Answers Pamphlet for

MATHEMATICS FOR ECONOMISTS

Carl P. Simon
Lawrence Blume

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Chapter 2

2.1 *i*) $y = 3x - 2$ is increasing everywhere, and has no local maxima or minima. See figure.*

ii) $y = -2x$ is decreasing everywhere, and has no local maxima or minima. See figure.

iii) $y = x^2 + 1$ has a global minimum of 1 at $x = 0$. It is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. See figure.

iv) $y = x^3 + x$ is increasing everywhere, and has no local maxima or minima. See figure.

v) $y = x^3 - x$ has a local maximum of $2/3\sqrt{3}$ at $-1/\sqrt{3}$, and a local minimum of $-2/3\sqrt{3}$ at $1/\sqrt{3}$, but no global maxima or minima. It increases on $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$ and decreases in between. See figure.

vi) $y = |x|$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$. It has a global minimum of 0 at $x = 0$. See figure.

2.2 Increasing functions include production and supply functions. Decreasing functions include demand and marginal utility. Functions with global critical points include average cost functions when a fixed cost is present, and profit functions.

2.3 1, 5, -2, 0.

2.4 *a*) $x \neq 1$; *b*) $x > 1$; *c*) all x; *d*) $x \neq \pm 1$; *e*) $-1 \leq x \leq +1$;
f) $-1 \leq x \leq +1, x \neq 0$.

2.5 *a*) $x \neq 1$, *b*) all x, *c*) $x \neq -1, -2$, *d*) all x.

2.6 The most common functions students come up with all have the nonnegative real numbers for their domain.

2.8 *a*) 1, *b*) -1, *c*) 0, *d*) 3.

2.8 *a*) The general form of a linear function is $f(x) = mx + b$, where b is the y-intercept and m is the slope. Here $m = 2$ and $b = 3$, so the formula is $f(x) = 2x + 3$.

b) Here $m = -3$ and $b = 0$, so the formula is $f(x) = -3x$.

*All figures are included at the back of the pamphlet.

c) We know m but need to compute b. Here $m = 4$, so the function is of the form $f(x) = 4x + b$. When $x = 1$, $f(x) = 1$, so b has to solve the equation $1 = 4 \cdot 1 + b$. Thus, $b = -3$ and $f(x) = 4x - 3$.

d) Here $m = -2$, so the function is of the form $f(x) = -2x + b$. When $x = 2$, $f(x) = -2$, thus b has to solve the equation $-2 = -2 \cdot 2 + b$, so $b = 2$ and $f(x) = -2x + 2$.

e) We need to compute m and b. Recall that given the value of $f(x)$ at two points, m equals the change in $f(x)$ divided by the change in x. Here $m = (5-3)/(4-2) = 1$. Now b solves the equation $3 = 1 \cdot 2 + b$, so $b = 1$ and $f(x) = x + 1$.

f) $m = [3 - (-4)]/(0 - 2) = -7/2$, and we are given that $b = 3$, so $f(x) = -(7/2)x + 3$.

2.9 *a*) The slope is the **marginal revenue**, that is, the rate at which revenue increases with output.

b) The slope is the **marginal cost**, that is, the rate at which the cost of purchasing x units increases with x.

c) The slope is the rate at which demand increases with price.

d) The slope is the **marginal propensity to consume**, that is, the rate at which aggregate consumption increases with national income.

e) The slope is the **marginal propensity to save**, that is, the rate at which aggregate savings increases with national income.

2.10 *a*) The slope of a secant line through points with x-values x and $x + h$ is $[m(x+h) - mx]/h = mh/h = m$.

b) For $f(x) = x^3$,

$$\begin{aligned}\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} \\ &= \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2.\end{aligned}$$

For $f(x) = x^4$,

$$\begin{aligned}f'(x) &= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = 4x^3.\end{aligned}$$

2.11 *a*) $-21x^2$, *b*) $-24x^{-3}$, *c*) $-(9/2)x^{-5/2}$, *d*) $1/4\sqrt{x}$,

e) $6x - 9 + (14/5)x^{-3/5} - (3/2)x^{-1/2}$, *f*) $20x^4 - (3/2)x^{-1/2}$,

g) $4x^3 + 9x^2 + 6x + 3$,

h) $(1/2)(x^{-1/2} - x^{-3/2})(4x^5 - 3\sqrt{x}) + (x^{1/2} + x^{-1/2})(20x^4 - (3/2)x^{-1/2})$,

i) $2/(x+1)^2$, *j*) $(1 - x^2)/(1 + x^2)^2$, *k*) $7(x^5 - 3x^2)^6(5x^4 - 6x)$,

l) $(10/3)(x^5 - 6x^2 + 3x)^{-1/3}(5x^4 - 12x + 3)$,

m) $3(3x^2 + 2)(x^3 + 2x)^2(4x + 5)^2 + 8(x^3 + 2x)^3(4x + 5)$.

2.12 *a*) The slope of the tangent line $l(x) = mx + b$ to the graph of $f(x)$ at x_0 is $m = f'(x_0) = 2x_0 = 6$. The tangent line goes through the point $(x_0, f(x_0)) = (3, 9)$, so b solves $9 = 6 \cdot 3 + b$. Thus $b = -9$ and $l(x) = 6x - 9$.

b) Applying the quotient rule, $f'(x) = (2 - x^2)/(x^2 + 2)^2$. Evaluating this at $x_0 = 1$, $m = 1/9$. The tangent line goes through the point $(1, 1/3)$. Solving for b, $l(x) = (1/9)x + 2/9$.

2.13
$$\begin{aligned}(f+g)'(x_0) &= \lim_{h\to 0}\frac{(f(x_0+h)+g(x_0+h)-\big(f(x_0)+g(x_0)\big)}{h}\\ &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0}\frac{g(x_0+h)-g(x_0)}{h}\\ &= f'(x_0)+g'(x_0)\end{aligned}$$

and similarly for $(f - g)'(x_0)$.

$$\begin{aligned}(kf)'(x_0) &= \lim_{h\to 0}\frac{k\,f(x_0+h)-k\,f(x_0)}{h}\\ &= k\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} = k\,f'(x_0).\end{aligned}$$

2.14 Let $F(x) = x^{-k} = 1/x^k$. Apply the quotient rule with $f(x) = 1$ and $g(x) = x^k$. Then $f'(x_0) = 0$, $g'(x_0) = kx_0^{k-1}$, and

$$F'(x_0) = -kx_0^{k-1}/x_0^{2k} = -kx_0^{-k-1}.$$

2.15 For positive x, $|x| = x$, so its derivative is 1. For negative x, $|x| = -x$, so its derivative is -1.

2.16, 17 *a*) $f'(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x & \text{if } x < 0. \end{cases}$

As x converges to 0 both from above and below, $f'(x)$ converges to 0, so the function is C^1. See figure.

b) This function is not continuous (and thus not differentiable). As x converges to 0 from above, $f(x)$ tends to 1, whereas x tends to 0 from below, $f(x)$ converges to -1. See figure.

c) This function is continuous, since $\lim_{x \to 1} f(x) = 1$ no matter how the limit is taken. But it is not differentiable at $x = 1$, since $\lim_{h \downarrow 0}[f(1 + h) - f(1)]/h = 3$ and $\lim_{h \uparrow 0}[f(1 + h) - f(1)]/h = 1$. See figure.

d) This function is C^1 at $x = 1$. No matter which formula is used, the value for the derivative of $f(x)$ at $x = 1$ is 3. See figure.

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 1, \\ 3 & \text{if } x \geq 1. \end{cases}$$

2.18 The interesting behavior of this function occurs in a neighborhood of $x = 0$. Computing, $[f(0 + h) - f(0)]/h = h^{-1/3}$, which converges to $+\infty$ or $-\infty$ as h converges to 0 from above or below, respectively. Thus $f(x) = x^{2/3}$ is not differentiable at $x = 0$. It is continuous at $x = 0$, since $\lim_{x \to 0} f(x) = 0$. This can easily be seen by plotting the function. See figure.

2.19 See figure.

2.20 *a*) $-42x$, *b*) $72x^{-4}$, *c*) $(45/4)x^{-7/2}$, *d*) $x^{-3/2}/8$,

e) $6 - (52/25)x^{-8/5} + (3/4)x^{-3/2}$, *f*) $80x^3 + (3/4)x^{-3/2}$,

g) $12x^2 + 18x + 6$,

h) $(-x^{-3/2}/4 + 3x^{-5/2}/4)(4x^5 - 3\sqrt{x}) + (x^{-1/2} - x^{-3/2})(20x^4 - 3x^{-1/2}/2) + (x^{1/2} + x^{-1/2})(80x^3 + 3x^{-3/2}/4)$,

i) $-4/(x + 1)^3$, *j*) $(2x^3 - 6x)/(x^2 + 1)^3$,

k) $42(5x^4 - 6x)^2(x^5 - 3x^2)^5 + 7(20x^3 - 6)(x^5 - 3x^2)^6$,

l) $(-10/9)(5x^4 - 12x + 3)^2(x^5 - 6x^2 + 3x)^{-4/3} + (10/3)(20x^3 - 12)(x^5 - 6x^2 + 3x)^{-1/3}$,

m) $12(x + 1)^2(4x + 5)^2(x^2 + 2x) + 96(x + 1)(4x + 5)(x^2 + 2x)^2 + 6(4x + 5)^2(x^2 + 2x)^2 + 32(x^2 + 2x)^3$.

2.21 *a*) $f'(x) = (5/3)x^{2/3}$, so $f(x)$ is C^1. But $x^{2/3}$ is not differentiable at $x = 0$, so f is not C^2 at $x = 0$. Everywhere else it is C^∞.

b) This function is a step function: $f(x) = k$ for $k \leq x < k + 1$, for every integer k. It is C^∞ except at integers, since it is constant on every interval $(k, k + 1)$. At integers it fails to be continuous.

2.22 $C'(20) = 86$, so $C(21) - C(20) \approx C'(20) \cdot 1 = 86$. Direct calculation shows that $C(21) = 1020$, so $C(21) - C(20) = 88$.

2.23 $C'(x) = 0.3x^2 - 0.5x + 300$. Then $C(6.1) - C(6) \approx C'(6) \cdot 0.1 = 30.78$.

2.24 $F'(t) = 8/(+2)^2$. Thus $F'(0) = 2$ and the population increase over the next half-year is $F'(0) \cdot 0.5 = 1$.

2.25 *a*) $f(x) = \sqrt{x}$, and $f'(x) = 1/2\sqrt{x}$. $f(50) = f(49) + f'(49) \cdot (1.0) = 7 + 1/14$.

b) $f(x) = x^{1/4}$, and $f'(x) = 1/(4x^{3/4})$. Then $f(9,997) \approx f(10{,}000) + f'(10{,}000) \cdot (-3.0) = 10 - 3/4{,}000 = 9.99925$.

c) $f(x) = x^5$, and $f'(x) = 5x^4$. $f(10.003) = f(10) + f'(10) \cdot 0.003 = 100{,}000 + 50{,}000 \cdot 0.003 = 100{,}150$.

Chapter 3

3.1 *a*) $f'(x) = 3x^2 + 3$, so $f'(x)$ is always positive and $f(x)$ is increasing throughout its domain. $f(0) = 0$, so the graph of f passes through the origin. See figure.

b) Early versions of the text have $f(x) = x^4 - 8x^3 + 18x - 11$ here, with $f'(x) = 4x^3 - 24x^2 + 18$. This itself is a complicated function. $f''(x) = 12x^2 - 48x$. Thus, $f'(x)$ has critical points at $x = 0$ and $x = 4$. The point $x = 0$ is a local maximum of f', and $x = 4$ is a local minimum. Evaluating, $f'(x)$ is positive at the local max and negative at the local min. This means it crosses the x-axis three times, so the original function f has three critical points. Since $f'(x)$ is negative for small x and positive for large x, the critical points of f are, from smallest to largest, a local minimum, a local maximum, and a local minimum.

Later versions have $f(x) = x^4 - 8x^3 + 18x^2 - 11$. Its y-intercept is at $(0, -11)$; $f'(x) = 4x^3 - 24x^2 + 36x = 4x(x^2 - 6x + 9) = 4x(x - 3)^2$.

Critical points are at $x = 0, 3$, i.e., (0,11) and (3,16).

$f' > 0$ (and f increasing) for $0 < x < \infty$ $(x \neq 3)$; $f' < 0$ (and f decreasing) for $-\infty < x < 0$. See figure.

c) $f'(x) = x^2 + 9$. This function is always positive, so f is forever increasing. A little checking shows its root to be between $-1/3$ and $-1/2$. See figure.

d) $f'(x) = 7x^6 - 7$, which has roots at $x = \pm 1$. The local maximum is at $(-1, 6)$, and the local minimum is at $(1, -6)$. The function is decreasing between these two points and increasing elsewhere. The y-intercept is at $(0, 0)$. See figure.

e) $f(0) = 0$; $f'(x) = (2/3)x^{-1/3}$. $f'(x) < 0$ (and f decreasing) for $x < 0$; $f'(x) > 0$ (and f increasing) for $x > 0$. As $x \to 0$, the graph of f becomes infinitely steep. See figure.

f) $f'(x) = 12x^5 - 12x^3 = 12x^3(x^2 - 1)$. The first derivative has roots at -1, 0 and 1. $f'(x)$ is negative for $x < -1$ and $0 < x < 1$, and positive for $-1 < x < 0$ and $x > 1$. Thus $f(x)$ is shaped like a w. Its three critical points are, alternately, a min, a max, and a min. Its values at the two minima are both -1, and its value at the maximum is $+2$. See figure.

3.2 Since f is differentiable at x_0, for small h, $[f(x_0 + h) - f(x_0)]/h < 0$. This means that for small positive h, $f(x_0 + h) < f(x_0)$ and, for small negative h, $f(x_0 + h) > f(x_0)$. Thus, f is decreasing near x_0.

3.3 *a*) $f''(x) = 6x$. The function is concave (concave down) on the negative reals and convex (concave up) on the positive reals.

b) $f''(x) = 12x^2 - 48x$, which is negative for $0 < x < 4$ and positive outside this interval. Thus f is concave on the interval $(0, 4)$ and convex elsewhere.

c) $f''(x) = 2x$, so f is concave on the negative reals and convex on the positive reals.

d) $f''(x) = 42x^5$, so f is concave on the negative reals and convex on the positive reals.

e) $f''(x) = -2x^{-4/3}/9$. This number is always less than 0 for $x \neq 0$. f is concave on $(0, \infty)$ and on $(-\infty, 0)$. It is not globally concave.

f) $f''(x) = 60x^4 - 36x^2$, which is negative on the interval $(-\sqrt{3/5}, \sqrt{3/5})$, and positive outside it. Thus, f is concave on this interval and convex elsewhere.

3.4–5 See figures.

3.6 There is a single vertical asymptote at $x = 2$. $f'(x) = -16(x + 4)/(x - 2)^3$ and $f''(x) = 32(x + 7)/(x - 2)^4$. Consequently there is a critical point at $x = -4$, where the function takes the value $-4/3$. $f''(-4) > 0$, so this is a local minimum. There is an inflection point at $x = -7$. f is decreasing to the right of its asymptote and to the left of $x = -4$, and increasing on $(-4, 2)$. See figure.

3.7 *a*) The leading monomial is x^{-1}, so $f(x)$ converges to 0 as x becomes very positive or very negative. It also has vertical asymptotes at $x = -1$ and $x = 1$. $f'(x) = -(x^2+1)/(x^2-1)^2$; so for $x < -1$ and $x > 1$, $f(x)$ is decreasing. (In fact it behaves as $1/x$.) It is also decreasing between the asymptotes. Thus, it goes from 0 to $-\infty$ as x goes from $-\infty$ to -1, from $+\infty$ to $-\infty$ as x goes from -1 to 1, and from $+\infty$ to 0 as x goes from 1 to $+\infty$. See figure.

b) $f(x)$ behaves as $1/x$ for x very large and very small. That is, as $|x|$ grows large, $f(x)$ tends to 0. $f'(x) = (1-x^2)/(x^2+1)^2$; so there are critical points at -1 and 1. These are, respectively, a minimum with value $-1/2$ and a maximum with value $1/2$. Inflection points are at $(-\sqrt{3}, -\sqrt{3}/4)$, $(0, 0)$, and $(\sqrt{3}, \sqrt{3}/4)$. See figure.

c) This function has a vertical asymptote at $x = -1$. The lead monomial is $x^2/x = x$, so in the tails it is increasing as $x \to +\infty$ and decreasing as $x \to -\infty$. As x converges to -1 from below, $f(x)$ tends to $-\infty$; as x converges to -1 from above, $f(x)$ converges to $+\infty$. See figure.

d) The lead monomial is $x^2/x^2 = 1$, so $f(x)$ converges to 1 as $|x|$ becomes large. It has vertical asymptotes at $x = 1$ and $x = -1$. In fact, f can be rewritten as $f(x) = 1 + (3x+1)/(x^2-1)$. Since $f'(x) = -(3x^2+2x+3)/(x^2-1)^2 < 0$, f is always decreasing. So, its general shape is that of the function in part 7*a*. See figure.

e) The lead monomial is $x^2/x = x$, so this function is increasing in x when $|x|$ is large. When x is small near its vertical asymtote at $x = 0$, it behaves as $1/x$. $f''(x) = 1 - 1/x^2$, which is 0 at ± 1. $x = -1$ is a local maximum and $x = 1$ is a local minimum. See figure.

f) This function is bell shaped. It is always positive, tends to 0 when $|x|$ is large, and has a maximum at $x = 0$ where it takes the value 1. See figure.

3.8 See figures.

3.9 *a*) No global max or min on D_1; max at 1 and min at 2 on D_2.

b) No max or min on D_1; min at 0 and max at 1 on D_2.

c) Min at -4, max at -2 on D_1; min at $+1$, no max on D_2.

d) Min at 0, max at 10 on D_1; min at 0, no max on D_2.

e) Min at -2, max at $+2$ on D_1; min at $-\sqrt{2}$, max at $+\sqrt{2}$ on D_2.

f) Min at 1, no max on D_1; max at -1, no min on D_2.

g) No min or max on D_1; max at 1 and min at 2 on D_2.

h) No max or min on D_1; max at 1 and min at 5 on D_2.

3.10 In this exercise x is the market price, which is a choice variable for the firm. $\pi(x) = x(15 - x) - 5(15 - x)$. This function is concave, and its first derivative is $\pi'(x) = -2x + 20$. $\pi'(x) = 0$ at $x = 10$.

3.11 From the information given, the demand function must be computed. The function is linear, and the slope is -1. It goes through the point $(10, 10)$, so the function must be $f(x) = 20 - x$. Then the profit function (as a function of price) must be $\pi(x) = (x - 5)(20 - x)$. $\pi'(x) = -2x + 25$, so profit is maximized at $x = 12.5$.

3.12 One can translate the proof of Theorem 3.4*a* in the text. Here is another idea. If ℓ is a secant line connecting (x_0, y_0) and (x_1, y_1) on the graph of a convex function $f(x)$, then the set of points $\big(x, f(x)\big)$ for $x \notin (x_0, x_1)$ lies *above* ℓ. Taking limits, the graph of a convex function always lies above each tangent line (except where they touch). If $f'(x_0) = 0$, then the tangent line is of the form $\ell(x) = f(x_0) = b$. Since $f(x)$ is convex near x_0, $f(x)$ must be at least as big as b for x near x_0, and so x_0 is a min.

3.13 Suppose $y_0 < x_0$; f is decreasing just to the right of y_0 and increasing just to the left of x_0. It must change from decreasing to increasing somewhere between y_0 and x_0, say at w_0. Then, w_0 is an interior critical point of f — contradicting the hypothesis that x_0 is the only critical point of f_0.

3.15 $AC(x) = x^2 + 1 + 1/x$. $MC(x) = 3x^2 + 1$. $MC(x_0) = AC(x_0)$ when $2x_0^2 = 1/x_0$, that is, at $x_0 = 2^{-1/3}$. $AC'(x) = 2x - 1/x^2$, so $AC(x)$ has a critical point at $x = 2^{-1/3}$. Thus c is satisfied. $MC(x)$ is increasing, $AC(x)$ is convex, and the two curves intersect only once at x_0, so it must be that to the left of x_0, $AC(x) > MC(x)$, and hence to the right, $AC(x) < MC(x)$. See figure.

3.16 Suppose $C(x) = \sqrt{x}$. Then $MC(x) = 1/2\sqrt{x}$, which is decreasing. $\pi(x) = px - \sqrt{x}$. $\pi'(x) = p - 1/2\sqrt{x}$. The equation $\pi'(x) = 0$ will have a solution, but $\pi''(x) = -1/4x^{3/2}$, which is always negative on the positive reals. Setting price equal to marginal cost gives a local min. Profit can always be increased by increasing output beyond this point.

3.17 *a*) Locate x^* correctly at the intersection of the MR and MC curves. Revenue at the optimum is described by the area of the rectangle with height $AR(x^*)$ and length x^*.

b) The rectangle with height $AC(x^*)$ and length x^*.

c) The rectangle with height $AR(x^*) - AC(x*)$ and length x^*.

3.18 For demand curve $x = a - bp$, the elasticity at $(a - bp, p)$ is $\varepsilon = -bp/(a - bp)$. Then, $\varepsilon = -1 \Longleftrightarrow bp = a - bp \Longleftrightarrow 2bp = a \Longleftrightarrow p = a/b$.

3.19 $\varepsilon = \dfrac{F'(p) \cdot p}{F(p)} = \dfrac{-rkp^{-r-1} \cdot p}{kp^{-r}} = -r$, constant.

3.20 x^* and p^* both increase.

3.21 The rectangle with height $p^* - AC(x^*)$ and length x^*.

3.22 First, compute the inverse demand: $p = a/b - (1/b)x$. Then revenue is $R(x) = (a/b)x - (1/b)x^2$ and $MR(x) = a/b - (2/b)x$. $MC(x) = 2kx$, so x^* solves $a/b - (2/b)x = 2kx$. The solution is $x^* = a/(2kb + 2)$, and the price will be $p^* = 2kab/(2kb^2 + 2b)$.

Chapter 4

4.1 *a*) $(g \circ h)(z) = (5z - 1)^2 + 4, \quad (h \circ g)(x) = 5x^2 + 19.$

b) $(g \circ h)(z) = (z - 1)^3(z + 1)^3, \quad (h \circ g)(x) = (x^3 - 1)(x^3 + 1).$

c, d) $(g \circ h)(z) = z, \quad (h \circ g)(x) = x.$

e) $g \circ h(z) = 1/(z^2 + 1), \quad h \circ g(x) = 1/x^2 + 1.$

4.2 *a*) Inside $y = 3x^2 + 1$, outside $z = y^{1/2}$.

b) Inside $y = 1/x$, outside $z = y^2 + 5y + 4$.

c) Inside $y = 2x - 7$, outside $z = \cos y$.

d) Inside $y = 4t + 1$, outside $z = 3^y$.

4.3 *a*) $(g \circ h)'(z) = 2(5z - 1)5 = 50z - 10, \quad (h \circ g)'(x) = 5 \cdot 2x = 10x.$

b) $(g \circ h)'(z) = 3[(z - 1)(z + 1)]^2(2z) = 6z(z - 1)^2(z + 1)^2$, $(h \circ g)'(x) = 2x^3 \cdot 3x^2 = 6x^5$.

c) $(g \circ h)'(z) = 1, \quad (h \circ g)'(x) = 1.$

d) $(g \circ h)'(z) = 1, \quad (h \circ g)'(x) = 1.$

e) $(g \circ h)'(z) = -2z/(z^2 + 1)^2, \quad (h \circ g)'(x) = -2/x^3.$

4.4 *a*) $(g \circ h)'(x) = \frac{1}{2}(3x^2 + 1)^{-1/2} \cdot 6x = 3x/\sqrt{3x^2 + 1}$

b) $(g \circ h)'(x) = [2(1/x) + 5]\left(-1/x^2\right).$

c) $(g \circ h)'(x) = -2\sin(2x - 7)$.

d) $(g \circ h)'(t) = (4\log 3)3^{4t+1}$.

4.5 *a*) $(g \circ h)'(x) = \cos(x^4) \cdot 4x^3$.

b) $(g \circ h)'(x) = \cos(1/x) \cdot (-1/x^2)$.

c) $(g \circ h)'(x) = \cos x/(2\sqrt{\sin x})$.

d) $(g \circ h)'(x) = (\cos\sqrt{x})/2\sqrt{x}$.

e) $(g \circ h)'(x) = (2x + 3)\exp(x^2 + 3x)$.

f) $(g \circ h)'(x) = -x^{-2}\exp(1/x)$.

g) $(g \circ h)'(x) = 2x/(x^2 + 4)$.

h) $(g \circ h)'(x) = 4x(x^2 + 4)\cos((x^2 + 4)^2)$.

4.6 $x'(t) = 2$ and $C'(x) = 12$, so $(d/dt)C\big(x(t)\big) = 2 \cdot 12 = 24$.

4.8 *a*) $g(y) = (y - 6)/3$, $-\infty < y < +\infty$.

b) $g(y) = 1/y - 1$, $-\infty < y < +\infty$, $y \neq 0$.

c) The range of $f(x) = x^{2/3}$ is the nonnegative reals, so this is the domain of the inverse. But notice that $f(x)$ is not one-to one from $\mathbf{R}$ to its range. It is one-to-one if the domain of f is restricted to $\mathbf{R}_+$. In this case the inverse is $g(y) = y^{3/2}$, $0 \leq y < \infty$. If the domain of f is restricted to $\mathbf{R}_-$, the inverse is $-g(y)$.

d) The graph of $f(x)$ is a parabola with a global minimum at $x = 1/2$, and is one-to-one on each side of it. Thus there will be two inverses. For a given y they are the solutions to $x^2 + x + 2 = y$. The two inverses are $z = \frac{1}{2}(-1 + \sqrt{4y - 7})$ and $z = \frac{1}{2}(-1 - \sqrt{4y - 7})$, with domain $y \geq 7/4$. If $y < 7/4$, the equation has no solution; there is no value of x such that $f(x) = y$.

