.12-6}$$

For lossless transverse electromagnetic (TEM) mode transmission lines, the velocity at which a constant phase, v_P, moves on the line and the velocity at which a packet of information, such as a modulation envelope, moves, v_G, are identical and simply equal to the velocity of propagation, v.

This produces a convenient means for determining the characteristic impedance of the TEM mode for a lossless line. Since

$$Z_0 = \sqrt{\frac{L}{C}}$$

and

$$v = \frac{1}{\sqrt{LC}}$$

Therefore,

$$Z_0 = \frac{1}{vC} \tag{4.12-7}$$

This equation provides a convenient expression for determining the characteristic impedance of a TEM mode, lossless transmission line if its distributed capacitance and velocity of propagation are known.

However, for transmission lines having losses and for waveguides v_P does vary with frequency, dispersion occurs, and the signal waveform at the input will be distorted as its separate frequency components travel along the line at different velocities. If the dispersion is significant, it may not be meaningful to speak of a single propagation delay for a signal waveform. However, if the dispersion is moderate over the signal frequency bandwidth, an approximate *group velocity* v_G can be defined [4]. To determine its value, suppose that an incident voltage wave, v_1, consists of two frequencies that are traveling on the line, one lower and one higher in frequency than ω_0, respectively, having a combined value at $x = 0$ given by

$$v_I(x = 0, t) = A[\sin(\omega_0 - d\omega)t + \sin(\omega_0 + d\omega)t] \qquad (4.12\text{-}8)$$

Then, at a general point x on the line

$$v_I(x, t) = A\{\sin[(\omega_0 - d\omega)t - (\beta_0 - d\beta)x] + \sin[(\omega_0 + d\omega)t - (\beta_0 + d\beta)x]\}$$

in which β is assumed to be a function of ω and hence is the reason for the differential $d\beta$. Rearranging terms in the arguments gives

$$v_I(x, t) = A\{\sin[(\omega_0 t - \beta_0 x) - (d\omega\, t - d\beta\, x)] + \sin[(\omega_0 t - \beta_0 x) + (d\omega\, t - d\beta\, x)]\} \qquad (4.12\text{-}9)$$

Next, recognizing that the two sine terms are of the form $\sin(A - B)$ and $\sin(A + B)$, expanding them and simplifying gives

$$v_I(x, t) = 2A \cos(d\omega\, t - d\beta\, x) \sin(\omega_0 t - \beta_0 x) \qquad (4.12\text{-}10)$$

This reveals that the total voltage on the line corresponds to a high frequency sinusoid varying at ωt rate whose amplitude is modulated by

$$\cos[(d\omega)t - (d\beta)x]$$

which varies with both time and distance. The modulation, itself, has the properties of a traveling wave. A constant phase of the modulation occurs for

$$(d\omega)t = (d\beta)x \qquad (4.12\text{-}11)$$

from which a group velocity, $v_G = x/t$, can be defined as

$$v_G = \frac{d\omega}{d\beta} \qquad (4.12\text{-}12)$$

This can be related to the phase velocity $v_P = \omega/\beta$ as follows:

$$v_G = \frac{d\omega}{d\beta} = \frac{d\omega}{d(\omega/v_P)} = \frac{1}{(d/d\omega)(\omega/v_P)} = \frac{1}{[v_P - \omega(dv_P/d\omega)]/v_P^2}$$
$$v_G = \frac{v_P}{1 - (\omega/v_P)(dv_P/d\omega)} \qquad (4.12\text{-}13)$$

This result was derived for only two frequencies, but using similar reasoning it can be argued that a signal comprising many individual frequencies will travel with an envelope velocity approximately equal to v_G, if the dispersion is not too great. However, this will not be true for lines having large dispersion, but in those cases it is inappropriate to assign any velocity to such a modulation because its shape would be changing so rapidly as it propagates. The concept of group velocity also is not meaningful when $dv_P/d\omega$ is positive, a situation called *anomalous dispersion*, [4, p. 48] because then, as can be seen from (4.12-13) group velocity would appear to be infinite or negative.

4.13 REFLECTION COEFFICIENT AND IMPEDANCE

We can evaluate the reflection coefficient in terms of the load impedance by dividing (4.10-5a) by (4.11-3). For this evaluation let us choose $x = 0$ at the load, then at the load location the exponential functions equal one and the load impedance Z_L is given by

$$Z_L = \frac{V(0)}{I(0)} = Z_0 \frac{V_I + V_R}{V_I - V_R} = Z_0 \frac{1 + V_R/V_I}{1 - V_R/V_I} \tag{4.13-1}$$

where V_I and V_R are phasor quantities. But $\Gamma(x) = V_R(x)/V_I(x)$ is the definition of the reflection coefficient and therefore

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{4.13-2}$$

where, in general, Γ and Γ_L are complex numbers. Equivalently,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Y_0 - Y_L}{Y_0 + Y_L} \tag{4.13-3}$$

where $Y_0 = 1/Z_0$. From this we see *that there will be no reflected wave* if $Z_L = Z_0$, that is, when the load impedance is equal to the characteristic impedance of the line. Under this condition, all of the power incident on the line will be absorbed in the load.

Since the choice of $x = 0$ is arbitrary, (4.13-2) must apply for any x. It follows that

$$Z(x) = Z_0 \frac{1 + \Gamma(x)}{1 - \Gamma(x)} \tag{4.13-4}$$

and

$$\Gamma(x) = \frac{Z(x) - Z_0}{Z(x) + Z_0} = \frac{Y_0 - Y(x)}{Y_0 + Y(x)} \tag{4.13-5}$$

for all x.