4.9 *a*) $(f^{-1})'(f(1)) = 1/3, \quad 1/f'(1) = 1/3$.

b) $(f^{-1})'(1/2) = -4, \quad 1/f'(1) = -(1 + 1)^2 = -4$.

c) $(f^{-1})'f(1) = 3/2, \quad 1/f'(1) = 3/2$.

d) $(f^{-1})'f(1) = 1/3, \quad 1/f'(1) = 1/3$.

4.10 To prove Theorem 4.4, let $f(x) = x^{1/-n} = 1/x^{1/n}$ for n a positive integer. Applying the quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 \cdot x^{1/n} - 1 \cdot (1/n)x^{(1/n)-1}}{x^{2/n}} \\ &= -(1/n)x^{-(1/n)-1}. \end{aligned}$$

For Theorem 4.5, the proof in the text applies to the case where both m and n are negative. To prove the remaining case, let $f(x) = x^{-m/n}$ where m, n are positive integers. Applying the quotient rule,

$$\begin{aligned} f'(x) &= \frac{0 \cdot x^{m/n} - 1 \cdot (m/n)x^{(m/n)-1}}{x^{2m/n}} \\ &= -(m/n)x^{(m/n)-1-(2m/n)} \\ &= -(m/n)x^{-(m/n)-1}. \end{aligned}$$

Chapter 5

5.1 *a*) 8, *b*) $1/8$, *c*) 2, *d*) 4, *e*) $1/4$, *f*) 1, *g*) $1/32$, *h*) 125, *i*) $1/3125$.

5.2 See figures.

5.3 By calculator to three decimal places: *a*) 2.699, *b*) 0.699, *c*) 3.091, *d*) 0.434, *e*) 3.401, *f*) 4.605, *g*) 1.099, *h*) 1.145. See figures.

5.4 *a*) 1, *b*) -3, *c*) 9, *d*) 3, *e*) 2, *f*) -1, *g*) 2, *h*) $1/2$, *i*) 0.

5.5 *a*) $2e^{6x} = 18 \Longrightarrow e^{6x} = 9 \Longrightarrow 6x = \ln 9 \Longrightarrow x = (\ln 9)/6$;

b) $e^{x^2} = 1 \Longrightarrow x^2 = \ln 1 = 0 \Longrightarrow x = 0$;

c) $2^x = e^5 \Longrightarrow \ln 2^x = \ln e^5 \Longrightarrow x \ln 2 = 5 \Longrightarrow x = 5/(\ln 2)$;

d) $2 + \ln 5/\ln 2$; *e*) $e^{5/2}$; *f*) 5.

5.6 Solve $3A = A \exp rt$. Dividing out A and taking logs, $\ln 3 = rt$, and $t = \ln 3/r$.

5.7 Solve $600 = 500 \exp(0.05t)$. $t = \ln(6/5)/0.05 = 3.65$.

5.8 *a*) $f'(x) = e^{3x} + 3xe^{3x}$, $f''(x) = 6e^{3x} + 9xe^{3x}$.

b) $f'(x) = (2x+3)e^{x^2+3x-2}$, $f''(x) = (4x^2 + 12x + 11)e^{x^2+3x-2}$.

c) $f'(x) = 8x^3/(x^4+2)$, $f''(x) = (48x^2 - 8x^6)/(x^4+2)^2$.

d) $f'(x) = (1-x)/e^x$, $f''(x) = (x-2)/e^x$.

e) $f'(x) = 1/(\ln x) - 1/[(\ln x)]^2$, $f''(x) = 2/[x(\ln x)^3] - 1/[x(\ln x)^2]$.

f) $f'(x) = (1 - \ln x)/x^2$, $f''(x) = (2\ln x - 3)/x^3$.

g) $f'(x) = x/(x^2+4)$, $f''(x) = (-x^2+4)/(x^2+4)^2$.

5.9 *a*) $f(x) = xe^x \Longrightarrow f'(x) = e^x(x+1)$ is positive (f increasing) for $x > -1$ and negative (f decreasing) for $x < -1$. $f''(x) = e^x(x+2)$ is positive (f convex) for $x > -2$, and negative (f concave) for $x < -2$. As x tends to $-\infty$, $f(x)$ goes to 0, and as x gets large, $f(x)$ behaves as e^x. Thus, the function has a horizontal asymptote at 0 as $x \to -\infty$, grows unboundedly as $x \to +\infty$, has a global minimum at -1, and has an inflection point at -2. See figure.

b) $y = xe^{-x}$; $y' = (1-x)e^{-x}$ is positive (f increasing) for $x < 1$, and negative (f decreasing) for $x > 1$. $y'' = (x-2)e^{-x}$ is positive (f convex) for $x > 2$, and negative (f concave) for $x < 2$. As x gets large, $f(x)$ tends to 0. As x goes to $-\infty$, so does $f(x)$. The inflection point is at $x = 2$. See figure.

c) $y = \frac{1}{2}(e^x + e^{-x})$; $y' = \frac{1}{2}(e^x - e^{-x})$ is positive (y increasing) if $x > 0$, and negative (y decreasing) if $x < 0$. $y'' = y$ is always positive (y always convex). See figure.

5.10 Let $f(x) = \mathrm{Log}(x)$, and let $g(x) = 10^{f(x)} = x$. Then, by the Chain Rule and Theorem 5.3,

$$g'(x) = (\ln 10)10^{f(x)} f'(x) = 1.$$

So,

$$f'(x) = \frac{1}{(\ln 10)10^{\mathrm{Log}(x)}} = \frac{1}{x \ln 10}.$$

5.11 The present value of the first option is $215/(1.1)^2 = 177.69$. The present value of the second option is $100/(1.1) + 100/(1.1)^2 = 173.56$. The present value of the third option is $100 + 95/(1.1)^2 = 178.51$.

5.12 By equation (14), the present value of the 5-year annuity is

$$500\frac{1 - e^{-0.5}}{e^{0.1} - 1} = 1870.62.$$

Equation (15) gives the present value of the infinitely lived annuity: $500/(e^{0.1} - 1) = 4754.17$.

5.13 $\ln B(t) = \sqrt{t}\ln 2$. Differentiating,

$$\frac{B'(t)}{B(t)} = \frac{\ln 2}{2\sqrt{t}}.$$

The solution to $B'(t)/B(t) = r$ is $t = 100(\ln 2)^2 = 48.05$.

5.14 $\ln V(t) = K + \sqrt{t}$. Differentiating, $V'(t)/V(t) = 1/2(\sqrt{t})$. The solution to $V'(t)/V(t) = r$ is $t = 1/(4r^2)$, which is independent of K. A check of the second order conditions shows this to be a maximum.

5.15 $\ln V(t) = \ln 2000 + t^{1/4}$. Differentiating, we find that

$$\frac{V'(t)}{V(t)} = \frac{1}{4}t^{-3/4}.$$

The solution to $V'(t)/V(t) = r$ is $t = (4r)^{-4/3}$. When $r = 0.1$, $t = 3.39$.

5.16 *a*) $\ln f(x) = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+4)$. Differentiating,

$$\frac{f'(x)}{f(x)} = \frac{x}{x^2+1} - \frac{x}{x^2+4}.$$

So,

$$f'(x) = f(x)\left(\frac{x}{x^2+1} - \frac{x}{x^2+4}\right) = \frac{3x}{(x^2+1)^{1/2}(x^2+4)^{3/2}}.$$

b) $\ln f(x) = 2x^2 \ln x$. Differentiating, $f'(x)/f(x) = 4x\ln x + 2x = 2x(1 + \ln x^2)$, so $f'(x) = 2x(1+\ln x^2)(x^2)^{x^2}$.

5.17 Let $h(x) = f(x)g(x)$, so $\ln h(x) = \ln f(x) + \ln g(x)$. Differentiating, $h'(x)/h(x) = f'(x)/f(x) + g'(x)/g(x)$. Multiplying both sides of the equality by x proves the claim.

Chapter 6

6.1

$$S = 0.05(100{,}000)$$
$$F = 0.4(100{,}000 - S).$$

Multiplying out the system gives

$$S = 5{,}000$$
$$F + 0.4S = 40{,}000.$$

Thus $S = 5{,}000$ and $F = 38{,}000$, and after-tax profits are \$57,000. Including contributions, after-tax profits were calculated to be \$53,605, so the \$5,956 contribution really cost only $\$57,000 - \$53,605 = \$3,395$.

6.2 Now $S = 0.05(100{,}000 - C - F)$, so the equations become

$$\begin{aligned} C + 0.1S + 0.1F &= 10{,}000 \\ 0.05C + S + 0.05F &= 5{,}000 \\ 0.4C + 0.4S + F &= 40{,}000. \end{aligned}$$

The solution is $C = 6{,}070$, $S = 2{,}875$, and $F = 36{,}422$.

6.3 $x_1 = 0.5x_1 + 0.5x_2 + 1$, $x_2 = 0x_1 + 0.25x_2 + 3$. The solution is $x_1 = 6$, $x_2 = 4$.

6.4 Solving the system of equations $x_1 = 0.5x_1 + 0.5x_2 + 1$ and $x_2 = 0.875x_1 + 0.25x_2 + 3$ gives $x_1 = -36$ and $x_2 = -38$; this is infeasible.

6.5 $0.002 \cdot 0.9 + 0.864 \cdot 0.1 = 0.0882$, and $0.004 \cdot 0.8 + 0.898 \cdot 0.2 = 0.1828$.

6.6 For black females, $\left\{\begin{array}{l} x_{t+1} = 0.993x_t + 0.106y_t \\ y_{t+1} = 0.007x_t + 0.894y_t \end{array}\right\}$.
To find the stationary distribution, set $x_{t+1} = x_t = x$ and $y_{t+1} = y_t = y$: $x = 0.9381$ and $y = 0.0619$.
For white females, $\left\{\begin{array}{l} x_{t+1} = 0.997x_t + 0.151y_t \\ y_{t+1} = 0.003x_t + 0.849y_t \end{array}\right\}$.
Stationary solution: $x = 0.9805$ and $y = 0.0195$.

6.7 The equation system is

$$\begin{aligned} 0.16Y - 1500r &= 0 \\ 0.2Y + 2000r &= 1000. \end{aligned}$$

The solution is $r = 0.2581$ and $Y = 2419.35$.

Chapter 7

7.1 *a* and *e*.

7.2 *a*) The solution is $x = 5$, $y = 6$, $z = 2$.

b) The solution is $x_1 = 1$, $x_2 = -2$, $x_3 = 1$.

7.3 *a*) $x = 17/3$, $y = -13/3$.

b) $x = 2$, $y = 1$, $z = 3$.

c) $x = 1$, $y = -1$, $z = -2$.

7.4 Start with system (*):

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
\vdots \qquad\quad & \quad \vdots \\
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
\vdots \qquad\quad & \quad \vdots \\
a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \\
\vdots \qquad\quad & \quad \vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n.
\end{aligned}$$

1) Change system (*) to

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
\vdots \qquad\quad & \quad \vdots \\
a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\
\vdots \qquad\quad & \quad \vdots \\
(ra_{i1} + a_{j1})x_1 + \cdots + (ra_{in} + a_{jn})x_n &= (rb_i + b_j) \\
\vdots \qquad\quad & \quad \vdots \\
a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n.
\end{aligned}$$

2) Change the *i*th equation of system (*) to $ra_{i1}x_1 + \cdots + ra_{in}x_n = rb_i$.

3) Change system (*) to

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\quad & \vdots \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \\ \vdots \qquad\quad & \vdots \\ a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ \vdots \qquad\quad & \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n. \end{aligned}$$

Reverse operations:

1) Subtract r times equation i from equation j, leaving other $n-1$ equations intact.
2) Multiply the ith equation through by $1/r$.
3) Interchange the ith and jth equations again.

7.5 The system to solve is

$$\begin{aligned} 0.20Y + 2000r &= 1000 \\ 0.16Y - 1500r &= 0. \end{aligned}$$

Solving the second equation for Y in terms of r gives $Y = 9375r$. Substituting into the first equation, $3875r = 1000$, so $r = 0.258$ and $Y = 2419.35$.

7.6 *a*) The system to solve is:

$$\begin{aligned} sY + ar &= I^0 \\ mY - hr &= 0. \end{aligned}$$

Solving the second equation for Y in terms of r gives $Y = (h/m)r$. Substituting into the first equation gives $(sh + am)r/m = I^0$, so $r = mI^0/(sh + am)$ and $Y = hI^0/(sh + am)$.

b, c) Differentiate the solutions with respect to s:

$$\frac{\partial r}{\partial s} = -\frac{hmI_0}{(sh + am)^2} < 0 \quad \text{and} \quad \frac{\partial Y}{\partial s} = -\frac{h^2 I_0}{(sh + am)^2} < 0.$$

7.7 Solving for y in terms of x in the second equation gives $y = -x - 10$. Substituting this into the first equation gives a new equation that must be satisfied by all solutions. This equation is $-30 = 4$. Since this is never satisfied, there are no solutions to the equation system.

7.8 If $a_{22} \neq 0$, then $x_2 = (b_2 - a_{21}x_1)/a_{22}$. Substituting into the first equation gives

$$b_1 = \frac{a_{11} - a_{12}a_{21}}{a_{22}} x_1 + \frac{a_{12}}{a_{22}} b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

A similar calculation solving the first equation for x_1 ends up at the same point if $a_{21} \neq 0$. The division that gives this answer is possible only if $a_{11}a_{22} - a_{12}a_{21} \neq 0$. In this case,

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

7.9 (1) Add 0.2 times row 1 to row 2. (2) Add 0.5 times row 1 to row 3. (3) Add 0.5 times row 2 to row 3.

7.10 $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -14 \\ 0 & 1 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0.3 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$

7.11 *a*) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\left(\begin{array}{cc|c} 3 & 3 & 4 \\ 1 & -1 & 10 \end{array}\right), \quad \left(\begin{array}{cc|c} 3 & 3 & 4 \\ 0 & -2 & 26/3 \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & 0 & 17/3 \\ 0 & 1 & -13/3 \end{array}\right).$$

The solution is $x = 17/3$, $y = -13/3$.

b) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 6 & 3 & -5 & | & 0 \\ 1 & 1 & 2 & | & 9 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 & -3 & | & 1 \\ 0 & 1/2 & 11/4 & | & 35/4 \\ 0 & 0 & -1/2 & | & -3/2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & 3 \end{pmatrix}.$$

The solution is $x = 2, y = 1, z = 3$.

c) The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 1 & 1 & 1 & | & -2 \\ 2 & -4 & 3 & | & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & -1 & | & 2 \\ 0 & -6 & 4 & | & -2 \\ 0 & 0 & 3/2 & | & -3 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & -2 \end{pmatrix}.$$

The solution is $x = 1, y = -1, z = -2$.

7.12 The original system, the row echelon form, and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 1 & 3 & -2 & | & 0 \\ 2 & 3 & 7 & -2 & | & 9 \\ 3 & 5 & 13 & -9 & | & 1 \\ -2 & 1 & 0 & -1 & | & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 3 & -2 & | & 0 \\ 0 & 1 & 1 & 2 & | & 9 \\ 0 & 0 & 2 & -7 & | & -17 \\ 0 & 0 & 0 & -1/2 & | & -3/2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & 0 & | & 2 \\ 0 & 0 & 0 & 1 & | & 3 \end{pmatrix}.$$

The solution is $w = -1, x = 1, y = 2, z = 3$.

7.13 *a*) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, *b*) $\begin{pmatrix} 1 & 3 & 4 \\ 0 & -1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$,

c) $\begin{pmatrix} -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

7.14 The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} -4 & 6 & 4 & | & 4 \\ 2 & -1 & 1 & | & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \frac{5}{4} & | & \frac{5}{4} \\ 0 & 1 & \frac{3}{2} & | & \frac{3}{2} \end{pmatrix}.$$

The solution set is the set of all (x, y, z) triples such that $x = \frac{5}{4} - \frac{5}{4}z$ and $y = \frac{3}{2} - \frac{3}{2}z$ as z ranges over all the real numbers.

7.15 The original system and the row echelon form are, respectively,

$$\begin{pmatrix} 1 & 1 & | & 1 \\ 1 & -k & | & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & | & 1 \\ 0 & -(k+1) & | & 0 \end{pmatrix}.$$

If $k = -1$, the second equation is a multiple of the first. In the row echelon form this appears as the second equation $0 + 0 = 0$. Any solution to the first equation solves the second equation as well, and so there are infinitely many solutions. For all other values of k there is a unique solution, with $x_1 = 1$ and $x_2 = 0$.

7.16 *a*) The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 2 & 1 & -1 & | & 1 \\ 3 & -1 & -1 & 2 & | & 3 \\ 0 & -1 & 1 & -1 & | & 1 \\ 2 & 3 & 3 & -3 & | & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & \frac{3}{11} & | & \frac{12}{11} \\ 0 & 1 & 0 & -\frac{1}{11} & | & -\frac{4}{11} \\ 0 & 0 & 1 & -\frac{12}{11} & | & \frac{7}{11} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

The variable z is free and the rest are basic. The solution is

$$\begin{aligned} w &= \tfrac{12}{11} - \tfrac{3}{11}z \\ x &= -\tfrac{4}{11} + \tfrac{1}{11}z \\ y &= \tfrac{7}{11} + \tfrac{12}{11}z. \end{aligned}$$

b) The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & -1 & 3 & -1 & | & 0 \\ 1 & 4 & -1 & 1 & | & 3 \\ 3 & 7 & 1 & 1 & | & 6 \\ 3 & 2 & 5 & -1 & | & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & | & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & | & \frac{3}{5} \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

The variables y and z are free, while w and x are basic. The solution is

$$\begin{aligned} w &= \tfrac{3}{5} - \tfrac{11}{5}y + \tfrac{3}{5}z \\ x &= \tfrac{3}{5} + \tfrac{4}{5}y - \tfrac{2}{5}z. \end{aligned}$$

c) The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 2 & 3 & -1 & | & 1 \\ -1 & 1 & 2 & 3 & | & 2 \\ 3 & -1 & 1 & 2 & | & 2 \\ 2 & 3 & -1 & 1 & | & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & | & 17/65 \\ 0 & 1 & 0 & 0 & | & 7/65 \\ 0 & 0 & 1 & 0 & | & 22/65 \\ 0 & 0 & 0 & 1 & | & 32/65 \end{pmatrix}.$$

All variables are basic. There are no free variables. The solution is $w = 17/65$, $x = 7/65$, $y = 22/65$, $z = 32/65$.

d) The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 1 & -1 & 2 & | & 3 \\ 2 & 2 & -2 & 4 & | & 6 \\ -3 & -3 & 3 & -6 & | & -9 \\ -2 & -2 & 2 & -4 & | & -6 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & -1 & 2 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

Variable w is basic and the remaining variables are free. The solution is $w = 3 - x + y - 2z$.

7.17 *a*) The original system and the reduced row echelon form are, respectively,

$$\begin{pmatrix} 1 & 1 & 1 & | & 13 \\ 1 & 5 & 10 & | & 61 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & -\frac{5}{4} & | & 1 \\ 0 & 1 & \frac{9}{4} & | & 12 \end{pmatrix}.$$

To have x and y integers, z should be an even multiple of 2, i.e., 4, 8, 16, ... To have $y \geq 0$, $z \leq 16/3$. So, $z = 4$, $x = 6$, $y = 3$.

b) 4 pennies, 6 nickels, 6 dimes! 16 coins worth 94 cents.

7.18 The reduced row echelon form of the system is

$$\begin{pmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \\ 0 & 0 & | & 8 + a \end{pmatrix}.$$

The last equation has solutions only when $a = -8$. In this case $x = y = 1$.

7.19 *a*) The second equation is -1 times the first equation. When the system is row-reduced, the second equation becomes $0x + 0y = 0$; that is, it is redundant. The resulting system is

$$\begin{pmatrix} 1 & 1 & | & 1 \\ 0 & 1-q+p & | & 1-q \\ 0 & 0 & | & 0 \end{pmatrix}.$$

This system has no solution if and only if $1 - q + p = 0$ and $1 - q \neq 0$. This happens if and only if $p = -(1 - q)$. With the nonnegativity constraints $p, q \geq 0$, this can never happen unless $q > 1$. So the equation system always has a solution. If $q = 1$ and $p = 0$, the equation system has infinitely many solutions with $x = 1 - y$; otherwise it has a unique solution.

b) If $q = 2$ and $p = 1$, the system contains the two equations $x + y = 1$ and $x + y = 0$, which cannot simultaneously be satisfied. More generally, if $q \neq 1$ and $p = q - 1$, the equation system has no solution.

7.20 *a*) A row echelon form of this matrix is

$$\begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix},$$

so its rank is 1.

b) A row echelon form of this matrix is

$$\begin{pmatrix} 2 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix},$$

so its rank is 2.

c) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

so its rank is 3.

d) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

so its rank is 3.

e) A row echelon form of this matrix is

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix},$$

so its rank is 3.

7.21 *a*) *i*) Rank $M = \#rows = \#cols$, so there is a unique solution $(0,0)$.

ii) Rank $M = \#rows < \#cols$, so there are infinitely many solutions.

iii) Rank $M = \#cols$, so there is a unique solution $(0,0)$.

iv) Rank $M = \#rows = \#cols$, so there is a unique solution $(0,0,0)$.

v) Rank $M < \#rows = \#cols$, so there are infinitely many solutions.

b) *i*) Rank $M = \#rows = \#cols$, so there is a unique solution.

ii) Rank $M = \#rows < \#cols$, so there are infinitely many solutions.

iii) Rank $M = \#cols$, so there are either zero solutions or one solution.

iv) Rank $M = \#rows = \#cols$, so there is a unique solution.

v) Rank $M < \#rows = \#cols$, so there are zero or infinitely many solutions.

7.22 *a*) Rank $M = 1 < \#rows = \#cols$, so the homogeneous system has infinitely many solutions and the general system has either 0 or infinitely many solutions.

b) Rank $M = 2 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

c) Rank $M = 3 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

d) Rank $M = 3 < \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has either zero or infinitely many solutions.

c) Rank $M = 3 = \#rows < \#cols$, so the homogeneous system has infinitely many solutions and the general system has infinitely many solutions.

7.23 Checking the reduced row echelon forms, only *c* has no nonzero rows.

7.24 Let A be an $n \times n$ matrix with row echelon form R. Let $a(j)$ be the number of leading zeros in row j of R. By definition of R,

$$0 \leq a(1) < a(2) < a(3) < \cdots$$

until one reaches k so that $a(k) = n$; then $a(j) = n$ for all $j \geq k$.
It follows that $a(j) \geq j - 1$ for all j.
If A is nonsingular, $a(n) < n$. Since $a(n) \geq n-1$, $a(n) = n-1$. This means $a(j) = j - 1$ for all j, and so the jth entry in row j (diagonal entry) is not zero.
Conversely, if every diagonal entry of R is not zero, $a(j) < j$ for all j.
Since $a(j) \geq j - 1$, $a(j) = j - 1$ for all j. Since $a(n) = n - 1$, A has full rank, i.e., is nonsingular.

7.25 *i*) The row-reduced row echelon form of the matrix for this system in the variables x, y, z, and w is

$$\begin{pmatrix} 1 & 2 & 0 & -1 & | & \frac{3}{4} \\ 0 & 0 & 1 & 0 & | & \frac{1}{4} \end{pmatrix}.$$

The rank of the system is 2. Thus, two variables can be endogenous at any one time: z and one other. For example, the variables x and z can be solved for in terms of w and y, and the solution is $x = 3/4 - 2y - w$ and $z = 1/4$.

ii) The row-reduced row echelon form of the matrix for this system is

$$\begin{pmatrix} 1 & 0 & -1 & 0 & | & 1 \\ 0 & 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

This matrix has rank 3, so three variables can be solved for in terms of the fourth. In particular, x, y, and w can be solved for in terms of z. One solution is $x = 1 + z$, $y = -z$, and $w = 0$.

7.26

$$\begin{aligned} C + 0.1S + 0.1F - 0.1P &= 0 \\ 0.05C + S \qquad\quad - 0.05P &= 0 \\ 0.4C + 0.4S + F - 0.4P &= 0. \end{aligned}$$

The reduced row-echelon form for the matrix representing the system is

$$\begin{pmatrix} 1 & 0 & 0 & -0.0595611 & | & 0 \\ 0 & 1 & 0 & -0.0470219 & | & 0 \\ 0 & 0 & 1 & -0.357367 & | & 0 \end{pmatrix}.$$

Thus the solution is $C = 0.0595611P$, $S = 0.0470219P$, and $F = 0.357367P$.

7.27 The equation system is

$$0.2Y + 2000r + 0M_s = 1000$$
$$0.16Y - 1500r - M_s = -M^0.$$

The reduced row echelon form of the matrix is

$$\begin{pmatrix} 1 & 0 & -3.22581 & | & 2419.36 + 3.22581M^0 \\ 0 & 1 & 0.000322581 & | & 0.258065 - 0.000322581M^0 \end{pmatrix}.$$

Thus, a solution is $Y = 2419.36 + 3.22581M^0 + 3.22581M_s$ and $r = 0.258065 - 0.000322581M^0 - 0.000322581M_s$.

7.28 *a*) Row reduce the matrix

$$\begin{pmatrix} s & a & | & I_0 + G \\ m & -a & | & M_s - M^0 \end{pmatrix}.$$

b) The solution is

$$Y = \frac{h(I^* + G) + a(M_s - M^*)}{sh + am}$$
$$r = \frac{m(I^* + G) + s(M_s - M^*)}{sh + am}.$$

c) Increases in I^*, G and M_s increase Y.
Increases in I^*, G and M° increase r.
Increases in $M^\circ \Longrightarrow$ decreases in Y.
Increases in $M^s \Longrightarrow$ decreases in r.

7.29 *a*) Here is one possibility. Row reducing the matrix associated with the system gives

$$\begin{pmatrix} 1 & 0 & \frac{11}{5} & 0 & | & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & 0 & | & \frac{3}{5} \\ 0 & 0 & 0 & 1 & | & 0 \end{pmatrix}.$$

A solution is then $w = \frac{3}{5} - \frac{11}{5}y$, $x = \frac{3}{5} + \frac{4}{5}y$, and $z = 0$.

b) If $y = 0$, then a solution is $w = \frac{3}{5}$, $x = \frac{3}{5}$, and $z = 0$.

c) Trying to solve the system in terms of z will not work. To see this, take z over to the right-hand side. The coefficient matrix for the resulting 3×3 system has rank 2. The system has infinitely many solutions.

7.30 The rank of the associated matrix is 2; twice the second equation plus the first equation equals the third equation. The reduced row echelon form is

$$\begin{pmatrix} 1 & 0 & \frac{11}{5} & -\frac{3}{5} & | & \frac{3}{5} \\ 0 & 1 & -\frac{4}{5} & \frac{2}{5} & | & \frac{3}{5} \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}.$$

In this case w and x can be solved for in terms of y and z. However, there is no successful decomposition involving three endogenous variables because no matrix of rank 2 can have a submatrix of rank 3.

Chapter 8

8.1 *a*) $A + B = \begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{pmatrix}$, $A - D$ undefined, $3B = \begin{pmatrix} 0 & 3 & -3 \\ 12 & -3 & 6 \end{pmatrix}$,

$DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$, $B^T = \begin{pmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}$, $A^T C^T = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}$,

$C + D = \begin{pmatrix} 3 & 3 \\ 4 & 0 \end{pmatrix}$, $B - A = \begin{pmatrix} -2 & -2 & -2 \\ 4 & 0 & 0 \end{pmatrix}$, AB undefined,

$CE = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $-D = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$, $(CE)^T = \begin{pmatrix} -1 & 4 \end{pmatrix}$,

$B + C$ undefined, $D - C = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$, $CA = \begin{pmatrix} 2 & 1 & 5 \\ 6 & 10 & 1 \end{pmatrix}$,

EC undefined, $(CA)^T = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}$, $E^T C^T = (CE)^T = \begin{pmatrix} -1 & 4 \end{pmatrix}$.

b) $DA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 4 \\ 2 & 2 & 3 \end{pmatrix}$

$A^T D^T = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{pmatrix} = (DA)^T$.

c) $CD = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix}$, $DC = \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix}$.

8.3 If A is 2×2 and B is 2×3, then AB is 2×3, so $B^T A^T = (AB)^T$ is 3×2. But A^T is 2×2 and B^T is 3×2, so $A^T B^T$ is not defined.

8.5 *a*) $AB = \begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix} = BA.$

b) $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ra + 0c & rb + 0d \\ 0a + rc & 0b + rd \end{pmatrix},$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} ar + b0 & a0 + br \\ cr + d0 & c0 + dr \end{pmatrix}.$

Both equal $\begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix}$. More generally, if $B = rI$ then $AB = A(rI) = r(AI) = rA$. $BA = (rI)A = r(IA) = rA$, too.

8.6 The 3×3 identity matrix is an example of everything except a row matrix and a column matrix. The book gives examples of each of these.

8.7 $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix},$ and

$\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -1 & -2 \end{pmatrix}.$

8.8 *a*) Suppose that U^1 and U^2 are upper triangular; i.e., each $U^k_{ij} = 0$ for $i > j$. Then, $[U^1 + U^2]_{ij} = U^1_{ij} + U^2_{ij} = 0$ if $i > j$. For multiplication, the (i, j)th entry of U^1U^2 is

$$[U^1U^2]_{ij} = \sum_{k<i} U^1_{ik}U^2_{kj} + \sum_{k\geq i} U^1_{ik}U^2_{kj}.$$

The first term is 0 because U^1 is upper triangular. If $i > j$, the second term is 0 because U^2 is upper triangular. Thus, if $i > j$, $[U^1U^2]_{ij} = 0$, and so the product is upper triangular.

If L^1 and L^2 are lower triangular, then $(L^1)^T$ and $(L^2)^T$ are upper triangular. By the previous paragraph, $(L^1)^T + (L^2)^T$ is upper triangular, and so $L^1 + L^2 = [(L^1)^T + (L^2)^T]^T$ is lower triangular. Similarly, $L^1L^2 = [(L^2)^T(L^1)^T]^T$ is lower triangular.

If D is both lower and upper triangular, and if $i > j$, $D_{ij} = 0$ (lower) and $D_{ji} = 0$ (upper), so D is diagonal. Conversely, if D is diagonal, it is obviously both upper and lower triangular. Consequently, if D^1 and D^2 are diagonal, then $D^1 + D^2$ and D^1D^2 are both upper and lower triangular, and hence diagonal.

b) Clearly, $D \subset U$; so $D \cap U = D$. If M is a matrix in $S \cap U$, then for $i < j$, $M_{ij} = 0$ (upper). Thus, $M_{ji} = 0$ (symmetric), so M is diagonal. If M is diagonal, then for $i \neq j$, $M_{ij} = 0 = M_{ji}$; so M is symmetric. Hence $D \subset S$.

c)
$$\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$$
$$= \begin{pmatrix} a_1b_1 & 0 & \cdots & 0 \\ 0 & a_2b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_nb_n \end{pmatrix}$$
$$= \begin{pmatrix} b_1a_1 & 0 & \cdots & 0 \\ 0 & b_2a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_na_n \end{pmatrix}$$
$$= \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}.$$

(This also shows D is closed under multiplication.) Not true for U. For example,

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 23 & 18 \end{pmatrix};$$
$$\begin{pmatrix} 4 & 0 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 17 & 18 \end{pmatrix}.$$

Symmetric matrices generally do not commute. Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $B = \begin{pmatrix} d & e \\ e & f \end{pmatrix}$. Then, $(AB)_{12} = ae + bf$ and $(BA)_{12} = bd + ec$. These two terms are generally not equal.

8.9 There are n choices for where to put the 1 in the first row, $n - 1$ choices for where to put the one in the second row, etc. There are $n \cdot (n-1) \cdots 1 = n!$ permutation matrices.

8.10 Not closed under addition: The identity matrix is a permutation matrix, but $I + I = 2I$ is not.

Closed under multiplication: Suppose P and Q are two $n \times n$ permutation matrices. First, show that each row of PQ has exactly one 1 and $n - 1$ 0s in it. The entries in row i of PQ are calculated by multiplying row i of P by the various columns of Q. If $P_{ij} = 1$, then $(PQ)_{ik} = 0$ unless column k of Q has its 1 in row j. Since Q is a permutation matrix, one and only one column of Q has a 1 in row j. So, there is one k such that $(PQ)_{ik} = 1$ and $n-1$ k's with $(PQ)_{ik} = 0$; that is, row i of PQ has one 1 and $n-1$ 0s. The transpose of a permutation matrix is a permutation matrix. So the same argument shows that each row of $Q^T P^T$ has one 1 and $n-1$ 0s. But each row of $Q^T P^T$ is a column of PQ. So, every row and every column of PQ contains only one 1 and $n-1$ 0s. Thus, PQ is a permutation matrix.

8.12 The three kinds of elementary $n \times n$ matrices are the E_{ij}'s, the $E_i(r)$'s, and the $E_{ij}(r)$'s in the notation of this section. Theorem 8.2 gives the proof for the E_{ij}'s. For the $E_i(r)$'s, a generic element e_{hj} of $E_i(r)$ is

$$\begin{cases} e_{hj} = 0 & \text{if } h \neq j, \\ e_{hh} = 1 & \text{if } h \neq i, \\ e_{ii} = r. \end{cases}$$

The (k,m)th entry of $E_i(r) \cdot A$ is

$$\sum_{j=1}^{n} e_{kj} \cdot a_{jm} = e_{kk} a_{km} = \begin{cases} a_{km} & \text{if } k \neq i \\ r a_{km} & \text{if } k = i. \end{cases}$$

So, $E_i(r) \cdot A$ is A with its ith row multiplied by r.

We now work with $E_{ij}(r)$, the result of adding r times row i to row j in the identity matrix I. The only nonzero entry in row i is the 1 in column i. So row j of $E_{ij}(r)$ has an r in column i, in addition to the 1 in column j. In symbols,

$$\begin{aligned} e_{hh} &= 1 \quad \text{for all } h \\ e_{ji} &= r \\ e_{hk} &= 0 \quad \text{for } h \neq k \text{ and } (h,k) \neq (j,i). \end{aligned}$$

Since the elements in the hth row of $E_{ij}(r) \cdot A$ are the products of row h of $E_{ij}(r)$ and the columns of A, rows of $E_{ij}(r) \cdot A$ are the same as the rows of A, except for row j. The typical mth entry in row j of $E_{ij}(r) \cdot A$ is

$$\sum_{k=1}^{n} e_{jk} \cdot a_{km} = e_{jj} a_{jm} + e_{ji} a_{im} = a_{jm} + r a_{im},$$

since the other e_{jk}'s are zero. But this states that row j of $E_{ij}(r) \cdot A$ is (row j of A) + r(row i of A).

8.13 We saw in Chapter 7 that by using a finite sequence of elementary row operations, one can transform any matrix A to its (reduced) row echelon form (RREF) U. Suppose we apply row operations $R_1, \ldots, R_m$ in that order to reduce A to RREF U. By Theorem 8.3, the same affect can be achieved by premultiplying A by the corresponding elementary matrices $E_1, \ldots, E_m$ so that

$$E_m \cdot E_{m-1} \cdots E_2 \cdot E_1 \cdot A = U.$$

Since U is in echelon form, each row has more leading zeros than its predecessor; i.e., U is upper triangular.

8.14 *a*) Permutation matrix P arises by permuting the rows of the $m \times m$ identity matrix I according to the permutation $s : \{1, \ldots, m\} \to \{1, \ldots, m\}$, so that row i of P is row $s(i)$ of I:

$$p_{ij} = \begin{cases} 1 & \text{if } j = s(i) \\ 0 & \text{otherwise.} \end{cases}$$

The (i, k)th entry of PA is:

$$\sum_{j=1}^{m} p_{ij} a_{jk} = p_{is(i)} a_{s(i)k} = a_{s(i)k},$$

the $(s(i), k)$th entry of A. Row i of PA is row $s(i)$ of A.

b) $AP = [[AP]^T]^T = [P^T A^T]^T$. If $P_{ij} = 1$, then $P^T_{ji} = 1$. Applying part *a* shows that $[AP]^T_{jk} = A^T_{ik}$, so $AP_{kj} = A_{ki}$.

8.15 Carry out the multiplication. In the first case,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The computation for the second case is carried out in a similar fashion.

8.16 Carry out the multiplication.

8.17

$$\left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$
$$\to \left(\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right).$$

Since $a = 0$,

$$\begin{pmatrix} -\dfrac{d}{bc} & \dfrac{1}{c} \\ \dfrac{1}{b} & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ -\dfrac{c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix}.$$

8.18 Carry out the multiplication.

8.19 *a*)

$$\begin{pmatrix} 2 & 1 & \vdots & 1 & 0 \\ 1 & 1 & \vdots & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & \vdots & \frac{1}{2} & 0 \\ 1 & 1 & \vdots & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & \frac{1}{2} & \vdots & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \vdots & -\frac{1}{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & \vdots & 1 & 1 \\ 0 & 1 & \vdots & -1 & 2 \end{pmatrix}.$$

The inverse is $\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$.

b) The inverse is $\begin{pmatrix} 4/6 & -5/6 \\ -2/6 & 4/6 \end{pmatrix}$.

c) Singular.

d)

$$\begin{pmatrix} 2 & 4 & 0 & \vdots & 1 & 0 & 0 \\ 4 & 6 & 3 & \vdots & 0 & 1 & 0 \\ -6 & -10 & 0 & \vdots & 0 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & -2 & 3 & \vdots & -2 & 1 & 0 \\ 0 & 2 & 0 & \vdots & 3 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & \vdots & \frac{1}{2} & 0 & 0 \\ 0 & 1 & -\frac{3}{2} & \vdots & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & \vdots & -\frac{5}{2} & 0 & -1 \\ 0 & 1 & 0 & \vdots & \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

e) The inverse is $\begin{pmatrix} -\frac{5}{2} & 0 & -1 \\ \frac{3}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

f) The inverse is $\begin{pmatrix} 2 & \frac{9}{2} & -\frac{15}{2} & \frac{11}{2} \\ \frac{1}{3} & -\frac{7}{3} & \frac{13}{3} & -\frac{8}{3} \\ -\frac{1}{4} & \frac{3}{4} & -1 & \frac{3}{4} \\ -1 & 1 & -1 & 1 \end{pmatrix}$.

8.20 *a*) $A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

b) $A^{-1} = \begin{pmatrix} -6 & 3/2 & -1 \\ 13 & -3 & 2 \\ 5/2 & -1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 4 \\ 20 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

c) $A^{-1} = \begin{pmatrix} -5/2 & 0 & -1 \\ 3/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

8.21 A $n \times n$ and AB defined implies B has n rows.
A $n \times n$ and BA defined implies B has n columns.

8.22 $A^4 = \begin{pmatrix} 34 & 21 \\ 21 & 13 \end{pmatrix}$, $A^3 = \begin{pmatrix} 13 & 8 \\ 8 & 5 \end{pmatrix}$, $A^{-2} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$.

8.23 To prove that $E_{ij} \cdot E_{ij} = I$, write the (h, k)th entry of E_{ij} as

$$e_{hk} = \begin{cases} 1 & \text{if } h = i, k = j, \\ 1 & \text{if } h = j, k = i, \\ 1 & \text{if } h \neq i, j \text{ and } h = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let a_{hk} denote the (h, k)th entry of $E_{ij} \cdot E_{ij}$:

$$a_{hk} = \sum_{r=1}^{n} e_{hr} e_{rk} = \begin{cases} e_{hj}e_{jk} & \text{if } h = i, \\ e_{hi}e_{ik} & \text{if } h = j, \\ e_{hh}e_{hk} & \text{if } h \neq i, j. \end{cases}$$

If $h = i$, case 1 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } k \neq h \\ 1 & \text{if } k = h. \end{cases}$$

If $h = j$, case 2 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k. \end{cases}$$

If $h \neq i, j$, case 3 tells us that

$$a_{hk} = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k. \end{cases}$$

In other words, (a_{hk}) is the identity matrix.

To see that $E_i(r) \cdot E_i(1/r) = I$, one easily checks that the inverse of $\text{diag}\{a_1, a_2, \dots, a_n\}$ is $\text{diag}\{1/a_1, \dots, 1/a_n\}$, where the entries listed are the diagonal entries of the diagonal matrix.

To see that $E_{ij}(r) \cdot E_{ij}(-r) = I$, write e_{hk} for the (h, k)th entry of $E_{ij}(r)$:

$$e_{hk} = \begin{cases} 1 & \text{if } h = k, \\ r & \text{if } (h, k) = (j, i), \\ 0 & \text{otherwise,} \end{cases}$$

as in Exercise 8.12. Let f_{hk} be the (h, k)th entry of $E_{ij}(-r)$, with $-r$ replacing r in case 2. Then, the (h, k)th entry of $E_{ij}(r) \cdot E_{ij}(-r)$ is

$$a_{hk} = \sum_{l=1}^{n} e_{hl} f_{lk} = \begin{cases} e_{hh} \cdot f_{hk} & \text{if } h \neq j \\ e_{jj} f_{jk} + e_{ji} f_{ik} & \text{if } h = j. \end{cases}$$

If $h \neq j$, $a_{hk} = e_{hh} f_{hk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k. \end{cases}$

If $h = j$, $\quad a_{hk} = \begin{cases} e_{jj} f_{ji} + e_{ji} f_{ii} = -r + r \quad = 0 & \text{if } k \neq j \\ e_{jj} f_{jj} + e_{ji} f_{ij} = 1 \cdot 1 + r \cdot 0 = 1 & \text{if } k = j. \end{cases}$

So, $\quad a_{hk} = \begin{cases} 1 & \text{if } h = k \\ 0 & \text{if } h \neq k, \end{cases}$

and $E_{ij}(r) \cdot E_{ij}(-r) = I$.

8.24 *a*) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ is invertible $\iff ad - bc = ad \neq 0 \iff a \neq 0$ and $d \neq 0$.

b) $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad} \begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$, lower triangular.

c) $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \dfrac{1}{ad} \begin{pmatrix} d & -b \\ 0 & a \end{pmatrix}$, upper triangular.

8.25 *a*) Part *a* holds since $A^{-1} \cdot A = A \cdot A^{-1} = I$ implies that A is the inverse of A^{-1}.

To prove *b*, compute that $I = I^T = (AA^{-1})^T = (A^{-1})^T \cdot A^T$. So, $(A^T)^{-1} = (A^{-1})^T$.

To prove *c*, observe that $(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$.

Similarly, $(B^{-1}A^{-1}) \cdot (AB) = I$.

b) Since

$$\begin{aligned}(A_1\cdots A_k)&(A_k^{-1}\cdot A_{k-1}^{-1}\cdots A_1^{-1})\\ &=(A_1\cdots A_{k-1})(A_kA_k^{-1})(A_{k-1}^{-1}\cdots A_1^{-1})\\ &=(A_1\cdots A_{k-1})(A_{k-1}^{-1}\cdots A_1^{-1})\\ &=(A_1\cdots A_{k-2})(A_{k-1}\cdot A_{k-1}^{-1})(A_{k-2}^{-1}\cdots A_1^{-1})\\ &=\cdots=A_1A_1^{-1}=I.\end{aligned}$$

So, $(A_1\cdots A_k)^{-1}=A_k^{-1}\cdot A_{k-1}^{-1}\cdots A_1^{-1}$.

c) For example, $A=\begin{pmatrix}1&\frac{1}{2}\\ \frac{1}{2}&0\end{pmatrix}$ and $B=\begin{pmatrix}0&\frac{1}{2}\\ \frac{1}{2}&1\end{pmatrix}$, or just take an invertible A and let $B=-A$.

d) Even for 1×1 matrices, $\dfrac{1}{a+b}\neq\dfrac{1}{a}+\dfrac{1}{b}$, in general.

8.26 *a*) One can use the statement and/or method of Exercise 8.25*b* with $A_1=\cdots=A_k=A$.

b) $A^r\cdot A^s=\underbrace{(A\cdots A)}_{r\text{ times}}\cdot\underbrace{(A\cdots A)}_{s\text{ times}}=\underbrace{A\cdots A\cdot A\cdots A}_{r+s\text{ times}}$.

c) $(rA)\cdot\left(\dfrac{1}{r}A^{-1}\right)=r\cdot\dfrac{1}{r}\cdot A\cdot A^{-1}=1\cdot I=I$.

8.27 *a*) Applying $AB=BA$ $(k-1)$ times, we easily find

$$\begin{aligned}AB^k&=A\cdot BB^{k-1}=BAB^{k-1}=BAB\cdot B^{k-2}=B^2AB^{k-2}\\ &=\cdots=B^{k-1}\cdot A\cdot B=B^{k-1}\cdot B\cdot A=B^kA.\end{aligned}$$

Use induction to prove $(AB)^k=(BA)^k$ if $AB=BA$. It is true for $k=1$ since $AB=BA$. Assume $(AB)^{k-1}=(BA)^{k-1}$ and prove it true for k:

$$\begin{aligned}(AB)^k&=(AB)^{k-1}AB\\ &=A^{k-1}B^{k-1}AB, &&\text{by inductive hypothesis}\\ &=A^{k-1}AB^{k-1}B, &&\text{by first sentence in } a\\ &=A^kB^k.\end{aligned}$$

b) Let $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and $B=\begin{pmatrix}1&0\\1&1\end{pmatrix}$. Then

$$(AB)^2=\begin{pmatrix}5&3\\3&2\end{pmatrix}\quad\text{but}\quad A^2B^2=\begin{pmatrix}5&2\\2&1\end{pmatrix}.$$

More generally, suppose that A and B are non-singular. If $ABAB = A^2B^2$, then premultiplying by A^{-1} and postmultiplying by B^{-1} give $AB = BA$.

c) $(A+B)^2 = A^2+AB+BA+B^2$. $(A+B)^2-(A^2+2AB+B^2) = BA-AB$. This equals 0 if and only if $AB = BA$.

8.28 $$D^{-1} = \begin{pmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{pmatrix}.$$

8.29 $\begin{pmatrix} a & b \\ b & d \end{pmatrix}^{-1} = \dfrac{1}{ac-b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$, a symmetric matrix.

8.30 Let U be an $n \times n$ upper-triangular matrix with (i, j)th entry u_{ij}. Let $B = U^{-1}$ with (i, j)th entry b_{ij}. Let $I = (e_{ij})$ be the identity matrix. Since U is upper triangular, $u_{ij} = 0$ for all $i > j$. Now $I = BU$; therefore,

$$1 = e_{11} = \sum_k b_{1k}u_{k1} = b_{11}u_{11}$$

since $u_{21} = \cdots = u_{n1} = 0$. Therefore, $u_{11} \neq 0$ and $b_{11} = 1/u_{11}$. For $h > 1$,

$$0 = e_{h1} = \sum_k b_{hk}u_{k1} = b_{h1}u_{11}.$$

Since $u_{11} \neq 0$, $b_{h1} = 0$ for $h > 1$.

Now, work with column 2 of B.

$$1 = e_{22} = \sum_k b_{2k}u_{k2} = b_{21}u_{12} + b_{22}u_{22} = b_{22}u_{22}$$

since $b_{21} = 0$. Therefore, $u_{22} \neq 0$ and $b_{22} = 1/u_{22}$.

For $h > 2$,

$$0 = e_{h2} = \sum_k b_{hk}u_{k2} = b_{h1}u_{12} + b_{h2}u_{22} = 0 + b_{h2}u_{22}.$$

Since $u_{22} \neq 0$, $b_{h2} = 0$. We conclude that $b_{h2} = 0$ for all $h > 2$. This argument shows $b_{hj} = 0$ for all $h > j$; that is, B too is upper triangular.

The second part follows by transposing the first part and Theorem 8.10*b*.

8.31 The (i, j)th entry of $P^T P$ is the product of the ith row of P^T and the jth column of P, that is, the product of the ith column of P and the jth column of P. This product is 0 if $i \neq j$ and 1 if $i = j$; that is, $P^T P = I$.

8.32 The criterion for invertability is

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{31} - a_{11}a_{21}a_{33} + a_{11}a_{22}a_{31} - a_{12}a_{21}a_{31} \neq 0.$$

See Section 26.1.

8.33 *a*) Suppose a $k \times l$ matrix A has a left inverse L (which must be $k \times k$) and a right inverse R (which must be $l \times l$). Then $LAR = (LA)R = IR = R$ and $LAR = L(AR) = LI = L$, so $R = L$. This is impossible since the two matrices are of different sizes.

b, c) Suppose A is $m \times n$ with $m < n$. If A has rank m, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every right-hand side $\mathbf{b}$, by Fact 7.11*a*. Let $\mathbf{e_i} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in the ith entry. Let $\mathbf{c_i}$ be one of the (infinitely many) solutions of $A\mathbf{x} = \mathbf{e}_i$. Then,

$$A \cdot [\mathbf{c}_1 \cdots \mathbf{c}_m] = [\mathbf{e}_1 \cdots \mathbf{e}_m] = I.$$

So, $C = [\mathbf{c}_1 \quad \mathbf{c}_2 \ \cdots \ \mathbf{c}_m]$ is one of the right inverses of A. Conversely, if A has a right inverse C, then the solution of $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$ since $A(C\mathbf{b}) = (AC)\mathbf{b} = \mathbf{b}$. By Fact 7.7, A must have rank m = number of rows of A.

d) If A is $m \times n$ with $m > n$, apply the previous analysis to A^T.

8.34 *a*) $(I - A)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ 8 \\ 8 \end{pmatrix}$; *b*) $(I - A)^{-1} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 14 \\ 14 \end{pmatrix}$;

c) $(I - A)^{-1} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 16 \\ 18 \end{pmatrix}$.

8.35 For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d > 0$, $a + c < 1$, and $b + d < 1$,

$$(I - A)^{-1} = \begin{pmatrix} 1 - a & -b \\ -c & 1 - d \end{pmatrix}^{-1} = \frac{1}{(1 - a)(1 - d) - bc} \begin{pmatrix} 1 - d & b \\ c & 1 - a \end{pmatrix}.$$

Since $a + c < 1$, $c < (1 - a)$; since $b + d < 1$, $b < (1 - d)$. Therefore, $0 < bc < (1 - a)(1 - d)$ and $(1 - a)(1 - d) - bc > 0$. So, $(I - A)^{-1}$ is a positive matrix.

8.36 Let $a_{\cdot 1}$ = # of columns of A_{11} = # of columns of A_{21}.
Let $a_{\cdot 2}$ = # of columns of A_{12} = # of columns of A_{22}.
Let $c_{1\cdot}$ = # of rows of C_{11} = # of rows of C_{12} = # of rows of C_{13}.
Let $c_{2\cdot}$ = # of rows of C_{21} = # of rows of C_{22} = # of rows of C_{23}.
Then, $a_{\cdot 1} = c_{1\cdot}$ and $a_{\cdot 2} = c_{2\cdot}$.

8.37 C should be written as $\begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$. In the notation of the previous problem

$$\begin{aligned} a_{\cdot 1} &= c_{1\cdot} = 2 \\ a_{\cdot 2} &= c_{2\cdot} = 1 \\ a_{\cdot 3} &= c_{3\cdot} = 3. \end{aligned}$$

8.38 $A^{-1} = \begin{pmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{nn}^{-1} \end{pmatrix}$.

8.39 In the notation of Exercise 8.36, A_{11} is of size $a_{1\cdot} \times a_{\cdot 1}$; $A_{12}A_{22}^{-1}A_{21}$ is also of size $a_{1\cdot} \times a_{\cdot 1}$, so D is well defined.

$$\begin{aligned} &\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} D^{-1} & -D^{-1}A_{12} \cdot A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D^{-1} & A_{22}^{-1}(I + A_{21}D^{-1}A_{12}A_{22}^{-1}) \end{pmatrix} \\ &= \begin{pmatrix} A_{11}D^{-1} - A_{12}A_{22}^{-1}A_{21}D^{-1} & -A_{11}D^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1}A_{21}D^{-1}A_{12}A_{22}^{-1} \\ A_{21}D^{-1} - A_{22}A_{22}^{-1}A_{21}D^{-1} & -A_{21}D^{-1}A_{12}A_{22}^{-1} + A_{22}A_{22}^{-1} + A_{22}A_{22}^{-1}A_{21}D^{-1}A_{12}A_{22}^{-1} \end{pmatrix}. \end{aligned}$$

Write $(1,1)$ as $(A_{11} - A_{12}A_{22}^{-1}A_{21})D^{-1} = DD^{-1} = I$. Write $(1,2)$ as

$$\begin{aligned} -(A_{11} - A_{12}A_{22}^{-1}A_{21})D^{1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} &= -DD^{-1}A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} \\ &= -A_{12}A_{22}^{-1} + A_{12}A_{22}^{-1} = 0. \end{aligned}$$

Write $(2,1)$ as $(A_{21}D^{-1} - IA_{21}D^{-1}) = 0$.
Write $(2,2)$ as $-A_{21}D^{-1}A_{21}A_{22}^{-1} + I + A_{21}D^{-1}A_{12}A_{22}^{-1} = I$.

So the product is the identity matrix $\begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & I \end{pmatrix}$.

8.40 $A^{-1} = \begin{pmatrix} A_{11}^{-1}(I + A_{12}C^{-1}A_{21}A_{11}^{-1}) & -A_{11}^{-1}A_{12}C^{-1} \\ -C^{-1}A_{21}A_{11}^{-1} & C^{-1} \end{pmatrix}$,

where $C = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

8.41 *a*) A_{11} and A_{22} nonsingular.

b) A_{11} and $A_{11} - (1/a_{22})A_{12}A_{21}$ nonsingular.

c) A_{22} invertible and $\mathbf{p}^T A_{22}^{-1}\mathbf{p}$ nonzero.

8.42 *a*) $E_{12}(3)$.

b) $E_{12}(-3)$, $E_{13}(2)$, $E_{23}(-1)$.

c) $E_{12}(-2)$, $E_{13}(3)$, $E_{23}(1)$.

d) $E_{12}(-3)$, $E_{14}(-2)$, $E_{23}(1)$, $E_{34}(-1)$.

8.43 *a*) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 0 & -1 \end{pmatrix}$,

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix}$,

c) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 0 & 1 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 3 & 8 \end{pmatrix}$,

d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 9 \\ 0 & 0 & 0 & -6 \end{pmatrix}$.

8.44 Suppose we can write A as $A = L_1U_1 = L_2U_2$ where L_1 and L_2 are lower triangular with only 1s on the diagonals. The proof and statement of Exercise 8.30 show that L_1 and L_2 are invertible and that L_1^{-1} and L_2^{-1} are lower triangular. Since $U_1 = L_1^{-1}A$ and $U_2 = L_2^{-1}A$, U_1 and U_2 are invertible too. Write $L_1U_1 = L_2U_2$ as $L_2^{-1}L_1 = U_2U_1^{-1}$. By Exercise 8.8, $L_2^{-1}L_1$ is lower triangular and $U_2U_1^{-1}$ is upper triangular. Therefore, $L_2^{-1}L_1$ and $U_2U_1^{-1}$ are both diagonal matrices. Since L_1 and L_2 have only 1s on the diagonal, L_2^{-1} and $L_2^{-1}L_1$ have only 1s on the diagonal. It follows that $L_2^{-1}L_1 = I$ and that $U_2U_1^{-1} = I$. Therefore, $L_2 = L_1$ and $U_2 = U_1$, and the LU decomposition of A is unique.

8.45 Suppose $A = L_1U_1 = L_2U_2$, as in the last exercise. First, choose U_2 to be a row echelon matrix of A. By rearranging the order of the variables, we can assume that each row of U_2 has exactly one more leading zero than the previous row and that its $U_{11} \neq 0$. Write $L_1U_1 = L_2U_2$ as $LU_1 = U_2$ where L is the lower-triangular matrix $L_2^{-1}L_1$ and has only 1s on its diagonal. Let $L = ((l_{ij}))$, $U_1 = ((v_{ij}))$, and $U_2 = ((u_{ij}))$.

$$0 \neq u_{11} = \sum_j l_{1j}v_{j1} = l_{11}v_{11} = v_{11} \Longrightarrow v_{11} \neq 0.$$

For $k > 1$, $u_{k1} = v_{k1} = 0$. So, $0 = u_{k1} = \sum_{j=1}^n l_{kj}v_{j1} = l_{k1}v_{11} \Longrightarrow l_{k1} = 0$ for $k > 1$.

This shows that the first column of L is $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Similar analysis shows that the jth column of L is $\begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$. We need only show that U_2 has no all-zero rows; this follows from the assumptions that U_2 is the row echelon matrix of A and A has maximal rank. It follows that L is the identity matrix and $U_1 = U_2$. Since every such U_i equals U_2, they equal each other. Since $I = L = L_2^{-1}L_1$, $L_2 = L_1$.

8.46 A simple example is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for all a.

8.47 As in Exercise 8.44, write A uniquely as $A = L_1U_1$ where L_1 is lower triangular and has only 1s on its diagonal. Decompose upper-triangular U_1 as

$$DU = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix}\begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & \cdots & u_{2n}/u_{22} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

(Since A is nonsingular, so is U_1, and so all its diagonal entries are nonzero.) Use the method of Exercise 8.44 to see that this DU decomposition is unique.

8.48 *a*) $\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

b) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{pmatrix}$.

d) Use answer to 8.43*d*:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 & 5/2 \\ 0 & 1 & 8/3 & 2/3 \\ 0 & 0 & 1 & -9/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

8.49 *i*) *a*) $\begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 1 \\ -3 & 4 & 1 \end{pmatrix} \xrightarrow[E_{13}(1)\cdot E_{12}(-2)]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & 1 \end{pmatrix} \xrightarrow[E_{23}]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

b) $P = P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

c) $PA = \begin{pmatrix} 3 & 2 & 0 \\ -3 & 4 & 1 \\ 6 & 4 & 1 \end{pmatrix} \xrightarrow[E_{13}(-2)\cdot E_{12}(1)]{} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

d) $PA = E_{12}^{-1} \cdot E_{13}^{-1} \cdot \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 0 & 6 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

ii) *a*) $\begin{pmatrix} 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix} \xrightarrow[E_{12}]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ -6 & -5 & -11 & -12 \\ 2 & 3 & -2 & 3 \end{pmatrix}$

$$\xrightarrow[E_{14}(-2)\cdot E_{13}(6)]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -6 & -1 \end{pmatrix} \xrightarrow[E_{24}(-1)\cdot E_{34}(-1)]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -7 & -5 \end{pmatrix}$$

$$\xrightarrow[E_{34}]{} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

b) P_{12} and P_{34}: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = P$.

c) $PA = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 2 & 3 & -2 & 3 \\ -6 & -5 & -11 & -12 \end{pmatrix} \underset{E_{14}(6)\cdot E_{13}(-2)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & -6 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

$\underset{E_{24}(-1)\cdot E_{23}(-1)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}$, as at end of part *a*.

d) $PA = (E_{24}(-1)\cdot E_{23}(-1)\cdot E_{14}(6)\cdot E_{13}(-2))^{-1}$

$$\cdot \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ -6 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -7 & -5 \\ 0 & 0 & 0 & -4 \end{pmatrix}.$$

8.50 *a*) In the general 2×2 case, a row interchange is required if $a_{11} = 0 \neq a_{21}$.

b) A row interchange is required if $a_{11} = 0$ and $a_{i1} \neq 0$ for some $i > 1$ or if $a_{11}a_{22} - a_{21}a_{12} = 0 \neq a_{11}a_{32} - a_{31}a_{13}$.

8.51 *b*) *i*) $\begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \equiv LU.$

$$L\mathbf{z} = \mathbf{b} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

$$U\mathbf{x} = \mathbf{z} \Longrightarrow \begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

ii) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ -4 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}.$

$$\begin{pmatrix} 2 & 4 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & 3 & 1 \\ -5 & -4 & 1 \\ -10 & -9 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

iii)
$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \\ -24 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix}.$$
$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

iv)
$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -14 \end{pmatrix} \Longrightarrow \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}.$$
$$\begin{pmatrix} 5 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Chapter 9

9.1 $a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$.

9.2 $a_{11} \cdot \det\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix} - a_{12} \det\begin{pmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$

$+a_{13} \det\begin{pmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{pmatrix} - a_{14} \det\begin{pmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$.

There are four terms, each consisting of a scalar multiple of the determinant of a 3×3 matrix. There are six terms in the expansion of the determinant of a 3×3 matrix, so the 4×4 expansion has $4 \cdot 6 = 24$ terms.

9.4 Row 2: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^3 a_{21}a_{12} + (-1)^4 a_{22}a_{11} = a_{11}a_{22} - a_{21}a_{12}$.

Column 1: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^2 a_{11}a_{22} + (-1)^3 a_{21}a_{12} = a_{11}a_{22} - a_{21}a_{12}$.

Column 2: $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (-1)^3 a_{12}a_{21} + (-1)^4 a_{22}a_{11} = a_{11}a_{22} - a_{21}a_{12}$.

9.5 Expand along column one:

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = a_{11}\cdot\det\begin{pmatrix} a_{22} & a_{23} \\ 0 & a_{33} \end{pmatrix} - 0\cdot\det\begin{pmatrix} a_{12} & a_{13} \\ 0 & a_{33} \end{pmatrix}$$
$$+ 0\cdot\det\begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix}$$
$$= a_{11}a_{22}a_{33} + 0 + 0.$$

9.6 $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc$, and $\det\begin{pmatrix} a & b \\ ra+c & rb+d \end{pmatrix} = rab+ad-rab-bc = ad-bc$.

9.7 *a*) $R = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $\det R = -1$, and $\det A = -1$.

b) $R = \begin{pmatrix} 2 & 4 & 0 \\ 0 & -8 & 3 \\ 0 & 0 & 3/4 \end{pmatrix}$, $\det R = -12$, and $\det A = -12$.

c) $R = \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & -6 \end{pmatrix}$, $\det R = -18$, and $\det A = 18$.

9.8 *a*) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. So, $\det = 3$.

b) One row echelon form is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -5 \end{pmatrix}$. So, $\det = -20$.

9.9 All nonsingular since $\det \neq 0$.

9.10 Carry out the calculation

$$\begin{pmatrix} 2 & 4 & 5 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -4 & -15 \\ 0 & -3 & 0 \\ -3 & 4 & 6 \end{pmatrix}\left(-\frac{1}{9}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

9.11 *a*) $\frac{1}{1}\begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}$.

$$b)\ \frac{1}{\det A}\cdot\begin{pmatrix} \begin{vmatrix}5&6\\0&8\end{vmatrix} & -\begin{vmatrix}2&3\\0&8\end{vmatrix} & \begin{vmatrix}2&3\\5&6\end{vmatrix} \\ -\begin{vmatrix}0&6\\1&8\end{vmatrix} & \begin{vmatrix}1&3\\1&8\end{vmatrix} & -\begin{vmatrix}1&3\\0&6\end{vmatrix} \\ \begin{vmatrix}0&5\\1&0\end{vmatrix} & -\begin{vmatrix}1&2\\1&0\end{vmatrix} & \begin{vmatrix}1&2\\0&5\end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{37}\cdot\begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & -2 & 5 \end{pmatrix}.$$

$$c)\ \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

9.12 $x_1 = 35/35 = 1, \quad x_2 = -70/35 = -2.$

9.13 *a*) $x_1 = -7/-7 = 1, \quad x_2 = 14/-7 = -2.$

b) $x_1 = -23/-23 = 1, \quad x_2 = 0/-23 = 0, \quad x_3 = -69/-23 = 3.$

9.14 *a*) $\det A = -1$, $\det B = -1$, $\det AB = +1$, $\det(A+B) = -4$;

b) $\det A = 24$, $\det B = 18$, $\det AB = 432$, $\det(A+B) = 56$;

c) $\det A = ad - bc$, $\det B = eh - fg$, $\det AB = (ad-bc)(eh-fg)$,

$\det(A+B) = \det A + \det B + ah - bg + de - cf.$

9.15

$$\frac{\partial Y}{\partial I_0} = \frac{\partial Y}{\partial G} = \frac{h}{sh+am}, \quad \frac{\partial r}{\partial I_0} = \frac{\partial r}{\partial G} = \frac{m}{sh+am}$$

$$\frac{\partial Y}{\partial M_s} = -\frac{\partial Y}{\partial M_0} = \frac{a}{sh+am}, \quad \frac{\partial r}{\partial M_s} = -\frac{\partial r}{\partial M_0} = \frac{-s}{sh+am}$$

$$\frac{\partial Y}{\partial m}, \quad \frac{\partial r}{\partial m} < 0$$

9.16

$$\frac{\partial Y}{\partial h} = \frac{(I^\circ + G)}{sh+am} - s\frac{(I^\circ + G)h + a(M_s - M^\circ)}{(sh+am)^2}$$

$$= \frac{a}{(sh+am)^2}[(I^\circ + G)m - (M_s - M^\circ)s]$$

$$= \frac{ar}{(sh+am)} > 0,$$

$$\frac{\partial r}{\partial m} = \frac{(I^\circ + G)sh + as(M_s - M^\circ)}{(sh + am)^2}$$

$$= \frac{sY}{(sh + am)} > 0,$$

$$\frac{\partial r}{\partial s} = -\frac{am(M_s - M^\circ) + mh(I^\circ + G)}{(sh + am)^2}$$

$$= -\frac{mY}{(sh + am)} < 0.$$

9.17

$$Y = \frac{h(I^0 + G) + a(M_s - M^0)}{(1 - t)sh + am}, \quad r = \frac{(I^0 + G)m - (1 - t)s(M_s - M^0)}{(1 - t)sh + am}$$

$$\frac{\partial Y}{\partial t} > 0, \quad \frac{\partial r}{\partial t} = \frac{msY}{am + (1 - t)hs} > 0$$

9.18 The IS curve is $[1 - a_0 - c_1(1 - t)]Y + (a + c_2)r = I^0 + G + c_0 - c_1 t_0$. The solution to the system is

$$Y = \frac{(a + c_2)(M_s - M_0) + h(G + I^0 + c_0 - c_1 t_0)}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m}$$

$$r = \frac{-[1 - a_0 - c_1(1 - t_1)(M_s - M_0)] + m(G + I^0 + c_0 - c_1 t_0)}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m}.$$

9.19 Under the obvious assumptions on parameter values, increasing I^0 increases both Y and r. Differentiating the solutions with respect to m,

$$\frac{\partial Y}{\partial m} = \frac{-(a + c_2)Y}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} < 0$$

$$\frac{\partial r}{\partial m} = \frac{c_0 - c_1 t_0 - (a + c_2)r}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0.$$

Similarly, differentiating the solutions with respect to c_0 gives

$$\frac{\partial Y}{\partial c_0} = \frac{h}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0,$$

$$\frac{\partial r}{\partial c_0} = \frac{m}{h[1 - a_0 - c_1(1 - t_1)] + (a + c_2)m} > 0.$$

9.20 The equation system is

$$\begin{aligned} C + cS + cF &= cP \\ sC + S \phantom{{}+cF} &= sP \\ fC + fS + F &= fP. \end{aligned}$$

The solution is

$$C = \frac{\begin{vmatrix} c & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sf - f - s)cP}{1 + sfc - fc - sc}$$

$$S = \frac{\begin{vmatrix} 1 & c & c \\ s & s & 0 \\ f & f & 1 \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{s(1 - c)P}{1 + sfc - fc - sc}$$

$$F = \frac{\begin{vmatrix} 1 & c & c \\ s & 1 & s \\ f & f & f \end{vmatrix} \cdot P}{\begin{vmatrix} 1 & c & c \\ s & 1 & 0 \\ f & f & 1 \end{vmatrix}} = \frac{(1 + sc - s - c)fP}{1 + sfc - fc - sc}.$$

Chapter 10

10.4 *a*) $(2, -1)$, *b*) $(-2, -1)$, *c*) $(2, 1)$, *d*) $(3, 0)$, *e*) $(1, 2, 4)$, *f*) $(2, -2, 3)$.

10.5 *a*) $(1, 3)$, *b*) $(-4, 12)$, *c*) undefined, *d*) $(0, 3, 3)$, *e*) $(0, 2)$, *f*) $(1, 4)$, *g*) $(1, 1)$, *h*) $(3, 7, 1)$, *i*) $(-2, -4, 0)$, *j*) undefined.

10.7 $-\mathbf{u}$ is what one adds to $\mathbf{u}$ to get $\mathbf{0}$. $(-1)\mathbf{u} + \mathbf{u} = (-1)\mathbf{u} + 1 \cdot \mathbf{u} = [(-1) + 1]\mathbf{u} = 0 \cdot \mathbf{u} = \mathbf{0}$. So $(-1)\mathbf{u} = -\mathbf{u}$.

10.8 $(r+s)\mathbf{u} = (r+s)(u_1,\ldots,u_n) = ((r+s)u_1,\ldots,(r+s)u_n)$

$= (ru_1,\ldots,ru_n) + (su_1,\ldots,su_n) = r\mathbf{u} + s\mathbf{u}.$

$r(\mathbf{u}+\mathbf{v}) = r\big((u+v)_1,\ldots,(u+v)_n\big) = (r(u+v)_1,\ldots,r(u+v)_n)$

$= (ru_1+rv_1,\ldots,ru_n+rv_n) = (ru_1,\ldots,ru_n) + (rv_1,\ldots,rv_n)$

$= r\mathbf{u} + r\mathbf{v}.$

10.10 *a*) 5, *b*) 3, *c*) $\sqrt{3}$, *d*) $3\sqrt{2}$, *e*) $\sqrt{2}$, *f*) $\sqrt{14}$, *g*) 2, *h*) $\sqrt{30}$, *i*) 3.

10.11 *a*) 5, *b*) 10, *c*) 4, *d*) $\sqrt{41}$, *e*) 6.

10.12 *a*) $\mathbf{u}\cdot\mathbf{v} = 2$, so the angle is acute; $\theta = \arccos(2/2\sqrt{2}) = 45°$.

b) $\mathbf{u}\cdot\mathbf{v} = 0$, so the angle is right; $\theta = 90°$.

c) $\mathbf{u}\cdot\mathbf{v} = 3$, so the angle is acute; $\theta = \arccos(\sqrt{3}/2) = 30°$.

d) $\mathbf{u}\cdot\mathbf{v} = -1$, so the angle is obtuse; $\theta = \arccos(-1/2\sqrt{3}) \approx 106.8°$.

e) $\mathbf{u}\cdot\mathbf{v} = 1$, so the angle is acute; $\theta = \arccos(1/\sqrt{5}) \approx 63.4°$.

10.13 Multiply each vector $\mathbf{u}$ by the scalar $1/\|\mathbf{u}\|$:

a) $(3/5, 4/5)$, *b*) $(1, 0)$, *c*) $\big(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}\big)$,

d) $\big(-1/\sqrt{14}, 2/\sqrt{14}, -3/\sqrt{14}\big)$.

10.14 Multiply each vector $\mathbf{v}$ found in Exercise 10.13 by the scalar -5:

a) $(-3,-4)$, *b*) $(-5, 0)$, *c*) $\big(-5/\sqrt{3}, -5/\sqrt{3}, -5/\sqrt{3}\big)$,

d) $\big(5/\sqrt{14}, -10/\sqrt{14}, 15/\sqrt{14}\big)$.

10.15 $\|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} - 2\,\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 - 2\,\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2.$

10.16 *a*) $\|\mathbf{u}\| = |u_1| + |u_2| \geq 0$, and equals 0 if and only if both terms in the sum equal 0.

$\|r\mathbf{u}\| = |ru_1| + |ru_2| = |r|(|u_1| + |u_2|) = r\|\mathbf{u}\|.$

$\|\mathbf{u}+\mathbf{v}\| = |u_1+v_1| + |u_2+v_2| \leq |u_1| + |u_2| + |v_1| + |v_2| = \|\mathbf{u}\| + \|\mathbf{v}\|.$

$\|\mathbf{u}\| = \max\{|u_1|, |u_2|\} \geq 0$, and equals 0 if and only if both terms in the max equal 0.

$$\|r\mathbf{u}\| = \max\{|ru_1|, |ru_2|\} = |r| \max\{|u_1|, |u_2|\} = |r|\|\mathbf{u}\|.$$

$$\|\mathbf{u} + \mathbf{v}\| = \max\{|u_1 + v_1|, |u_2 + v_2|\} \le \max\{|u_1| + |v_1|, |u_2| + |v_2|\}$$
$$\le \max\{|u_1|, |u_2|\} + \max\{|v_1|, |v_2|\} = \|\mathbf{u}\| + \|\mathbf{v}\|.$$

b) $|u_1| + |u_2| + \cdots + |u_n|$, called the l_1 norm;
$\max\{|u_1|, |u_2|, \ldots, |u_n|\}$, known as the l_∞ or sup norm.

10.17 *a*) $\mathbf{u} \cdot \mathbf{v} = \sum_i u_i v_i = \sum_i v_i u_i = \mathbf{v} \cdot \mathbf{u}$.

b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \sum_i u_i(v_i + w_i) = \sum_i u_i v_i + u_i w_i = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.

c) $\mathbf{u} \cdot (r\mathbf{v}) = \sum_i u_i(rv_i) = r \sum_i u_i v_i = r\mathbf{u} \cdot \mathbf{v}$. A similar calculation proves the other assertion.

d) $\mathbf{u} \cdot \mathbf{u} = \sum_i u_i^2 \ge 0$.

e) Every term in the sum $\mathbf{u} \cdot \mathbf{u}$ is nonnegative, so $\mathbf{u} \cdot \mathbf{u} = 0$ iff every term is 0; this is true iff each $u_i = 0$.

f) $(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} + 2(\mathbf{u} \cdot \mathbf{v}) + \mathbf{v} \cdot \mathbf{v}$ by part *a*.

10.19 Putting one vertex of the box at the origin, the long side is the vector $(4, 0, 0)$ and the diagonal is the vector $(4, 3, 2)$. Then, $\theta = \arccos(\mathbf{u} \cdot \mathbf{v}/\|\mathbf{u}\|\|\mathbf{v}\|) = \arccos\left(4/\sqrt{29}\right) \approx 42.03°$.

10.20 The two diagonals are $(\mathbf{u} + \mathbf{v})$ and $(\mathbf{v} - \mathbf{u})$. Their inner product is $\mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{u} = \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2$. This equals 0 if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

10.21 *a*) $\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} - 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = 2\mathbf{u} \cdot \mathbf{u} + 2\mathbf{v} \cdot \mathbf{v} = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

b) Calculating as in part *a*, $\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2 = 4\mathbf{u} \cdot \mathbf{v}$.

10.22 $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u} \cdot \mathbf{v}$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$. When applied to (perpendicular) vectors in $\mathbf{R}^2$, this is exactly Pythagoras' theorem.

10.23 *a*) Interchanging $\mathbf{u}$ and $\mathbf{v}$ interchanges the rows in the three matrices in the definition of cross product and thus changes the sign of all three determinants.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) \\
&= -\left(\begin{vmatrix} v_2 & v_3 \\ u_2 & u_3 \end{vmatrix}, -\begin{vmatrix} v_1 & v_3 \\ u_1 & u_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ u_1 & u_2 \end{vmatrix} \right) \\
&= -\mathbf{v} \times \mathbf{u}.
\end{aligned}$$

$b)$ $$\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = u_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0.$$

$c)$ $$\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0.$$

Alternatively, $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = -\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{u}) = 0.$

$d)$ $$\begin{aligned}(r\mathbf{u})\times\mathbf{v} &= \left(\begin{vmatrix} ru_2 & ru_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} ru_1 & ru_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} ru_1 & ru_2 \\ v_1 & v_2 \end{vmatrix}\right) \\ &= r\left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right) \\ &= r(\mathbf{u}\times\mathbf{v}).\end{aligned}$$

A similar calculation proves the remaining assertion.

$e)$ $$\begin{aligned}(\mathbf{u}+\mathbf{w})\times\mathbf{v} &= \left(\begin{vmatrix} u_2+w_2 & u_3+w_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1+w_1 & u_3+w_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1+w_1 & u_2+w_2 \\ v_1 & v_2 \end{vmatrix}\right) \\ &= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right) \\ &\quad + \left(\begin{vmatrix} w_2 & w_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} w_1 & w_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} w_1 & w_2 \\ v_1 & v_2 \end{vmatrix}\right) \\ &= (\mathbf{u}\times\mathbf{v}) + (\mathbf{w}\times\mathbf{v}).\end{aligned}$$

$f)$ $$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\|^2 &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}^2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}^2 \\ &= (u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_3)^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \mathbf{u}\cdot\mathbf{v}\end{aligned}$$

$g)$ $$\begin{aligned}\|\mathbf{u}\times\mathbf{v}\|^2 &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta) \\ &= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta.\end{aligned}$$

h) $\mathbf{u} \times \mathbf{u} = \left(\begin{vmatrix} u_2 & u_3 \\ u_2 & u_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ u_1 & u_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ u_1 & u_2 \end{vmatrix} \right) = (0, 0, 0)$. Alternatively, $\mathbf{u} \times \mathbf{u} = \|\mathbf{u}\|^4 \sin^2 0 = 0$.

i)
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

10.24
$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \mathbf{e}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right) = \mathbf{u} \times \mathbf{v}.$$

10.25 *a*) $\mathbf{u} \times \mathbf{v} = (-1, 0, 1)$.

b) $\mathbf{u} \times \mathbf{v} = (-7, 3, 5)$.

10.26 *a*) The area of the parallelogram is $\|\mathbf{v}\|h = \|\mathbf{v}\|\|\mathbf{u}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$.

b) By *a*, the area of the triangle with vertices *A*, *B*, *C* is $\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$.

$$\begin{aligned} \overrightarrow{AB} &= (0, 1, 3) - (1, -1, 2) = (-1, 2, 1) \\ \overrightarrow{AC} &= (2, 1, 0) - (1, -1, 2) = (1, 2, -2) \\ \overrightarrow{AB} \times \overrightarrow{AC} &= (-6, 1, -4) \\ \tfrac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| &= \tfrac{1}{2}\sqrt{36 + 1 + 16} = \tfrac{1}{2}\sqrt{53}. \end{aligned}$$

10.27 If $\mathbf{z} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$, then $\mathbf{x} - \mathbf{z} = \frac{1}{2}(\mathbf{x} - \mathbf{y}) = \mathbf{z} - \mathbf{y}$, so $\|\mathbf{x} - \mathbf{z}\| = \|\mathbf{y} - \mathbf{z}\|$.

10.28 *a*) $\mathbf{x}(t) = (3 + 2t, 0)$, and its midpoint is $(4, 0)$.

b) $\mathbf{x}(t) = (1 - t, t)$, and its midpoint is $(0.5, 0.5)$.

c) $\mathbf{x}(t) = (1 + t, t, 1 - t)$, and its midpoint is $(1.5, 0.5, 0.5)$.

10.29 No. If it were, then judging from the first coordinate, the point would occur at $t = 2$. But then the last coordinate of the point would be $8 \cdot 2 + 4 = 20$, not 16.

10.30 *a*) Solve both equations for t and equate:

$$t = \frac{x_1 - 4}{-2} \quad \text{and} \quad t = \frac{x_2 - 3}{6} \quad \Longrightarrow \quad x_2 = -3x_1 + 15.$$

b) Alternative method: set $t = 0$ and $t = 1$ to get the two points $(3, 5)$ and $(4, 4)$. Use these two points to deduce equation $x_2 = -x_1 + 8$.

c) $x_2 = 5$ (x_1 takes on any value).

10.31 *a*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 5/2 \end{pmatrix} + \begin{pmatrix} 1 \\ 3/2 \end{pmatrix} t.$ *b*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 7 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t.$

c) $\mathbf{x}(t) = \begin{pmatrix} 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t.$

10.32 No. Solving the first two coordinate equations for s and t, $s = -2$ and $t = 3$. But with these values for s and t, the third coordinate should be 1, not 2.

10.33 *a*) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 4 \end{pmatrix}, x_2 = 2x_1.$

b) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 3 \\ 9 \end{pmatrix}, x_2 = 3x_1 - 2.$

c) $\mathbf{x}(t) = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \end{pmatrix}, x_2 = -\frac{4}{3}x_1 + 4.$

10.34 *a*) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} t.$

b) $\mathbf{x}(t) = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} t.$

c) $\mathbf{x}(s, t) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} t + \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix} s.$

d) $\mathbf{x}(s, t) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} t + \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} s.$

10.35 *a*) $y = -\frac{1}{2}x + \frac{5}{2}$. *b*) $y = (x/2) + 1$. *c*) $-7x + 2y + z = -3$. *d*) $y = 4$.

10.36 *a*) $\mathbf{x}(s, t) = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ -6 \\ 0 \end{pmatrix} t + \begin{pmatrix} -6 \\ 0 \\ 3 \end{pmatrix} s, \quad x - y + 2z = 6.$

b) $\mathbf{x}(s,t) = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} s, \quad x + 2y + 3z = 12.$

10.37 *a*) Rewriting the symmetric equations, $y = y_0 - (b/a)x_0 + (b/a)x$ and $z = z_0 - (c/a)x_0 + (c/a)x$. Then

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} 1 \\ b/a \\ c/a \end{pmatrix} t \quad \text{or} \quad \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} t.$$

b) The two planes are described by any two distinct equalities in system (20). For example,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} \qquad \text{and} \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

In other words, $-bx + ay = -bx_0 + ay_0$ and $cy - bz = cy_0 - bz_0$.

c) *i*) $\dfrac{x_1 - 2}{-1} = \dfrac{x_2 - 3}{4} = \dfrac{x_3 - 1}{5}$, *ii*) $\dfrac{x_1 - 1}{4} = \dfrac{x_2 - 2}{5} = \dfrac{x_3 - 3}{6}$.

d) *i*) $4x_1 + x_2 = 11$ and $5x_2 - 4x_3 = 11$.

ii) $5x_1 - 4x_2 = -3$ and $6x_2 - 5x_3 = -3$.

10.38 *a*) Normals $(1, 2, -3)$ and $(1, 3, -2)$ do not line up, so planes intersect.

b) Normals $(1, 2, -3)$ and $(-2, -4, 6)$ do line up, so planes do not intersect.

10.39 *a*) $(x - 1, y - 2, z - 3) \cdot (-1, 1, 0) = 0$, so $x - y = -1$.

b) The line runs through the points $(4, 2, 6)$ and $(1, 3, 11)$, so the plane must be orthogonal to the difference vector $(-3, 1, 5)$. Thus $(x - 1, y - 1, z + 1) \cdot (-3, 1, 5) = 0$, or $-3x + y + 5z = -7$.

c) The general equation for the plane is $\alpha x + \beta y + \gamma z = \delta$. The equations to be satisfied are $a = \delta/\alpha$, $b = \delta/\beta$, and $c = \delta/\gamma$. A solution is $\alpha = 1/a$, $\beta = 1/b$. $\gamma = 1/c$ and $\delta = 1$, so $(1/a)x + (1/b)y + (1/c)z = 1$.

10.40 Plug $x = 3 + t$, $y = 1 - 7t$, and $z = 3 - 3t$ into the equation $x + y + z = 1$ of the plane and solve for t: $(3 + t) + (1 - 7t) + (3 - 3t) = 1 \Longrightarrow t = 2/3$. The point is $\big(11/3, -11/3, 1\big)$.

10.41 $\begin{pmatrix} 1 & 1 & -1 & \vdots & 4 \\ 1 & 2 & 1 & \vdots & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & -3 & \vdots & 5 \\ 0 & 1 & 2 & \vdots & -1 \end{pmatrix} \Longrightarrow \begin{matrix} x = 5 + 3z \\ y = -1 - 2z. \end{matrix}$

Taking $z = t$, write the line as $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$.

10.42 $IS : [1 - c_1(1 - t_1) - a_0]Y + (a + c_2)r = c_0 - c_1 t_0 + I^* + G$

$LM : mY - hr = M_s - M^*$.

I^* rises $\Longrightarrow$ IS moves up $\Longrightarrow$ Y^* and r^* increase.

M_s rises $\Longrightarrow$ LM moves up $\Longrightarrow$ Y^* decreases and r^* increases.

m rises $\Longrightarrow$ LM becomes steeper $\Longrightarrow$ Y^* decreases and r^* increases.

h rises $\Longrightarrow$ LM flatter $\Longrightarrow$ Y^* increases and r^* decreases.

a_0 rises $\Longrightarrow$ IS flatter (with same r-intercept) $\Longrightarrow$ r^* and Y^* rise.

c_0 rises $\Longrightarrow$ IS moves up $\Longrightarrow$ Y^* and r^* increase.

t_1 or t_2 rises $\Longrightarrow$ IS steeper (with same r-intercept) $\Longrightarrow$ r^* and Y^* decrease.

Chapter 11

11.1 Suppose $\mathbf{v}_1 = r_2\mathbf{v}_2$. Then $1v_1 - r_2\mathbf{v}_2 = r_2\mathbf{v}_2 - r_2\mathbf{v}_2 = \mathbf{0}$.

Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. Suppose that $c_i \neq 0$. Then $c_i\mathbf{v}_i = -c_j\mathbf{v}_j$, so $\mathbf{v}_i = (c_j/c_i)\mathbf{v}_j$.

11.2 *a*) Condition (3) gives the equation system

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ c_1 + 2c_2 &= 0. \end{aligned}$$

The only solution is $c_1 = c_2 = 0$, so these vectors are independent.

b) Condition (3) gives the equation system

$$\begin{aligned} 2c_1 + c_2 &= 0 \\ -4c_1 - 2c_2 &= 0. \end{aligned}$$

One solution is $c_1 = -2$, $c_2 = 1$, so these vectors are dependent.

c) Condition (3) gives the equation system

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ c_1 + c_2 \qquad &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

Clearly $c_1 = c_2 = 0$ is the unique solution.

d) Condition (3) gives the equation system

$$\begin{aligned} c_1 \quad\quad + c_3 &= 0 \\ c_1 + c_2 \quad\quad &= 0 \\ c_2 + c_3 &= 0. \end{aligned}$$

The only solution is $c_1 = c_2 = c_3 = 0$, so these vectors are independent.

11.3 *a*) The coefficient matrix of the equation system of condition 3 is $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. The rank of this matrix is 3, so the homogeneous equation system has only one solution, $c_1 = c_2 = c_3 = 0$. Thus these vectors are independent.

b) The coefficient matrix of the equation system of condition 3 is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The rank of this matrix is 2, so the homogeneous equation system has an infinite number of solutions aside from $c_1 = c_2 = c_3 = 0$. Thus these vectors are dependent.

11.4 If $c_1\mathbf{v}_1 = \mathbf{v}_2$, then $c_1\mathbf{v}_1 + (-1)\mathbf{v}_2 = \mathbf{0}$, and condition (4) fails to hold. If condition (4) fails to hold, then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$ has a nonzero solution in which, say, $c_2 \neq 0$. Then $\mathbf{v}_2 = (c_1/c_2)\mathbf{v}_1$, and v_2 is a multiple of v_1.

11.5 *a*) The negation of "$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ implies $c_1 = c_2 = c_3 = 0$" is "There is some nonzero choice of c_1, c_2, and c_3 that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$."

b) Suppose that condition (5) fails and that $c_1 \neq 0$. Then $\mathbf{v}_1 = -(c_2/c_1)\mathbf{v}_1 - (c_3/c_1)\mathbf{v}_3$.

11.6 Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a collection of vectors such that $\mathbf{v}_1 = \mathbf{0}$. Then $1\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n = \mathbf{0}$ and the vectors are linearly dependent.

11.7 $A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, so the columns of A are linearly independent if and only if the equation system $A \cdot \mathbf{c} = \mathbf{0}$ has no nonzero solution.

11.8 The condition of Theorem 2 is necessary and sufficient for the equation system of Theorem 1 to have a unique solution, which must be the trivial solution. Thus A has rank n, and it follows from Theorem 9.3 that $\det A \neq 0$.

11.9 *a*) $(2,2) = 3(1,2) - 1(1,4)$.

b) Solve the equation system whose augmented matrix is $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{pmatrix}$.

The solution is $c_1 = 0$, $c_2 = 1$, and $c_3 = 2$.

11.10 No they do not. Checking the equation system $Ax = b$, we see that A has rank 2; this means that the equation system does not have a solution for general b.

11.11 For any column vector $\mathbf{b}$, if the equation system $Ax = \mathbf{b}$ has a solution x^*, then $x_1^*\mathbf{v}_1 + \cdots + x_n^*\mathbf{v}_n = \mathbf{b}$. Consequently, if the equation system has a solution for every right-hand side, then every vector $\mathbf{b}$ can be written as a linear combination of the column vectors $\mathbf{v}_i$. Conversely, if the equation system fails to have a solution for some right-hand side $\mathbf{b}$, then $\mathbf{b}$ is not a linear combination of the $\mathbf{v}_i$, and the $\mathbf{v}_i$ do not span $\mathbf{R}^\mathbf{n}$.

11.12 The vectors in *a* are not independent. The vectors in *b* are a basis. The vectors in *c* are not independent. The vectors in *d* are a basis.

11.13 *a*) $\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0$, so these vectors span $\mathbf{R}^\mathbf{2}$ and are independent.

b) $\det \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1 \neq 0$, so these vectors span $\mathbf{R}^\mathbf{2}$ and are independent.

d) $\det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = 2 \neq 0$, so these vectors span $\mathbf{R}^\mathbf{2}$ and are independent.

11.14 Two vectors cannot span $\mathbf{R}^\mathbf{3}$, so the vectors in *a* are not a basis. More than three vectors in $\mathbf{R}^\mathbf{3}$ cannot be independent, so the vectors in *e* fail to be a basis. The matrix of column vectors in *b* and *c* have rank less than 3, so neither of these collections of vectors is a basis. The 3×3 matrix A whose column vectors are the vectors of *d* has rank 3, so $\det A \neq 0$. Thus they are a basis according to Theorem 11.8.

11.15 Suppose *a* is true. Then the $n \times n$ matrix A whose columns are the vectors $\mathbf{v}_i$ has rank n, and therefore $Ax = b$ has a solution for every right-hand side b. Thus the column vectors $\mathbf{v}_i$ span $\mathbf{R}^\mathbf{n}$. Since they are linearly independent and span $\mathbf{R}^\mathbf{3}$, they are a basis. Since the rank of A is n, $\det A \neq 0$. Thus *a* implies b, c and d. Finally suppose d is true. Since $\det A \neq 0$, the solution $x = 0$ to $Ax = 0$ is unique. Thus the columns of A are linearly independent; d implies a.

Chapter 12

12.1 *a*) $x_n = n$. *b*) $x_n = 1/n$. *c*) $x_n = 2^{(-1)^{n-1}(n-1)}$.

d) $x_n = (-1)(n-1)(n-1)/n$. *e*) $x_n = (-1)^n$. *f*) $x_n = (n+1)/n$.

g) x_n is the truncation of π to n decimal places.

h) x_n is the value of the nth decimal place in π.

12.2 *a*) 1/2 comes before 3/2 in original sequence.

b) Not infinite.

c) $2/1$ is not in original sequence.

12.3 If x and y are both positive, so is $x + y$, and $x + y = |x| + |y| = |x + y|$.
If x and y are both negative, so is $x + y$, and $-(x + y) = -x - y$; that is, $|x + y| = |x| + |y|$.
If x and y are opposite signs, say $x > 0$ and $y \leq 0$, then $x + y \leq x = |x| \leq |x| + |y|$ and $-x - y \leq -y = |y| \leq |x| + |y|$. Since $x + y \leq |x| + |y|$ and $-(x + y) \leq |x| + |y|$, $|x + y| \leq |x| + |y|$.
It follows that $|x| = |y + (x - y)| \leq |y| + |x - y|$; so $|x| - |y| \leq |x - y|$. Also, $|y| = |x + (y - x)| \leq |x| + |y - x| = |x| + |x - y|$; so $|y| - |x| \leq |x - y|$. Therefore, $||x| - |y|| \leq |x - y|$.

12.4 If x and y are ≥ 0, so is xy and $|x||y| = xy = |xy|$.
If x and y are ≤ 0, $xy = |xy|$.
If $x \geq 0$ and $y \leq 0$, $xy \leq 0$ and $|xy| = -xy$. Then, $|x||y| = x(-y) = -xy = |xy|$. Similarly, for $x \leq 0$ and $y \geq 0$.

12.5 $|x + y + z| = |(x + y) + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$.

12.6 Follow the proof of Theorem 12.2, changing the last four lines to

$$\begin{aligned} |(x_n - y_n) - (x - y)| &= |(x_n - x) - (y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

12.7 *a*) There exists $N > 0$ such that $n \geq N \Longrightarrow |x_n - x_0| < \frac{1}{2}|x_0|$.
Then, $|x_0| - |x_n| \leq |x_0 - x_n| < \frac{1}{2}|x_0|$, or $\frac{1}{2}|x_0| \leq |x_n|$, for all $n \geq N$.
Let $B = \min\left\{\frac{1}{2}|x_0|, |x_1|, |x_2|, \ldots, |x_N|\right\}$.
For $n \leq N$, $|x_n| \geq \min\{|x_j| : 1 \leq j \leq N\} \geq B$.
For $n \geq N$, $|x_n| \geq \frac{1}{2}|x_0| \geq B$.

b) Let B be as in part *a*). Let $\varepsilon > 0$. Choose N such that for $n \geq N$, $|x_n - x_0| \leq \varepsilon B|x_0|$. Then, for $n \geq N$,

$$\left|\frac{1}{x_n} - \frac{1}{x_0}\right| = \frac{|x_n - x_0|}{|x_n|\,|x_0|} \leq \frac{|x_n - x_0|}{B|x_0|}$$

(since $|x_n| \geq B$ for all $n \Longrightarrow 1/|x_n| \leq 1/B$ for all n)

$$\leq \frac{\varepsilon \cdot B|x_0|}{B|x_0|} = \varepsilon.$$

Therefore, $1/x_n \to 1/x_0$.

12.8 Suppose $y_n \to y$ with all y_ns and y nonzero. By the previous exercise, $1/y_n \to 1/y$. By Theorem 12.3

$$\frac{x_n}{y_n} = x_n \cdot \frac{1}{y_n} \to x \cdot \frac{1}{y} = \frac{x}{y}.$$

12.9 Let $\varepsilon > 0$. Choose N such that for $n \geq N$, $|x_n - 0| \leq \varepsilon \cdot B$ where $|y_n| \leq B$ for all n. Then, for $n \geq N$,

$$|x_n \cdot y_n - 0| = |x_n \cdot y_n| = |x_n||y_n| \leq \frac{\varepsilon}{B} \cdot B = \varepsilon.$$

12.10 Suppose that $x_n \geq b$ for all n and that $x < b$. Choose $\varepsilon > 0$ such that $0 < \varepsilon < b - x$. So, $\varepsilon + x < b$ and $I_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ lies to the left of b on the number line. There exists an $N > 0$ such that for all $n \geq N$, $x_n \in I_\varepsilon(x)$. For these x_ns, $x_n < x + \varepsilon < b$, a contradiction to the hypothesis that $x_n \geq b$ for all n.

12.11 The proof of Theorem 12.3 carries over perfectly, using the fact that $|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ holds in $\mathbf{R^n}$.

12.12 Suppose $\mathbf{x}_n \to \mathbf{a}$ and $\mathbf{b} \neq \mathbf{a}$ is an accumulation point. Choose $\varepsilon = \|\mathbf{a} - \mathbf{b}\|/4$. There exists an N such that $n \geq N \Longrightarrow \|\mathbf{x}_n - \mathbf{a}\| < \varepsilon$. Since $\mathbf{b}$ is an accumulation point, there exists an $m \geq N$ such that $\|\mathbf{x}_m - \mathbf{b}\| < \varepsilon$.

Then,

$$\begin{aligned}\|\mathbf{a}-\mathbf{b}\| &= \|\mathbf{a}-\mathbf{x}_m+\mathbf{x}_m-\mathbf{b}\| \\ &\le \|\mathbf{x}_m-\mathbf{a}\|+\|\mathbf{x}_m-\mathbf{b}\| \\ &< \varepsilon+\varepsilon = 2\varepsilon = \frac{\|\mathbf{a}-\mathbf{b}\|}{2}.\end{aligned}$$

Contradiction!

12.13 The union of the open ball of radius 2 around 0 with the open ball of radius 2 around 1 is open but not a ball. So is the intersection of these two sets.

12.14 Let $\mathbf{x} \in \mathbf{R^n}_+$. Choose $\varepsilon = \min\{x_i : i = 1, \ldots, n\} > 0$. Show $B_{\varepsilon/2}(\mathbf{x}) \subset \mathbf{R^n}_+$. Let $\mathbf{y} \in B_{\varepsilon/2}(\mathbf{x})$ and $j = 1, \ldots, n$.

$$|y_j - x_j| \le \sqrt{(y_1-x_1)^2+\cdots+(y_n-x_n)^2} = \|\mathbf{y}-\mathbf{x}\| < \varepsilon/2.$$

$$x_j - y_j \le |x_j - y_j| \Longrightarrow x_j - |x_j - y_j| \le y_j.$$

$$y_j \ge x_j - |x_j - y_j| \ge \varepsilon - \varepsilon/2 = \varepsilon/2 > 0.$$

12.15 In $\mathbf{R^1}$, $B_\varepsilon(x) = \{y : \|y-x\| < \varepsilon\} = \{y : |y-x| < \varepsilon\}$

$$= \{y : -\varepsilon < y - x < +\varepsilon\} = \{y : x-\varepsilon < y < x+\varepsilon\}$$

$$= (x-\varepsilon, x+\varepsilon).$$

12.16 If A is open, for all $x \in A$ there is an open ball B_x containing x and in A. Then $\cup_{x\in A} B_x \subset A \subset \cup_{x\in A} B_x$. Let A° denote the interior of A. By definition, for all $x \in A$, $B_x \subset A^\circ$, so $\cup_{x\in A} B_x \subset A^\circ \subset A \subset \cup_{x\in A} B_x \subset A^\circ$, and $A^\circ \subset A \subset A^\circ$. The two sets are identical.

12.17 $\mathbf{x} \in \operatorname{int} S \Longrightarrow$ there exists $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset S$. If $\mathbf{x}_n \to \mathbf{x}$, there exists an $N > 0$ such that $\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon$ for all $n \ge N$; that is, $\mathbf{x}_n \in B_\varepsilon(\mathbf{x})$ for all $n \ge N$.

12.18 Suppose $\{x_n\}$ is a sequence in $[a, b]$ that converges to x. By Theorem 12.4, $x \in [a, b]$ too.

12.19 Fix $\mathbf{z} \in \mathbf{R^n}$ and $\varepsilon > 0$. Let $F = \{\mathbf{x} \in \mathbf{R^n} : \|\mathbf{x}-\mathbf{z}\| \le \varepsilon\}$. Let $\mathbf{y}_n$ be a sequence in F with $\mathbf{y}_n \to \mathbf{y}$. Show $\mathbf{y} \in F$; that is, $\|\mathbf{y}-\mathbf{z}\| \le \varepsilon$. Suppose $\|\mathbf{y}-\mathbf{z}\| > \varepsilon$. Choose $\varepsilon_1 = \frac{1}{2}(\|\mathbf{y}-\mathbf{z}\| - \varepsilon) > 0$. Choose $\mathbf{y}_n$ in the sequence with $\|\mathbf{y}_n - \mathbf{y}\| < \varepsilon_1$. Then,

$$\begin{aligned} \|\mathbf{y} - \mathbf{z}\| &= \|(\mathbf{y} - \mathbf{y}_n) + (\mathbf{y}_n - \mathbf{z})\| \\ &\le \|\mathbf{y} - \mathbf{y}_n\| + \|\mathbf{y}_n - \mathbf{z}\| \\ &\le \varepsilon_1 + \varepsilon = \tfrac{1}{2}\|\mathbf{y} - \mathbf{z}\|, \end{aligned}$$

a contradiction.

12.20 Let F be a finite set of points $\{\mathbf{p}_1, \dots, \mathbf{p}_M\}$ in $\mathbf{R}^\mathbf{n}$. Let $d = \min\{\|\mathbf{p}_i - \mathbf{p}_j\| : i \neq j,\ i, j = 1, \dots, M\} > 0$. Suppose $\{\mathbf{x}_n\}$ is a sequence in F and $\mathbf{x}_n \to \mathbf{x}$. Show $\mathbf{x} \in F$. There exists an N such that $n \ge N \Longrightarrow \|\mathbf{x}_n - \mathbf{x}\| < d/2$. Let n_1, $n_2 \ge N$. Then, $\|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\| \le \|\mathbf{x}_{n_1} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_{n_2}\| < d$. But $\|\mathbf{x}_{n_1} - \mathbf{x}_{n_2}\| < d$ and $\mathbf{x}_{n_1}$, $\mathbf{x}_{n_2} \in F$ implies that $\mathbf{x}_{n_1} = \mathbf{x}_{n_2}$; that is, there exists a $\mathbf{p} \in F$ such that $\mathbf{x}_n = \mathbf{p}$ for all $n \ge N$. Then, $\lim \mathbf{x}_n = \mathbf{p} \in F$. Proof that the set of integers is a closed set is the same, taking $d = 1$.

12.21 *a*) Not open, because $(0, 0)$ is in the set but $(0, \varepsilon)$ is not in the set for any $\varepsilon \neq 0$.
Not closed, because $\{(n/(n+1), 0) : \text{integer } n\}$ is in the set, but its limit $(1, 0)$ is not.

b) Not open, since $(0, 0)$ is in the set, but $(0, \varepsilon)$ for all small $\varepsilon > 0$ is not. Closed; same argument as in Exercise 12.20*a*.

c) Not open, since no disk about $(1, 0)$ is in the set.
Closed, since if $\{(x_n, y_n)\}$ is a sequence in the set that converges to (x, y), then $x + y = \lim x_n + \lim y_n = \lim(x_n + y_n) = 1$.

d) Open, since if $x + y < 1$, then so is $(x + h) + (y + k)$ for all sufficiently small h and k.
Not closed, because if $x + y = 1$, then (x, y) is not in the set, but $(x - 1/n, y - 1/n)$ for all integers $n > 0$.

e) This set is the union of the x- and y-axes. Each axis is closed. For instance, if $(x_n, 0)$ converges to (x, y), then $y = 0$ and so (x, y) is on the x-axis. Thus the set is the union of two closed sets, and hence is closed. The set is not open. $(0, 0)$ is in the set, but $(\varepsilon, \varepsilon)$ fails to be in the set for all $\varepsilon > 0$.

12.22 The complement of any intersection of closed sets is a union of open sets. This union is open according to Theorem 12.8, so its complement is closed. The complement of a finite union of closed sets is a finite intersection of open sets. According to Theorem 12.8 this intersection is open, so its complement is closed.

12.23 If $x \in \operatorname{cl} S$, then there exists a sequence $\{x_n\} \subset S$ converging to x. Since there are points in S arbitrarily near to x, $x \notin \operatorname{int} T$. If $x \notin \operatorname{cl} S$, then for some $\varepsilon > 0$, $B_\varepsilon(x) \cap S = \emptyset$. (Otherwise it would be possible to construct a sequence in S converging to x.) Therefore $x \in \operatorname{int} T$.

12.24 Any accumulation point of S is in $\operatorname{cl} S$. To see this, observe that if x is an accumulation point of S, then for all $n > 0, B_{1/n}(x) \cap S \neq \emptyset$. For each integer n, choose x_n to be in this set. Then the sequence $\{x_n\}$ is in S and converges to x, so $x \in \operatorname{cl} S$. Conversely, if x is in $\operatorname{cl} S$, then there is a sequence $\{y_n\} \subset S$ with limit x. Therefore, for all $\varepsilon > 0$ there is a $y \in S$ such that $\|y - x\| < \varepsilon$. Consequently, for all $\varepsilon > 0, B_\varepsilon(x) \cap S \neq \emptyset$, so x is an accumulation point of S.

12.25 There is no ball around b contained in $(a, b]$, so this interval is not open. A sequence converging down to a from above and bounded above by b will be in the interval, but its limit a is not. Hence the interval is not closed.

The sequence converges to 0, but 0 is not in the set so the set is not closed. No open ball of radius less than 1/2 around the point 1 contains any elements of the set, so the set is not open.

A sequence in the line without the point converging to the point lies in the line without the point, but its limit does not. Hence the set is not closed. Any open ball around a point on the line will contain points not on the line, so the set is not closed.

12.26 Suppose that x is a boundary point. Then for all n there exist points $x_n \in S$ and y_n in S^c, both within distance $1/n$ of x. Thus x is the limit of both the $\{x_n\}$ and $\{y_n\}$ sequences, so $x \in \operatorname{cl} S$ and $x \in \operatorname{cl} S^c$. Conversely, if x is in both $\operatorname{cl} S$ and $\operatorname{cl} S^c$, there exist sequences $\{x_n\} \subset S$ and $\{y_n\} \subset S^c$ converging to x. Thus every open ball around x contains elements of both sequences, so x is a boundary point.

12.27 The whole space $\mathbf{R^m}$ contains open balls around every one of its elements, so it is open. Any limit of a sequence in $\mathbf{R^m}$ is, by definition, in $\mathbf{R^m}$, so $\mathbf{R^m}$ is closed. The empty set is the complement of $\mathbf{R^m}$. It is the complement of a closed and open set, and consequently is both open and closed. Checking directly, the empty set contains no points or sequences to falsify either definiton, and so it satisfies both.

12.28 The hint says it all. Let a be a member of S and b not. Let $l = \{x : x = x(t) \equiv ta + (1 - t)b, 0 \leq t < \infty\}$. Let $t*$ be the least upper bound of the set of all t such that $x(t) \in S$. Consider the point $x(t^*)$. It must be in S because there exists a sequence $\{t_n\}$ converging up to t^* such that each

$x(t_n) \in S$, $x(t*)$ is the limit of the $x(t_n)$, and S is closed. Since S is open, there exists a ball of positive radius around $x(t*)$ contained in S. Thus for $\varepsilon > 0$ sufficiently small, $x(t * + \varepsilon) \in S$ which contradicts the construction of $t*$.

Chapter 13

13.1 See figures.

13.3 Intersect the graph with the plane $z = k$ in $\mathbf{R}^3$, and project the resulting curve down to the $z = 0$ plane.

13.4 A map that gives various depths in a lake. Close level curves imply a steep drop off (and possibly good fishing).

13.6 *a*) Spheres around the origin.

b) Cylinders around the x_3-axis.

c) The intersection of the graph with planes parallel to the x_1x_3-plane are parabolas with a minimum at $x + 1 = 0$. The level of the parabolas shrinks as y grows.

d) Parallel planes.

13.9 See figures.

13.10 The graph of f is $G = \{(x, y) : y = f(x)\}$. Consider the curve $F(t) = (t, f(t))$. The point (x, y) is on the curve if and only if there is a t such that $x = t$ and $y = f(t)$; that is true if and only if $y = f(x)$; and this is true if and only if (x, y) is in the graph of f.

13.11 *a*) $\begin{pmatrix} 2 & -3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

b) $\begin{pmatrix} 2 & -3 \\ 1 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$,

c) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & -6 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

13.12 *a*) $\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, *b*) $(x_1 \quad x_2 \quad x_3)\begin{pmatrix} 5 & -5 \\ -5 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,

c) $(x_1 \quad x_2 \quad x_3)\begin{pmatrix} 1 & 2 & -3 \\ 2 & 2 & 4 \\ -3 & 4 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

13.13 $f(x) = \sin x$, $g(x) = e^x$, $h(x) = \log x$.

13.14 Suppose $f(\mathbf{x})$ is linear. Let $a_i = f(\mathbf{e}_i)$, and suppose $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k$. Then

$$\begin{aligned} f(\mathbf{x}) &= f(x_1\mathbf{e}_1 + \cdots + x_k\mathbf{e}_k) \\ &= \sum_i x_i f(\mathbf{e}_i) \\ &= \sum_i a_i x_i \\ &= (a_1, \ldots, a_k)(x_1, \ldots, x_k). \end{aligned}$$

13.15 Consider the line $\mathbf{x}(t) = \mathbf{x} + t\mathbf{y}$, and suppose f is linear. Then $f(\mathbf{x}(t)) = f(\mathbf{x} + t\mathbf{y}) = f(\mathbf{x}) + tf(\mathbf{y})$, so the image of the line will be a line if $f(\mathbf{y}) \neq \mathbf{0}$ and a point if $f(\mathbf{y}) = \mathbf{0}$.

13.16 Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence with limit $\mathbf{x}$. Since g is continuous, there exists an N such that for all $n \geq N$, $|g(\mathbf{x}_n) - g(\mathbf{x})| < g(\mathbf{x})/2$. Thus, for $n \geq N$, $g(\mathbf{x}_n) \neq 0$, since $g(\mathbf{x}) - g(\mathbf{x}_n) < g(\mathbf{x})/2$ implies that $0 < g(\mathbf{x})/2 < g(\mathbf{x}_n)$. Since convergence depends only on the "large N" behavior of the sequence, we can suppose without loss of generality that $g(\mathbf{x}_n)$ is never 0. Now, $f(\mathbf{x}_n)$ converges to $f(\mathbf{x})$ and $g(\mathbf{x}_n)$ converges to $g(\mathbf{x})$, so the result follows from Exercise 12.8.

13.17 Suppose not; that is, suppose that for every n there exists an $\mathbf{x}_n \in B_{1/n}(\mathbf{x}^*)$ with $f(\mathbf{x}_n) \leq 0$. Since $||\mathbf{x}_n - \mathbf{x}^*|| < 1/n$, $\mathbf{x}_n \to \mathbf{x}^*$. Since f is continuous, $f(\mathbf{x}_n) \to f(\mathbf{x}^*)$. Since each $f(\mathbf{x}_n) \leq 0$, $f(\mathbf{x}^*) = \lim f(\mathbf{x}_n) \leq 0$, but $f(\mathbf{x}^*) > 0$. (See Theorem 12.4).

13.18 From Theorem 12.5 conclude that a function $f : \mathbf{R^k} \to \mathbf{R^m}$ is continuous if and only if its coordinate functions are continuous. From Theorems 12.2 and 12.3 and this observation, conclude that if f and g are continuous, then the coordinate functions of $f + g$ and $f \cdot g$ are continuous.

13.19 $\Longrightarrow$: Suppose $f = (f_1, \dots, f_m)$ is continuous at $\mathbf{x}^*$. Let $\{\mathbf{x}_n\}$ be a sequence converging to $\mathbf{x}^*$. Then, $f(\mathbf{x}_n) \to f(\mathbf{x}^*)$. By Theorem 12.5, each $f_i(\mathbf{x}_n) \to f_i(\mathbf{x}^*)$. So each f_i is continuous.

$\Longleftarrow$: Reverse the above argument.

13.20 Suppose $\mathbf{x}_n = (x_{n1}, x_{n2}, \dots, x_{nk}) \to \mathbf{x}_0 = (x_{01}, x_{02}, \dots, x_{0k})$. By Theorem 12.5, $x_{ni} \to x_{0i}$ for each i. This says $h(x_1, \dots, x_k) = x_i$ is continuous. Any monomial $g(x_1, \dots, \mathbf{x}_k) = Cx_1^{n_1} \cdots x_k^{n_k}$ is a product of such hs (for positive integer n_is) and is continuous by Theorem 13.4. Any polynomial is a sum of monomials and is continuous by Theorem 13.4.

13.21 Let $\{t_n\}_{n=1}^{\infty}$ denote a sequence with limit t^*. Then $g(t_n) = f(t_n, a_2, \dots, a_k) \to f(t^*, a_2, \dots, a_k) = g(t^*)$, so g is continuous.

For $f(x, y) = xy^2/(x^4 + y^4)$, $f(t, a) = ta^2/(t^4 + a^4)$. If $a \neq 0$ this sequence clearly converges to $0/a^4 = 0$ as t converges to 0. If $a = 0$, $f(t, 0) = 0$ for all t. Similar arguments apply to continuity in the second coordinate. But $f(t, t) = t^3/(t^4 + t^4)$, which grows as $1/t$ as $t \to 0$.

13.22 *a*) If $y \in f(U_1)$, then there is an $x \in U_1$ such that $f(x) = y$. Since $U_1 \subset U_2$, $x \in U_2$, so $y \in f(U_2)$.

b) If $x \in f^{-1}(V_1)$, then there is a $y \in V_1$ such that $f(x) = y$. Now $y \in V_2$, so $x \in f^{-1}(V_2)$.

c) If $x \in U$, then there is a $y \in f(U)$ such that $f(x) = y$. Thus $x \in f^{-1}(f(U))$.

d) If $y \in f(f^{-1}(V))$, then there is an $x \in f^{-1}(V)$ such that $f(x) = y$, so $f(x) \in V$.

e) $x \in f^{-1}(V^c)$ if and only if there is a $y \in V^c$ such that $f(x) = y$. Since $f(x) = y \in V^c$, $f(x) \notin V$. Thus $x \notin f^{-1}(V)$, and so $x \in (f^{-1}(V))^c$.

13.23

	$f(x)$	*Domain*	*Range*	*One-to-one*	$f^{-1}(y)$	*Onto*
a)	$3x - 7$	**R**	**R**	Yes	$\frac{1}{3}(y + 7)$	Yes
b)	$x^2 - 1$	**R**	$[-1, \infty)$	No		No
c)	e^x	**R**	$(0, \infty)$	Yes	$\ln(y)$	No
d)	$x^3 - x$	**R**	**R**	No		Yes
e)	$x/(x^2 + 1)$	**R**	$[-\frac{1}{2}, \frac{1}{2}]$	No		No
f)	x^3	**R**	**R**	Yes	$y^{1/3}$	Yes
g)	$1/x$	$\mathbf{R} - \{\mathbf{0}\}$	$\mathbf{R} - \{\mathbf{0}\}$	Yes	$1/y$	No
h)	$\sqrt{x - 1}$	$[1, \infty)$	$[0, \infty)$	Yes	$y^2 + 1$	No
i)	xe^{-x}	**R**	$(-\infty, 1/e]$	No		No

13.24 *a*) $f(x) = \log x$, $g(x) = x^2 + 1$; *b*) $f(x) = x^2$, $g(x) = \sin x$;
c) $f(x) = (\cos x, \sin x)$, $g(x) = x^3$; *d*) $f(x) = x^3 + x$, $g(x) = x^2 y$.

Chapter 14

14.1 *a*) $\dfrac{\partial f}{\partial x} = 8xy - 3y^3 + 6, \qquad \dfrac{\partial f}{\partial y} = 4x^2 - 9xy^2.$

b) $\dfrac{\partial f}{\partial x} = y, \qquad \dfrac{\partial f}{\partial y} = x.$

c) $\dfrac{\partial f}{\partial x} = y^2, \qquad \dfrac{\partial f}{\partial y} = 2xy.$

d) $\dfrac{\partial f}{\partial x} = 2e^{2x+3y}, \qquad \dfrac{\partial f}{\partial y} = 3e^{2x+3y}.$

e) $\dfrac{\partial f}{\partial x} = \dfrac{-2y}{(x+y)^2}, \qquad \dfrac{\partial f}{\partial y} = \dfrac{2x}{(x+y)^2}.$

f) $\dfrac{\partial f}{\partial x} = 6xy - 7\sqrt{y}, \qquad \dfrac{\partial f}{\partial y} = 3x^2 - \dfrac{7x}{2\sqrt{y}}.$

14.2 For the Cobb-Douglas function,

$$\frac{\partial f}{\partial x_i} = k\alpha_i x_i^{\alpha_i - 1} x_j^{\alpha_j} = \alpha_i \frac{q}{x}.$$

For the CES function,

$$\frac{\partial f}{\partial x_i} = c_i hk(c_1 x_1^{-a} + c_2 x_2^{-a})^{-(h/a)-1} x_i^{-a-1}.$$

14.3 $(\partial T/\partial x)(x^*, y^*)$ is the rate of change of temperature with respect to an increase in x while holding y fixed at the point (x^*, y^*).

14.4 *a*) $Q = 5400$.

b, c) $Q(998, 216) = 5392.798$. The approximation gives $Q \approx 5392.8$, which is in error by -0.002.

$Q(1000, 217.5) = 5412.471$. The approximation gives $Q \approx 5412.5$, which is in error by -0.029.

d) For the approximation to be in error by more than 2.0, ΔL must be at least 58.475, or about 5.875% of L.

14.5 *a*) $\varepsilon_I = (I/Q_1)(\partial Q_1/\partial I) = (I/Q_1)(b_1 Q/I) = b_1$. Similarly the own and cross-price elasticities are, respectively, a_{11} and a_{12}.

b) We would expect a_{11} to be negative and b_1 to be positive, but in fact these constraints are not derivable from theoretical principles.

14.6 *a*) $Q = 9/2$.

b) $\Delta Q = \dfrac{-12p_2^{3/2}}{p_1^3}\Delta p_1 + \dfrac{9\sqrt{p_2}}{p_1^2}\Delta p_2$.

Evaluating at $p_1 = 6$, $\Delta p_1 = 1/4$, $p_2 = 9$, and $\Delta p_2 = -1/2$ gives $Q = -3/4$.

c) Evaluating the expression in part *b* when $\Delta p_1 = \Delta p_2 = 0.2$, gives $\Delta Q = -0.15$.

d) In case *b*, $Q = 3.80645$ and the estimate is $Q_e = 3.75$, with an error of $Q - Q_e = 0.05645$. In case *c*, $Q = 4.35562$ and the estimate is $Q_e = 4.35$, with an error of $Q - Q_e = 0.00562$.

14.7 *a*) $Q = 240$.

b) $\Delta Q = \dfrac{10x_2^{1/2}x_3^{1/6}}{3x_3^{5/6}}\Delta x_1 + \dfrac{5x_1^{1/3}x_3^{1/6}}{x_2^{1/2}}\Delta x_2 + \dfrac{5x_1^{1/3}x_2^{1/2}}{3x_3^{5/6}}\Delta x_3$.

Evaluating, $Q + \Delta Q = 238.046$.

c) The actual output at $x_1 = 27.1$, $x_2 = 15.7$, and $x_3 = 64$ is 238.032.

d) $Q + \Delta Q = 241.843$, and the actual output is 238.837.

14.8 *a*) $df = (4x^3 + 4xy^2 + y^4)\,dx + (10 + 4x^2 y + 4xy^3)\,dy$.

Taking $x = 10$, $y = 1$, $dx = 0.36$, and $y = 0.04$ gives $f \approx 11692.8$.

b) $df = \dfrac{4\sqrt{y}}{x^{1/3}}\,dx + \dfrac{3x^{2/3}}{\sqrt{y}}\,dy$.

Evaluating at $x = 1000$, $dx = -2$, $y = 100$, and $dy = 1.5$, $f = 6037$.

c)
$$df = \frac{dx}{4\sqrt{x}\sqrt{x^{1/2} + y^{1/3} + 5z^2}} + \frac{dy}{6y^{2/3}\sqrt{x^{1/2} + y^{1/3} + 5z^2}} + \frac{5z\,dz}{\sqrt{x^{1/2} + y^{1/3} + 5z^2}}.$$

Evaluating at $x = 4$, $dx = 0.2$, $y = 8$, $dy = -0.05$, $z = 1$, and $dz = 0.02$ gives $f \approx 3.04097$.

14.9 $Q = 3 * 100 * 5 = 1500$. $\partial Q/\partial K = 2/3 * 1500/1000 = 1$. $\partial Q/\partial L = 1/3 * 1500/125 = 4$.

$$Q(998, 128) \approx 1500 + 1 \cdot (-2) + 4 \cdot 3 = 1510.$$

14.10 $df = \dfrac{3x^2}{2\sqrt{x^3 - y^3 - z^3}}\, dx - \dfrac{3y^2}{2\sqrt{x^3 - y^3 - z^3}}\, dy - \dfrac{3z^2}{2\sqrt{x^3 - y^3 - z^3}}\, dz.$

Evaluating at $x = 4$, $dx = 0.1$, $y = 2$, $dy = -0.05$, $z = 1$, and $dz = 0.02$ gives $f \approx 6.5075$.

14.11 Here is the chain rule calculation:

$$\begin{aligned} \frac{d}{dt} f(x(t), y(t)) &= \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t) \\ &= [3y(t)^2 + 2](-6t) + [6x(t)y(t)](12t^2 + 1). \end{aligned}$$

14.12 The rate of change of output with respect to time is $2.5 \cdot 0.5 + 3 \cdot 2 = 7.25$.

14.13 $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial t} + \dfrac{\partial z}{\partial w} w'(t) + \dfrac{\partial z}{\partial x} x'(t) + \dfrac{\partial z}{\partial y} y'(t)$

$$= \frac{5t}{w^2 y} - \frac{5t^2 + 3xy}{w^3 y} e^t + \frac{3}{2w^2} 2t - \frac{10t^2}{4w^2 y^2} \frac{2t}{2\sqrt{t^2 + 1}}.$$

At $t = 0$, $x = 1$, $y = 1$, and $w = 2$. Substituting, $dz/dt = -3/8$.

14.14 $(3 \cdot 10{,}000^{-1/4} \cdot 625^{1/4}) \cdot (-10 \cdot 100 \cdot 100) + (10{,}000^{3/4} \cdot 625^{-3/4}) \cdot 250 = -148{,}000$.

14.15 At time $t = 1$, Bob the Bat's position will be $(2, 1, 2)$. The tangent vector to Bob's path is $(2, 2t, 2t) = (2, 2, 2)$. His displacement after 2 units of time from his time 1 position will be $(4, 4, 4)$, so his position at $t = 3$ will be $(2, 1, 2) + (4, 4, 4) = (6, 5, 6)$.

14.16 The car's velocity vector is $(e^t + 10t, 4t^3 - 4)$. This vector will be parallel to the x-axis when its y-coordinate is 0. This happens only when $t = 1$. At $t = 1$, the position vector is $(e + 5, -3)$.

14.17 $$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x}\\ &= -\frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}\\ \frac{\partial F}{\partial y} &= \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y}\\ &= \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}.\end{aligned}$$

14.18 $\Delta 4x^2y = (8xy, 4x^2)$. At $(2,3)$ this vector is $(48,16)$. Normalizing it to length 1, the length-1 vector of maximal ascent is $(3/\sqrt{10}, 1/\sqrt{10})$.

14.19 $\Delta y^2e^{3x} = (3y^2e^{3x}, 2ye^{3x})$. At $x = 0$, $y = 3$, this vector is proportional to $(9,2)$. Normalizing to length 1, the vector is $(9/\sqrt{85}, 2/\sqrt{85})$.

14.20 $\nabla f(x,y) = (y^2 + 3x^2y, 2xy + x^3)$. Evaluating at $(4,-2)$, $\nabla f(4,-2) = (-92, 48)$. Then $\nabla f(4,-2)\cdot(1/\sqrt{10}, 3/\sqrt{10}) = 52/\sqrt{10}$.

14.21 $$\begin{pmatrix}\partial q_1/\partial t & \partial q_1/\partial r\\ \partial q_2/\partial t & \partial q_2/\partial r\end{pmatrix} = \begin{pmatrix}-15/2 & 15/4 & 9/2\\ -100/9 & -200/27 & 80/3\end{pmatrix}\begin{pmatrix}1 & 0\\ 6 & 90\\ 1 & 20\end{pmatrix}.$$

14.22 $$\begin{aligned}\begin{pmatrix}\partial f_1/\partial x & \partial f_1/\partial y\\ \partial f_2/\partial x & \partial f_2/\partial y\end{pmatrix} &= \begin{pmatrix}1 & 1\\ 0 & 2v\end{pmatrix}\begin{pmatrix}2x & 0\\ 2y & 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}2 & 2\\ 0 & 4\end{pmatrix}.\end{aligned}$$

14.23 *a*) $\begin{pmatrix}8y & 8x - 9y^2\\ 8x - 9y^2 & -18xy\end{pmatrix}$, *b*) $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, *c*) $\begin{pmatrix}0 & 2y\\ 2y & 2x\end{pmatrix}$,

d) $\begin{pmatrix}4e^{2x+3y} & 6e^{2x+3y}\\ 6e^{2x+3y} & 9e^{2x+3y}\end{pmatrix}$,

e) $\begin{pmatrix}4y/(x-y)^3 & -2(x+y)/(x-y)^3\\ -2(x+y)/(x-y)^3 & 4x/(x-y)^3\end{pmatrix}$,

f) $\begin{pmatrix}6y & 6x - 7/2\sqrt{y}\\ 6x - 7/2\sqrt{y} & \frac{7x}{4y^{3/2}}\end{pmatrix}$.

14.24 $$\frac{\partial^3 Q}{\partial k^3} = \frac{15l^{3/4}}{16k^{9/4}}, \quad \frac{\partial^3 Q}{\partial k\,\partial k\,\partial l} = \frac{\partial^3 Q}{\partial k\,\partial l\,\partial k} = \frac{\partial^3 Q}{\partial l\,\partial k\,\partial k} = -\frac{9}{16k^{5/4}l^{1/4}},$$
$$\frac{\partial^3 Q}{\partial k\,\partial l\,\partial l} = \frac{\partial^3 Q}{\partial l\,\partial k\,\partial l} = \frac{\partial^3 Q}{\partial l\,\partial l\,\partial k} = -\frac{9}{16k^{1/4}l^{5/4}}, \quad \frac{\partial^3 Q}{\partial l^3} = \frac{15l^{9/4}}{16k^{3/4}}.$$

14.25 $$\frac{\partial^3 F}{\partial x_1\,\partial x_2\,\partial x_3} = \frac{\partial^2}{\partial x_1\,\partial x_2}\frac{\partial F}{\partial x_3} = \frac{\partial^2}{\partial x_2\,\partial x_1}\frac{\partial F}{\partial x_3} = \frac{\partial^3 F}{\partial x_2\,\partial x_1\,\partial x_3}$$
$$= \frac{\partial}{x_2}\frac{\partial^2 F}{\partial x_1\,\partial x_3} = \frac{\partial}{x_2}\frac{\partial^2 F}{\partial x_3\,\partial x_1} = \frac{\partial^3 F}{\partial x_2\,\partial x_3\,\partial x_1}$$
$$= \frac{\partial^2}{\partial x_2\,\partial x_3}\frac{\partial F}{\partial x_1} = \frac{\partial^2}{\partial x_3\,\partial x_2}\frac{\partial F}{\partial x_1} = \frac{\partial^3 F}{\partial x_3\,\partial x_2\,\partial x_1},$$

and so forth.

14.26 The Cobb-Douglas production function is $f(x_1, x_2) = Kx_1^{a_1}x_2^{a_2}$.

$$\frac{\partial f}{\partial x_1} = a_1 K x_1^{a_1-1}x_2^{a_2} > 0 \Rightarrow a_1 > 0;$$
$$\frac{\partial^2 f}{\partial x_1^2} = a_1(a_1-1)Kx_1^{a_1-2}x_2^{a_2} < 0 \Rightarrow a_1(a_1-1) < 0 \Rightarrow 0 < a_1 < 1.$$

Similarly, we want $0 < a_2 < 1$.

The CES production function is $f(x_1, x_2) = k(c_1 x_1^{-a} + c_2 x_2^{-a})^{-h/a}$.

$$\frac{\partial f}{\partial x_1} = c_1 hk x_1^{-1-a}\left(\frac{c_1}{x_1^a} + \frac{c_2}{x_2^a}\right)^{-1-(h/a)} > 0 \Rightarrow c_1 hk > 0;$$
$$\frac{\partial^2 f}{\partial x_1^2} = \frac{c_1 hk x_2^a\left(-(1+a)c_2 x_1^a - c_1(1-h)x_2^a\right)}{x_1^2\left(c_1 x_1^{-a} + c_2 x_2^{-a}\right)^{h/a}\left(c_2 x_1^a + c_1 x_2^a\right)^2} > 0$$
$$\Rightarrow (1+a)c_2 x_1^a + c_1(1-h)x_2^a > 0.$$

Thus $c_2(1+a) > 0$ and $c_1(1-h) > 0$. Similarly, $c_2 hk > 0$, $c_1(1+a) > 0$, and $c_2(1-h) > 0$. The Law of Diminishing Returns (absent the word "Marginal") is used to refer to the concavity of the production function. This imposes further constraints on the CES parameters, although not on the Cobb-Douglas parameters.

14.27 As in the previous exercise, $0 < 3/4 < 1$ implies diminishing marginal productivity. $\lambda > 1 \Rightarrow F(\lambda K, \lambda L) = (\lambda K)^{3/4}(\lambda L)^{3/4} = \lambda^{3/2}F(K, L)$. $\lambda^{3/2} > 1$ for $\lambda > 1 \Rightarrow$ increasing returns.

14.28 *a*) $x = 0, y \neq 0 \Rightarrow f(0, y) = 0 \Rightarrow (\partial f/\partial y)(0, y) = 0$.

$x \neq 0, y = 0 \Rightarrow f(x, 0) = 0 \Rightarrow (\partial f/\partial x)(x, 0) = 0$.

If $f \in C^1$, $(\partial f/\partial x)(0, 0) = \lim_{x\to 0}(\partial f/\partial x)(x, 0) = \lim 0 = 0$.

b) $\dfrac{\partial f}{\partial x} = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}; \dfrac{\partial f}{\partial y} = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^4}.$

c) $$\frac{\partial^2 f}{\partial y\,\partial x}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = \lim_{y\to 0}\frac{(\partial f/\partial x)(0,y) - (\partial f/\partial y)(x,0)}{y}$$

$$= \lim_{y\to 0}\frac{-y-0}{y} = -1.$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = \lim_{x\to 0}\frac{(\partial f/\partial y)(x,0) - (\partial f/\partial y)(0,0)}{x}$$

$$= \lim_{x\to 0}\frac{x-0}{x} = 1.$$

So, $\partial^2 f/\partial x\,\partial y \neq \partial^2 f/\partial y\,\partial x$ at $(0,0)$.

f) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$ for $(x,y) \neq (0,0)$.

g) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{x^6 + 9x^6 - 9x^6 - x^6}{(2x^2)^3} = 0.$

h) $\dfrac{\partial^2 f}{\partial x \partial y} \not\to \dfrac{\partial^2 f}{\partial y \partial x}$ as $x \to 0$. f is not C^2.

Chapter 15

15.1a *a*) $\left.\dfrac{\partial f}{\partial y}\right|_{(5,2)} = -3xy^2 + 5y^4\Big|_{(5,2)} = 20 \neq 0.$

b) $\dfrac{\Delta y}{\Delta x} \simeq -\dfrac{\partial f/\partial x}{\partial f/\partial y} = -\dfrac{2}{20} = -0.1;$

$\Delta x = -0.2 \Rightarrow \Delta y \approx 0.02 \Rightarrow y \approx 2.02.$

15.2 Let G be a C^1 function of (x,y) about (x_0,y_0). Suppose $G(x_0,y_0) = c$. If $(\partial G/\partial x)(x_0,y_0) \neq 0$, then there exists a C^1 function $y \longmapsto x(y)$ defined on an interval I about y_0 such that:

$$G(x(y),y) = c \quad \text{for all } y \in I,$$

$$x(y_0) = y_0,$$

$$x'(y_0) = -\frac{(\partial G/\partial y)(x_0,y_0)}{(\partial G/\partial x)(x_0,y_0)}.$$

15.3 $\dfrac{\Delta y}{\Delta x} \simeq -\dfrac{2x-3y}{-3x+3y^2} = \dfrac{1}{15}$ at $x = 4, y = 3$.

$$y(3.7) \approx y(4) + \frac{\Delta y}{\Delta x}(-0.3) = 3 + \frac{1}{15}(-0.3) = 2.98.$$

15.4 Let $G(x, y) = x^2 + 3xy + y^3 - 7$. $G(0, y) = 0$ implies $y = 7^{1/3}$. $DG(0, 7^{1/3}) = 3(-7^{1/3}, 7^{2/3})$. Since the partial derivative with respect to y is not 0, y can be described as a function of x around $x = 0$. Furthermore, $y'(0) = -G_x(0, 7^{1/3})/G_y(0, 7^{1/3}) = 7^{-1/3}$. Combining $y(0) = 7^{1/3}$ and $y'(0) = 7^{-1/3}$ yields

$$\begin{aligned} y(-0.1) &\approx y(0) + 7^{-1/3} \cdot (-0.1) \\ &= 1.86066\cdots \\ y(0.15) &\approx y(0) + 7^{-1/3} \cdot (0.15) \\ &= 1.99134\cdots. \end{aligned}$$

15.5 The implicit calculation is as follows: $G(x, y) = y^2 - 5xy + 4x^2$. $DG(x, y) = (8x - 5y, -5x + 2y)$, which is $(3, -3)$ at $x = y = 1$, so $y'(x) = 1$.

Explicitly, the equation $G(x, y) = G(1, 1)$ has solutions $y = -4x$ and $y = x$. The point $(1, 1)$ is on the second branch, so $y(x) = x$ and the derivative is $y'(x) = 1$.

15.6 *a*) $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3 = 0$; $x_1 = 6, x_2 = 3$ implies $27 + y^3 = 0$. So, $y = -3$.

b) $(\partial F/\partial y)(6, 3, -3) = 27 \neq 0$; so the equation implicitly defines y as a function of x around the point $(6, 3, -3)$.

c) $\dfrac{\partial y}{\partial x_1}(6,3) = -\dfrac{\partial F/\partial x_1}{\partial F/\partial y} = -\dfrac{2x_1}{3y^2} = -\dfrac{12}{27} = -\dfrac{4}{9}$ and

$\dfrac{\partial y}{\partial x_2}(6,3) = -\dfrac{\partial F/\partial x_2}{\partial F/\partial y} = \dfrac{2x_2}{3y^2} = \dfrac{6}{27} = \dfrac{2}{9}$.

d) $y(6.2, 2.9) \approx y(6, 3) - \frac{4}{9}(0.2) + \frac{2}{9}(-0.1) = -\frac{28}{9}$.

15.7 $G(x, p, w) = pf'(x) - w$. The Jacobian of G is $D_{(x,p,w)}G = (pf''(x), f'(x), -1)$. By hypothesis, $D_xG = pf''(x) < 0$, so the Implicit Function

Theorem implies that

$$D_{(p,w)}x(p,w) = \left(\frac{-f'(x)}{pf''(x)}, \frac{1}{pf''(x)}\right).$$

$$x \approx x_0 - \frac{f'(x_0)}{pf''(x_0)}\Delta p + \frac{1}{pf''(x_0)}\Delta w.$$

From the concavity assumption it follows that $D_p x(p,w) > 0$ and $D_w x(p,w) < 0$.

15.8 *a*) $G(x,y,z) = x^3 + 3y^2 + 4xz^2 - 3z^2y$. When $x = y = 1$, $G(1,1,z) = 1$ becomes $z^2 + 3 = 0$, which has no real solution; so z is not defined as a function of x and y.

b) The solution to $G(1,0,z) = 0$ is $z = 0$. At $(1,0,0)$, $D_zG(1,0,0) = 0$, so we cannot conclude from the Implicit Function Theorem that the equation system defines z as a function of x and y. Checking directly, we see that the only possible solutions for z are

$$z = \pm\frac{1}{\sqrt{3}}\sqrt{4x + \frac{3}{4x-3y} - \frac{16x^2}{4x-3y} - \frac{3x^3}{4x-3y} + 3y}.$$

In any neighborhood of $(x,y) = (1,0)$ the argument of the square root takes on negative values, so in fact z is not defined as a function of x and y in some neighborhood of $(1,0)$.

c) The solutions to $G(0.5,0,z) = 0$ are $z = \pm\sqrt{7}/4$. At $(0.5,0,\sqrt{7}/4)$, $\partial G/\partial z = \sqrt{7} \neq 0$, so conclude from the Implicit Function Theorem that the equation system defines z as a function of x and y.

$$\frac{\partial z}{\partial x} = -\frac{\partial G/\partial x}{\partial G/\partial z} = -\frac{3x^2 + 4z^2}{8xz - 3yz} = -\frac{5}{2\sqrt{7}} \qquad \text{at } (0.5, 0, \sqrt{7}/4).$$

$$\frac{\partial z}{\partial y} = -\frac{\partial G/\partial y}{\partial G/\partial z} = \frac{3\sqrt{7}}{16} \qquad \text{at } (0.5, 0, \sqrt{17}/4).$$

15.9 *a*) $3x^2yz + xyz^2 = 30$ at $(1,3,2)$.

$$\frac{\partial x}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial x} = -\frac{3x^2z + xz^2}{6xyz + yz^2} = -\frac{5}{24} \qquad \text{at } (1,3,2).$$

$$x \simeq x_0 + \frac{\partial x}{\partial y}\Delta y + \frac{\partial x}{\partial z}\Delta z = 1 - \frac{5}{24}(0.2) + 0 = \frac{23}{24}.$$

b) The equation to solve is $3yzx^2 + yz^2x - 30 = 0$. The solution has two branches. The branch that gives the correct value of x is

$$x = \frac{-yz^2 + \sqrt{y^3z^4 + 360yz}}{6yz}.$$

$$\begin{aligned}\frac{\partial x}{\partial y} &= \frac{-z^2 + 0.5(y^2z^4 + 360yz)^{-1/2}(2yz^4 + 360z)}{6yz} \\ &\quad - \frac{-yz^2 + (y^2z^4 + 360yz)^{1/2}}{6y^2z} \\ &= -5/24,\end{aligned}$$

after tedious arithmetic.

15.11 *a*) $(f_x, f_y) \cdot (g_x, g_y) = f_x(x,y)g_x(x,y) + f_y(x,y)g_y(x,y) = 0$ for all (x, y).

b) If $f_x = g_y$ and $f_y = -g_x$, $(f_x, f_y) \cdot (g_x, g_y) = 0$.

15.12 *a*) $f_x = 2xe^y$ and $f_y = x^2e^y$, so $\nabla f(2,0) = (4,4)$ and slope $= -f_x/f_y = -1$.

b) $\nabla f(2,0)/\|\nabla f(2,0)\| = (1/\sqrt{2}, 1/\sqrt{2})$.

15.13 *a*) $Q = 60x^{2/3}y^{1/3}$; $x = 64$, $y = 27$.
$60 \cdot 64^{2/3} \cdot 27^{1/3} = 60 \cdot 16 \cdot 3 = 2880$.

b) $\nabla Q(64, 27) = (40x^{-1/3}y^{1/3}, 20x^{2/3}y^{-2/3})\big|_{(64,27)} = (30, 320/9)$. Normalizing the vector to unit length gives $(1/46.52)(30, 35.55\cdots)$.

c) $\Delta y = +1.5$. So Δx must solve $30\Delta x + (320/9)(1.5) = 0$; this implies that $\Delta x = -16/9$.

15.15 In the solution of (24),

$$\frac{\partial Y}{\partial M^s} = \frac{i_1}{c_2(1-b) + i_1c_1} > 0, \qquad \frac{\partial Y}{\partial T} = \frac{-bc_2}{c_2(1-b) + i_1c_1} < 0,$$

$$\frac{\partial r}{\partial M^s} = \frac{-(1-b)}{c_2(1-b) + i_1c_1} < 0, \qquad \frac{\partial r}{\partial T} = \frac{-bc_1}{c_2(1-b) + i_1c_1} < 0.$$

In the solution of (37),

$$\frac{\partial Y}{\partial M^s} = \frac{I'(r)}{D} > 0, \qquad \frac{\partial Y}{\partial T} = -\frac{1}{D}\frac{\partial M}{\partial r}C'(Y^* - T^*) < 0,$$

$$\frac{\partial r}{\partial M^s} = \frac{1 - C'(Y^* - T^*)}{D} < 0, \qquad \frac{\partial r}{\partial T} = \frac{1}{D}\frac{\partial M}{\partial Y}C'(Y^* - T^*) < 0.$$

15.16 $F(x,y) = x^3y - z$ and $G(x,y) = x + y^2 + z^3$. Computing derivatives,

$$\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_{(1,2,1)} = \begin{pmatrix} 3x^2y & x^3 \\ 1 & 2y \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 1 & 4 \end{pmatrix},$$

which is nonsingular.

$$\frac{\partial x}{\partial z} = -\frac{\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix}}{\begin{vmatrix} 6 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{7}{23}, \qquad \frac{\partial y}{\partial z} = -\frac{\begin{vmatrix} 6 & -1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 6 & 1 \\ 1 & 4 \end{vmatrix}} = -\frac{19}{23},$$

so $x \approx 1 + (7/23)(0.1) = 1.03\cdots$, $\quad y \approx 2 - (19/23)(0.1) = 1.917\cdots$.

15.17

$$\frac{\partial x}{\partial u} = -\frac{\det \dfrac{\partial(F,G)}{\partial(u,y)}}{\det \dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\det \begin{pmatrix} 4u & 2y - x \\ 2u & 4y + x \end{pmatrix}}{\det \begin{pmatrix} -y & 2y - x \\ y & 4y + x \end{pmatrix}}$$

$$= -\frac{\det \begin{pmatrix} 4 & 7 \\ 2 & 17 \end{pmatrix}}{\det \begin{pmatrix} -4 & 7 \\ 4 & 17 \end{pmatrix}} = \frac{9}{16}.$$

$$\frac{\partial x}{\partial v} = -\frac{\det \dfrac{\partial(F,G)}{\partial(v,y)}}{\det \dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\det \begin{pmatrix} -2 & 7 \\ -2 & 17 \end{pmatrix}}{\det \begin{pmatrix} -4 & 7 \\ 4 & 17 \end{pmatrix}}$$

$$= -\frac{20}{96} = -\frac{5}{24}.$$

Similarly,

$$\frac{\partial y}{\partial u} = -\frac{24}{96} = -\frac{1}{4} \quad \text{and} \quad \frac{\partial y}{\partial v} = \frac{16}{96} = \frac{1}{6}.$$

$$y \approx y_0 + \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v = 4 - \frac{1}{4}(-0.1) - \frac{1}{6}(-0.1) \quad = 4.008\cdots$$

$$x \approx x_0 + \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v = 1 + \frac{9}{16}(-0.1) - \frac{5}{24}(-0.1) = 0.96458\cdots.$$

15.18 We cannot. Since there are two equations, we must have two, not three, endogenous variables.

15.19

$$\frac{\partial z}{\partial x} = -\frac{\det \dfrac{\partial(F,G)}{\partial(v,x)}}{\det \dfrac{\partial(F,G)}{\partial(v,z)}} = -\frac{\begin{vmatrix} 4y^2v^2 & z^3 \\ yz^2 & z \end{vmatrix}}{\begin{vmatrix} 4y^2v^3 & 3xz^2 \\ yz^2 & x+2yvz \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} 4 & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}} = -\frac{1}{3}.$$

$$\frac{\partial z}{\partial y} = -\frac{\det \dfrac{\partial(F,G)}{\partial(v,y)}}{\det \dfrac{\partial(F,G)}{\partial(v,z)}} = -\frac{\begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}} = -\frac{2}{9}.$$

$$\frac{\partial v}{\partial x} = -\frac{\det \dfrac{\partial(F,G)}{\partial(x,z)}}{\det \dfrac{\partial(F,G)}{\partial(v,z)}} = -\frac{\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}} = 0.$$

$$\frac{\partial v}{\partial y} = -\frac{\det \dfrac{\partial(F,G)}{\partial(y,z)}}{\det \dfrac{\partial(F,G)}{\partial(v,z)}} = -\frac{\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix}} = -\frac{3}{9} = -\frac{1}{3}.$$

15.20 $\dfrac{\partial(F,G)}{\partial(u,v)} = \begin{pmatrix} 4u & 2v \\ 2u & 2v \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & -2 \end{pmatrix}$ has determinant -4. Yes, one can take (u,v) as endogenous.

$$\frac{\partial u}{\partial y} = -\frac{\det \dfrac{\partial(F,G)}{\partial(y,v)}}{\det \dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 7 & -2 \\ 17 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix}} = \frac{20}{4} = 5,$$

$$\frac{\partial v}{\partial y} = -\frac{\det \dfrac{\partial(F,G)}{\partial(u,v)}}{\det \dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} 4 & 7 \\ 2 & 17 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 2 & -2 \end{vmatrix}} = \frac{54}{4} = \frac{27}{2}.$$

$$u \approx u_0 + \frac{\partial u}{\partial y}\,\Delta y = 1 + 5(0.02) = 1.1,$$

$$v \approx v_0 + \frac{\partial v}{\partial y}\,\Delta y = -1 + \frac{27}{2}(0.02) = -0.73.$$

15.21 *a*)

$$\begin{aligned} (2z+y)\,dx + x\quad dy + (2x+1-z^{-1/2})\,dz &= 0 \\ (yz)\,dx + (xz)\,dy + (xy)\qquad\qquad dz &= 0, \end{aligned}$$

or

$$4\,dx + 3\,dy + 6(0.1) = 0$$
$$2\,dx + 3\,dy + 6(0.1) = 0.$$

Subtracting yields $dx = 0$ and $dy = -0.6/3 = -0.2$. So, $x \approx 3$, $y \approx 1.8$.

b) The vector x of partial derivatives of the equation system with respect to y and z is singular.

15.22 The Jacobian is $\begin{pmatrix} 1 & 2 & 1 \\ 6xyz & 3x^2z & 3x^2y \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 12 & 12 & 12 \end{pmatrix}$.

One cannot have x and z endogenous. If x and y are endogenous,

$$\frac{\partial x}{\partial z} = -\frac{\det \dfrac{\partial(F,G)}{\partial(z,y)}}{\det \dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} 1 & 2 \\ 12 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 12 & 12 \end{vmatrix}} = -1,$$

$$\frac{\partial y}{\partial z} = -\frac{\det \dfrac{\partial(F,G)}{\partial(x,z)}}{\det \dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} 1 & 1 \\ 12 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & 12 \end{vmatrix}} = 0.$$

So, $x \approx x_0 + (\partial x/\partial z)\Delta z = 2 - 0.25 = 1.75$, $y = y_0 + (\partial y/\partial z)\Delta z = 1$.

If y and z are endogenous,

$$\frac{\partial z}{\partial x} = -\frac{\det \dfrac{\partial(F,G)}{\partial(y,x)}}{\det \dfrac{\partial(F,G)}{\partial(y,z)}} = -\frac{\begin{vmatrix} 2 & 1 \\ 12 & 12 \end{vmatrix}}{\begin{vmatrix} 2 & 1 \\ 12 & 12 \end{vmatrix}} = -1, \qquad \frac{\partial y}{\partial x} = 0.$$

So, $z \approx z_0 + (\partial z/\partial x)\Delta x = 1 - 1(0.25) = 0.75$, $y = 1$.

15.23 *a*) $\det \dfrac{\partial(F,G)}{\partial(x,y)} = \det \begin{pmatrix} 1 & 2 \\ 6xyz & 3x^2z \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix} = -12 \neq 0.$

b) $\dfrac{\partial x}{\partial z} = -\dfrac{\det \dfrac{\partial(F,G)}{\partial(z,y)}}{\det \dfrac{\partial(F,G)}{\partial(x,y)}} = -\dfrac{\begin{vmatrix} 1 & 2 \\ 12 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 12 & 12 \end{vmatrix}} = -1, \qquad \dfrac{\partial y}{\partial z} = 0.$

So, $x \approx x_0 + (\partial x/\partial z)\Delta z = 2 + (-1)(.2) = 1.8$, $y \approx 1$.

15.24
$$\begin{aligned} a\ln x + b\ln y &= \ln z \\ x + y &= 125. \end{aligned}$$

$x = 25$, $y = 100$, $z = 50$, and $a = b = 0.5$.

b exogenous implies that x and y are endogenous.

$$\begin{aligned} \frac{a}{x}\,dx + \frac{b}{y}\,dy + \ln y\,db &= 0 \\ dx + dy &= 0. \end{aligned}$$

So,

$$\frac{0.5}{0.25}\,dx + \frac{0.5}{100}(-dx) + \ln 100\,(0.004) = 0$$
$$dx = -\frac{\ln 100 \cdot (0.004)}{(0.5/25) - (0.5/100)} = -1.228, \qquad dy = +1.228.$$

So, $x \approx 23.77$ and $y \approx 101.228$.

15.25 Take Y, C, I, and r as endogenous, with G, T, and M^s exogenous. The matrix of endogenous variables is

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -b & 1 & 0 & 0 \\ 0 & 0 & 1 & i_1 \\ c_1 & 0 & 0 & c_2 \end{pmatrix}$$

which has determinant $-[(1-b)c_2 + i_1 c_1] > 0$.

15.26 Taking u_i to be ln gives a log transformation of the Cobb-Douglas functional form.

15.27 Solve (45) for x_1 and (46) for y_1, and plug these expressions into (42); (44) results.

15.28 Suppose $x_1 > y_1$. Then, $u_1'(x_1) < u_1'(y_1)$ since $u_1'' < 0$, and $\alpha/(1-\alpha) > p$ by (41). Then, by (43) $u_2'(x_2) < u_2'(y_2)$ and $x_2 > y_2$. We cannot have $x_1 > y_1$, $x_2 > y_2$, and $x_1 + x_2 = y_1 + y_2 = 1$. A similar argument shows that $x_1 < y_1$ leads to a contradiction. We conclude $x_1 = y_1$ and $x_2 = y_2$. Then, $u_1'(x_1) = u_1'(y_1)$ and $p = \alpha/(1-\alpha)$. Finally, $p = \alpha/(1-\alpha)$, and $q = e_1 = 1$ and $x_1 = y_1$ in (42) imply $x_1 = y_1 = \alpha$; (45) and (46) imply $x_2 = y_2 = 1 - \alpha$.

15.30 $R_1 = R_2 = 1$ and $D = 1$ in (53).

$$\frac{\partial x_1}{\partial e_2} = \frac{\partial x_2}{\partial e_2} = 0, \quad \frac{\partial y_1}{\partial e_2} = \frac{\partial p}{\partial e_2} = 1, \quad \frac{\partial y_2}{\partial e_2} = 0.$$

15.31

$$\begin{pmatrix} -\frac{\alpha}{1-\alpha}u_1''(\alpha) & \frac{\alpha}{1-\alpha}u_1''(\alpha) & -u_1'(\alpha) \\ \frac{\alpha}{1-\alpha} & 1 & 1-\alpha \\ \frac{\alpha}{1-\alpha}u_2''(1-\alpha) & -\frac{\alpha}{1-\alpha}u_2''(1-\alpha) & -u_2'(1-\alpha) \end{pmatrix} \begin{pmatrix} dx_2 \\ dy_2 \\ dp \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{1-\alpha}u_1''(\alpha)de_1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} -\frac{\alpha u_1''(\alpha)}{u_1'(\alpha)} & \frac{\alpha u_1''(\alpha)}{u_1'(\alpha)} & -(1-\alpha) \\ \alpha & 1-\alpha & (1-\alpha)^2 \\ \frac{(1-\alpha)u_2''(1-\alpha)}{u_2'(1-\alpha)} & -\frac{(1-\alpha)u_2''(1-\alpha)}{u_2'(1-\alpha)} & -\frac{(1-\alpha)^2}{\alpha} \end{pmatrix} \begin{pmatrix} dx_2 \\ dy_2 \\ dp \end{pmatrix} = \begin{pmatrix} -\frac{\alpha u_1''(\alpha)}{u_1'(\alpha)}de_1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} r_1(\alpha) & -r_1(\alpha) & -(1-\alpha) \\ \alpha & 1-\alpha & (1-\alpha)^2 \\ -r_2(1-\alpha) & r_2(1-\alpha) & -\frac{(1-\alpha)^2}{\alpha} \end{pmatrix} \begin{pmatrix} dx_2 \\ dy_2 \\ dp \end{pmatrix} = \begin{pmatrix} r_1(\alpha)de_1 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of the matrix is still $D = R_1(1-\alpha)^2/\alpha + R_2(1-\alpha)$. For example,

$$\frac{\partial p}{\partial e_1} = \frac{R_1 R_2}{D} \quad \text{and} \quad \frac{\partial y_2}{\partial e_1} = \frac{R_1(1-R_2)(1-\alpha)^2}{D}.$$

15.33 The system is $x^2 - y^2 = a$, $2xy = b$.

$$x = \frac{b}{2y} \quad \text{implies} \quad x^2 = \frac{b^2}{4y^2};$$

so $\qquad b^2/4y^2 - y^2 = a \quad \text{and} \quad 4y^4 + 4ay^2 - b^2 = 0.$

Therefore,

$$y^2 = -\frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = -\frac{a \pm \sqrt{a^2 + b^2}}{2};$$

so, $\qquad y = \pm\sqrt{\frac{(a^2+b^2)^{1/2} - a}{2}},$

$$x^2 = a + y^2 = a - \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + b^2},$$

$$x = \pm\sqrt{\frac{(a^2+b^2)^{1/2}+a}{2}}.$$

Since xy must equal $b/2$, there are two preimages for each $(a,b) \neq (0,0)$.

15.34 If F is one-to-one, then for any $\mathbf{b}$ there is at most one $\mathbf{x}$ with $F(\mathbf{x}) = \mathbf{b}$, so there is no other $\mathbf{z} \neq \mathbf{x}$ with $F(\mathbf{z}) = F(\mathbf{x})$. Thus, $F(\mathbf{z}) = F(\mathbf{x})$ imples $\mathbf{z} = \mathbf{x}$.

15.35 Let $F : \mathbf{R} \to \mathbf{R}$ be a C^1 function with $F(x^*) = y^*$. If $F'(x^*) \neq 0$, there is an interval $(x^* - r, x^* + r)$ and an interval $(y^* - s_0, y^* + s_1)$ such that F is a one-to-one map from $(x^* - r_1, x^* + r)$ to $(y^* - s_0, y^* + s_1)$.

The natural map $F^{-1} : (y^* - s_0, y^* + s_1) \to (x^* - r_1, x^* + r)$ is C^1, and

$$\frac{dF^{-1}}{dy}(y^*) = \frac{1}{(dF/dx)(x^*)}.$$

15.36 $DF(x,y) = \begin{pmatrix} 1 & e^y \\ -e^{-x} & 1 \end{pmatrix}$ has determinant $1 + e^{y-x}$, which is positive for all (x,y). By Theorem 15.9, F is everywhere locally invertible.

15.37 $f'(t) = (-\sin t, \cos t)$ is never $(0,0)$. By Theorem 15.8*b*, f is locally one-to-one. But $f(0) = f(2\pi) = f(4\pi)$; that is, f is not globally one-to-one.

15.38 $DF(x,y) = \begin{pmatrix} -e^y \sin x & e^y \cos x \\ e^y \cos x & e^y \sin x \end{pmatrix}$ with determinant $= -e^y(\sin^2 x + \cos^2 x) = -e^y \neq 0$ for all (x,y). By Theorem 15.9, F is everywhere locally one-to-one and onto. But $F(x + 2\pi n, y) = F(x,y)$ for all (x,y) and all integers n, so F is not globally one-to-one.

15.39 $f'(x) = e^x \neq 0$. By Theorem 15.9, f is locally one-to-one and onto. But f itself can take on no negative values.

Chapter 16

16.1 *a*) The Leading Principal Minors (LPMs) are $2, 1$; positive definite.

b) The LPMs are $-3, -1$; indefinite.

c) The LPMs are $-3, 2$; negative definite.

d) The LPMs are $2, 0$. All principal minors ≥ 0; positive semidefinite.

e) The LPMs are $1, 0, -25$; indefinite.

f) The LPMs are $-1, 0, 0$. All 1×1 principal minors ≤ 0. All 2×2 principal minors ≥ 0; negative semidefinite.

g) The first three LPMs are $1, 2, -10$; indefinite.

16.2 Suppose $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. For $\mathbf{x} = \mathbf{e}_i = (0, \dots, 0, 1, 0, \dots)$, $\mathbf{e}_i^T A\mathbf{e}_i = a_{ii} > 0$. If A is positive semidefinite, we must have $a_{ii} = \mathbf{e}_i^T A\mathbf{e}_i \geq 0$. Similarly, if A is negative definite, then each $a_{ii} < 0$, and if A is negative semidefinite, then each $a_{ii} \leq 0$. Parts *b*, *e*, and *g* in the previous exercise show that these conditions are not sufficient.

16.3 For A positive definite, $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. Fix indices $i_1, \dots, i_k$, say $1, \dots, k$. Then, for all nonzero $\mathbf{x}$ such that $x_{k+1} = \cdots = x_n = 0$, $\mathbf{x}^T A\mathbf{x} > 0$. This means that the top leftmost $k \times k$ submatrix of A must be positive definite.

16.4 How many different ways can one choose k items from a list of n?

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

16.5
$$\begin{aligned}
&(x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= a_{11}x_1^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 + a_{22}x_2^2 + a_{33}x_3^2 \\
&= a_{11}\left(x_1^2 + \frac{2a_{12}}{a_{11}}x_1x_2 + \frac{2a_{13}}{a_{11}}x_1x_3\right) + 2a_{23}x_2x_3 + a_{22}x_2^2 + a_{33}x_3^2 \\
&= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 - \frac{a_{12}^2}{a_{11}}x_2^2 - \frac{a_{13}^2}{a_{11}}x_3^2 - \frac{2a_{12}a_{13}}{a_{11}}x_2x_3 \\
&\quad + 2a_{23}x_2x_3 + a_{22}x_2^2 + a_{33}x_3^2 \\
&= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + \left(\frac{a_{11}a_{22} - a_{12}^2}{a_{11}}\right)x_2^2 \\
&\quad + \frac{2(a_{11}a_{23} - a_{12}a_{13})}{a_{11}}x_2x_3 + \left(\frac{a_{11}a_{33} - a_{13}^2}{a_{11}}\right)x_3^2 \\
&= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 \\
&\quad + \left(\frac{a_{11}a_{22} - a_{12}^2}{a_{11}}\right)\left[x_2^2 + 2\left(\frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2}\right)x_2x_3 + \left(\frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2}\right)^2 x_3^2\right] \\
&\quad + \left[\frac{a_{11}a_{33} - a_{13}^2}{a_{11}} - \frac{(a_{11}a_{23} - a_{12}a_{13})^2}{(a_{11}a_{22} - a_{12}^2)a_{11}}\right]x_3^2
\end{aligned}$$

$$= a_{11}\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + \left(\frac{a_{11}a_{22} - a_{12}^2}{a_{11}}\right)\left[x_2 + \left(\frac{a_{11}a_{23} - a_{12}a_{13}}{a_{11}a_{22} - a_{12}^2}\right)x_3\right]^2$$

$$+ \frac{1}{a_{11}a_{22} - a_{12}^2}\left[(a_{11}a_{33} - a_{13}^2)(a_{11}a_{22} - a_{12}^2) - \frac{(a_{11}a_{23} - a_{12}a_{13})^2}{a_{11}}\right]x_3^2$$

$$= |A_1|\left(x_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{13}}{a_{11}}x_3\right)^2 + \frac{|A_2|}{|A_1|}\left(x_2 + \frac{a_{11}a_{23} - a_{12}a_{13}}{|A_2|}x_3\right)^2 + \frac{|A_3|}{|A_2|}x_3^2.$$

If $|A_1|$, $|A_2|$, and $|A_3| > 0$, $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$.

If $|A_1| < 0$, $|A_2| > 0$, and $|A_3| < 0$, $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.

16.6 *a*) $A = \begin{pmatrix} 0 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \vdots & 1 & 1 \\ 1 & \vdots & 1 & -1 \end{pmatrix}$ $n = 2$, $m = 1$, $|A| = 2 > 0$; negative definite.

b) $A = \begin{pmatrix} 0 & \vdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \vdots & 4 & 1 \\ 1 & \vdots & 1 & -1 \end{pmatrix}$ $n = 2$, $m = 1$, $|A| = -1 < 0$; positive definite.

c) $A = \begin{pmatrix} 0 & 0 & \vdots & 1 & 1 & 1 \\ 0 & 0 & \vdots & 1 & 1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & -1 & 2 \\ 1 & 1 & \vdots & -1 & 1 & 0 \\ 1 & -1 & \vdots & 2 & 0 & -1 \end{pmatrix}$ $n = 3$, $m = 2$, $|A| = 16 > 0$; positive definite.

d) $A = \begin{pmatrix} 0 & 0 & \vdots & 1 & 1 & 1 \\ 0 & 0 & \vdots & 1 & 1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \vdots & 1 & -1 & 2 \\ 1 & 1 & \vdots & -1 & 1 & 0 \\ 1 & -1 & \vdots & 2 & 0 & 1 \end{pmatrix}$ $n = 3$, $m = 2$, $|A| = 16 > 0$; positive definite.

e) $A = \begin{pmatrix} 0 & \vdots & 1 & 1 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \vdots & 1 & 2 & -3 \\ 1 & \vdots & 2 & 0 & 0 \\ -1 & \vdots & -3 & 0 & -1 \end{pmatrix}$ $n = 3$, $m = 1$, $|A_4| = 4$, $|A_3| = 3$; indefinite.

16.7 Write the leading principal minors of H_{n+m} as $H_1, H_2, \ldots, H_{n+m}$. According to Theorem 16.4, we look at the signs of the last $n-m$ minors:

$$H_{n+m}, H_{n+m-1}, H_{n+m-2}, \ldots, H_{n+m-(n-m)+1} = H_{2m+1}.$$

The patterns are for n variables and m constraints:

	Negative Definite	*Positive Definite*
H_{n+m}	$(-1)^n$	$(-1)^m$
H_{n+m-1}	$(-1)^{n-1}$	$(-1)^m$
$\vdots$	$\vdots$	$\vdots$
H_{2m+2}	$(-1)^{m+2}$	$(-1)^m$
H_{2m+1}	$(-1)^{m+1}$	$(-1)^m$

We see that this is the same pattern as:

Negative definite iff $\text{sign} H_{2m+1} = \text{sign}(-1)^{m+1}$ and signs alternate.

Positive definite iff $\text{sign} H_{2m+1} = \text{sign}(-1)^m$ and all signs the same.

16.8 To change the matrices in Example 8 to those in Example 7, first move the rows up and then the columns across. For every movement of the rows there is a corresponding movement of the columns. At the end there are an even number of row and column interchanges, so the determinant is unchanged.

16.9 Maximize $Q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_2 + 2a_{13}x_1x_3 + 2a_{23}x_{23}$ subject to $b_1x_1 + b_2x_2 + b_3x_3 = 0$.

$$x_1 = -\left(\frac{b_2}{b_1}x_2 + \frac{b_3}{b_1}x_3\right).$$

$$\begin{aligned}
Q &= a_{11}\left(\frac{b_2}{b_1}x_2 + \frac{b_3}{b_1}x_3\right)^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{23}x_2x_3 \\
&\quad - 2a_{12}\left(\frac{b_2}{b_1}x_2 + \frac{b_3}{b_1}x_3\right)x_2 - 2a_{13}\left(\frac{b_2}{b_1}x_2 + \frac{b_3}{b_1}x_3\right)x_3 \\
&= (x_2 \quad x_3)\begin{pmatrix} \frac{b_1^2a_{22} - 2a_{12}b_1b_2 + a_{11}b_2^2}{b_1^2} & \frac{a_{11}b_2b_3 - a_{12}b_3 - a_{13}b_2 + a_{23}b_1}{b_1} \\ \frac{a_{11}b_2b_3 - a_{12}b_3 - a_{13}b_2 + a_{23}b_1}{b_1} & \frac{a_{33}b_1^2 - 2a_{13}b_1b_3 + a_{11}b_3^2}{b_1^2} \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}.
\end{aligned}$$

The first leading principal minor A_1 of this 2×2 matrix is

$$\frac{1}{b_1^2}(a_{22}b_1^2 - 2a_{12}b_1b_2 + a_{11}b_2^2) = -\frac{1}{b_1^2} \cdot \det\begin{pmatrix} 0 & b_1 & b_2 \\ b_1 & a_{11} & a_{12} \\ b_2 & a_{12} & a_{22} \end{pmatrix}.$$

The second leading principal minor A_2 is

$$\frac{1}{b_1^2}\Bigg[(a_{22}b_1^2 - 2a_{12}b_1b_2 + a_{11}b_2^2)(a_{33}b_1^2 - 2a_{13}b_1b_3 + a_{11}b_3^2)\cdot\frac{1}{b_1^2}$$

$$- (a_{11}b_2b_3 - a_{12}b_3 - a_{13}b_2 + a_{23}b_1)^2\Bigg]$$

$$= -\frac{1}{b_1^2}\cdot\det\begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & a_{11} & a_{12} & a_{13} \\ b_2 & a_{12} & a_{22} & a_{23} \\ b_3 & a_{13} & a_{23} & a_{33}\end{pmatrix}.$$

For a constrained max (negative definite), we want $A_1 < 0$ and $A_2 > 0$. This is the same as $H_3 > 0$ and $H_4 < 0$ for the original 4×4 bordered Hessian, as Theorem 16.4 states.

16.10

$$\begin{pmatrix} I_n & \mathbf{0}_n \\ (A_n^{-1}\mathbf{a})^T & 1\end{pmatrix}\begin{pmatrix} A_n & \mathbf{0}_n \\ \mathbf{0}_n^T & d\end{pmatrix}\begin{pmatrix} I_n & A_n^{-1}\mathbf{a} \\ \mathbf{0}_n^T & 1\end{pmatrix}$$

$$= \begin{pmatrix} I_nA_n & \mathbf{0}_n \\ (A_N^{-1}\mathbf{a})^TA_n & d\end{pmatrix}\begin{pmatrix} I_n & A_n^{-1}\mathbf{a} \\ \mathbf{0}_n^T & 1\end{pmatrix}$$

$$= \begin{pmatrix} A_n & A_nA_n^{-1}\mathbf{a} \\ \mathbf{a}_1^TA_n^{-1}A_n & (A_n^{-1}\mathbf{a})^TA_nA_n^{-1}\mathbf{a} + d\end{pmatrix}$$

$$= \begin{pmatrix} A_n & \mathbf{a} \\ \mathbf{a}^T & \mathbf{a}^TA_n^{-1}\mathbf{a} + d\end{pmatrix}$$

$$= \begin{pmatrix} A_n & \mathbf{a} \\ \mathbf{a}^T & a_{n+1,n+1}\end{pmatrix}$$

because $d = a_{n+1,n+1} - \mathbf{a}^TA_n^{-1}\mathbf{a}$.

16.11 Again, we use induction on n. We suppose that the theorem is true for $n \times n$ matrices and prove it for $(n+1)\times(n+1)$ matrices. Let $A_1, A_2, \ldots, A_{n+1}$ be the leading principal submatrices of $(n+1)\times(n+1)$ matrix A. Suppose $|A_1| < 0$, $|A_2| > 0, \ldots$. By the inductive hypothesis, A_n is negative definite on $\mathbf{R^n}$. Partition A as in (17) and write it as $A = Q^TBQ$ as in (18). For any $(n+1)$ vector $\mathbf{z} = \begin{pmatrix}\mathbf{x} \\ x_{n+1}\end{pmatrix}$, $\mathbf{z}^TB\mathbf{z} = \mathbf{x}^TA_n\mathbf{x} + dx_{n+1}^2$. Since $d = \det A_{n+1}/\det A_n$ by (19), $d < 0$. So, $\mathbf{z}^TB\mathbf{z} < 0$ for all $\mathbf{z} \neq \mathbf{0}$ in $\mathbf{R^{n+1}}$. Therefore, B is negative definite. By Lemma 16.2, A is negative definite too.

To prove the converse, assume it true for $n \times n$ matrices: A negative definite $\Longrightarrow \det A_j$ has sign $(-1)^j$ for $j = 1, \ldots, n$. Since $0 > \begin{pmatrix}\mathbf{x} \\ 0\end{pmatrix}^T A \begin{pmatrix}\mathbf{x} \\ 0\end{pmatrix} =$

$\mathbf{x}^T A_n \mathbf{x}$, A_n is negative definite and $\det A_n$ has sign $(-1)^n$. As in (19), $\det A_{n+1} = \det A_n \cdot d$. As in (20), $d < 0$ if A is negative definite. Therefore, $\det A_{n+1}$ has sign $(-1)^{n+1}$.

17.1 *a*) $f(x, y) = xy^2 + x^3y - xy$.

$f_x = y^2 + 3x^2y - y = y(y + 3x^2 - 1)$ and $f_y = 2xy + x^3 - x = x(2y + x^2 - 1)$.

Case 1: $x = 0, y = 0$.

Case 2: $x = 0, y + 3x^2 - 1 = 0$; that is, $(x, y) = (0, 1)$.

Case 3: If $y = 0$ and $2y + x^2 - 1 = 0$, then $x^2 = 1$ and so $(x, y) = (1, 0)$ or $(-1, 0)$.

Case 4: If $y + 3x^2 - 1 = 0$ and $2y + x^2 - 1 = 0$, then $3x^2 = 1 - y$ and $3x^2 = 3 - 6y$, so $1 - y = 3 - 6y$. Therefore, $y = 2/5$ and $x^2 = (1 - y)/3 = 1/5$; so $x = \pm 1/\sqrt{5}$.

Six critical points: $(0, 0)$, $(0, 1)$, $(1, 0)$, $(-1, 0)$, $(+1/\sqrt{5}, 2/5)$, $(-1/\sqrt{5}, 2/5)$.

Hessian: $H = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}$.

At $(0, 0)$, $H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which is indefinite, so $(0, 0)$ is a saddle.

At $(0, 1)$, $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is indefinite, so $(0, 1)$ is a saddle.

At $(1, 0)$, $H = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ which is indefinite, so $(1, 0)$ is a saddle.

At $(-1, 0)$, $H = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$ which is indefinite, so $(-1, 0)$ is a saddle.

At $\left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$, $H = \begin{pmatrix} 12/5\sqrt{5} & 2/5 \\ 2/5 & 2/\sqrt{5} \end{pmatrix}$ which is positive definite, so $\left(\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$ is a local min.

At $\left(-\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$, $H = \begin{pmatrix} -12/5\sqrt{5} & 2/5 \\ 2/5 & -2/\sqrt{5} \end{pmatrix}$ which is negative definite, so $\left(-\frac{1}{\sqrt{5}}, \frac{2}{5}\right)$ is a local max.

b) $f(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5.$

$f_x = 2x - 6y + 10$ and $f_y = -6x + 4y + 2.$

Critical point: $(13/7, 16/7)$.

Hessian: $\begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}$ which is indefinite, so the point is a saddle.

c) $f(x,y) = x^4 + x^2 - 6xy + 3y^2.$

$f_x = 4x^3 + 2x - 6y$ and $f_y = -6x + 6y.$

$f_y = 0$ implies $x = y$. Then, $f_x = 0$ implies $4x^3 = 4x$. Therefore $x^3 = x$, so $x = \pm 1, 0$.

Three critical points: $(-1,-1), (0,0), (1,1)$.

Hessian: $H = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$

$H(-1,-1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}$ which is positive definite, so $(-1,-1)$ is a local min.

$H(1,1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}$ which is positive definite, so $(1,1)$ is a local min.

$H(0,0) = \begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix}$ which is indefinite, so $(0,0)$ is a saddle.

d) $f(x,y) = 3x^4 + 3x^2y - y^3.$

$f_x = 12x^3 + 6xy$ and $f_y = 3x^2 - 3y^2.$

$f_y = 0$ implies $x = \pm y$. $f_x(x,x) = 12x^3 + 6x^2 = 6x^2(2x+1)$ implies $x = 0, -1/2$.

$f_x(+x,-x) = 12x^3 - 6x^2 = 6x^2(2x-1)$ implies $x = 0, 1/2$.

Critical points: $(0,0), (-1/2,-1/2), (1/2,-1/2)$.

Hessian: $H = \begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}.$

$H(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which is indeterminate, but $f(0,y) = -y^3$, so $(0,0)$ is neither a max nor a min.

$H(-1/2,-1/2) = \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix}$ which is positive definite, so $(-1/2,-1/2)$ is a local min.

$H(1/2,-1/2) = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}$ which is positive definite, so $(1/2,-1/2)$ is a local min.

17.2 *a*) $f_x = 2x + 6y + 6$, $f_y = 6x + 2y - 3z + 17$, and $f_z = -3y + 8z - 2$.

$f_x = f_y = f_z = 0$ implies $(x, y, z) = \left(-369/137, -14/137, 29/137\right)$.

Hessian: $H = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$ which is indefinite, so the point is a saddle.

b) $f_x = 2x\,e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2x)\,e^{-(x^2+y^2+z^2)}$,
$f_y = 4y\,e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2y)\,e^{-(x^2+y^2+z^2)}$,
$f_z = 6z\,e^{-(x^2+y^2+z^2)} + (x^2 + 2y^2 + 3z^2)(-2z)\,e^{-(x^2+y^2+z^2)}$.
$f_x = 0$ implies $2x(x^2 + 2y^2 + 3z^2 - 1) = 0$.
$f_y = 0$ implies $2y(x^2 + 2y^2 + 3z^2 - 2) = 0$.
$f_z = 0$ implies $2z(x^2 + 2y^2 + 3z^2 - 3) = 0$.

Case 1: $x = y = z = 0$.

Case 2: $x^2 + 2y^2 + 3z^2 = 1$ and $y = z = 0$ implies $x = \pm 1$. So, $(1, 0, 0)$ and $(-1, 0, 0)$.

Case 3: $x^2 + 2y^2 + 3z^2 = 2$ and $x = z = 0$ implies $y = \pm 1$. So, $(0, 1, 0)$ and $(0, -1, 0)$.

Case 4: $x^2 + 2y^2 + 3z^2 = 3$ and $x = y = 0$ implies $z = \pm 1$. So, $(0, 0, 1)$ and $(0, 0, -1)$.

Hessian:

$$\mathrm{H} = \begin{pmatrix} 2A[(1-B) - 2x^2(2-B)] & -4xyA(3-B) & -4xzA(4-B) \\ -4xyA(3-B) & 2A[(2-B) - 2y^2(4-B)] & -4yzA(5-B) \\ -4xzA(4-B) & -4yzA(5-B) & 2A[(3-B) - 2z^2(6-B)] \end{pmatrix},$$

where $A = \exp(-(x^2 + y^2 + z^2))$ and $B = x^2 + 2y^2 + 3z^2$.

$H(0,0,0) = A\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ which is positive definite, so $(0, 0, 0)$ is a local min,

$H(\pm 1,0,0) = e^{-1}\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ which is indefinite, so $(\pm 1, 0, 0)$ are saddles,

$H(0,\pm 1,0) = e^{-1}\begin{pmatrix} -2 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ which is indefinite, so $(0, \pm 1, 0)$ are saddles,

$H(0,0,\pm 1) = e^{-1}\begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -12 \end{pmatrix}$ which is negative definite, so $(0, 0, \pm 1)$ are local maxes.

17.3 *a*) At local max $\left(-1/\sqrt{5}, 2/5\right)$, $f\left(-1/\sqrt{5}, 2/5\right) \approx 0.07155\cdots$.

At local min $\left(1/\sqrt{5}, 2/5\right)$, $f\left(1/\sqrt{5}, 2/5\right) \approx -0.07155$.

But $f(1,1) = 1$ and $f(-1,1) = -1$. So, neither is global.

b) $(0,0,0)$ is a global minimum.
$(0,0,\pm 1)$ are global maxima.

c) $H(x,y) = \begin{pmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{pmatrix}$ has determinant equal to $72x^2 - 24$, which is ≥ 0 for $x^2 > 1/3$.

So, f is convex on the open sets $\{(x,y) : x \geq 1/\sqrt{3}\} = A$ and $\{(x,y) : x < -1/\sqrt{3}\} = B$.

$f(1,1) = -1$ is a global min on A, and $f(-1,-1) = -1$ is a global max on B.

17.4 $\Pi(x,y) = x^{1/4}y^{1/4} - 4(x+y)$.

$\Pi_x = \frac{1}{4}x^{-3/4}y^{1/4} - 4 = 0, \qquad \Pi_y = \frac{1}{4}x^{1/4}y^{-3/4} - 4 = 0.$

$x^{-3/4}y^{1/4}/x^{1/4}y^{-3/4} = 4/4 = 1$, or $y = x = 1/256$.

$$H = \begin{pmatrix} -\frac{3}{16}x^{-7/4}y^{1/4} & \frac{1}{16}x^{-3/4}y^{-3/4} \\ \frac{1}{16}x^{-3/4}y^{-3/4} & -\frac{3}{16}x^{1/4}y^{-7/4} \end{pmatrix}.$$

$$\left.\begin{aligned} |H_1| &= -\frac{3}{16}x^{-7/4}y^{1/4} < 0, \\ |H_2| &= \frac{8}{256}x^{-6/4}y^{-6/4} > 0 \end{aligned}\right. \qquad \text{for all } (x,y) \in \mathbf{R^2}_{++}.$$

17.5 $\Pi(x,y) = px^ay^b - wx - ry$.

$\Pi_x = pax^{a-1}y^b - w = 0$, $\Pi_y = pbx^ay^{b-1} - r = 0$.

$$\frac{pax^{a-1}y^b}{pbx^ay^{b-1}} = \frac{w}{r} \quad \text{implies} \quad \frac{y}{x} = \frac{bw}{ar}; \qquad \text{so} \qquad y = \frac{bw}{ar}x.$$

Plug this into $\Pi_x = 0$ to solve for x (and then y).

$$H = \begin{pmatrix} pa(a-1)x^{a-2}y^b & pabx^{a-1}y^{b-1} \\ pabx^{a-1}y^{b-1} & pb(b-1)x^ay^{b-2} \end{pmatrix}.$$

$|H_1| = pa(a-1)x^{a-2}y^b < 0$ if and only if $0 < a < 1$. $|H_2| = p^2[a(a-1)b(b-1) - a^2b^2]x^{2a-2}y^{2b-2} > 0$ if and only if $ab\big[(a-1)(b-$

$1) - ab\big] = ab\big[1 - (a+b)\big] > 0$. If $0 < a < 1, 0 < b < 1$, and $a + b < 1$, the solution to the first order conditions is a global max.

17.6 Maximize $F(q_1, q_2) = q_1(10 - q_1) + q_2(16 - q_2) - (10 + (q_1 + q_2)^2)$

$$= 10q_1 - q_1^2 + 16q_2 - q_2^2 - 10 - (q_1 + q_2)^2.$$

$$F_{q_1} = 10 - 2q_1 - 2(q_1 + q_2) = 0 \quad \text{or} \quad 4q_1 + 2q_2 = 10,$$
$$F_{q_2} = 16 - 2q_2 - 2(q_1 + q_2) = 0 \quad \text{or} \quad 2q_1 + 4q_2 = 16.$$

$q_2 = 11/3$ and $q_1 = 2/3$; $p_2 = 16 - 11/3 = 37/3$ and $p_1 = 10 - 2/3 = 28/3$.

$\Pi = (2/3)(28/3) + (11/3)(37/3) - 10 - (13/3)^2 = 204/9 = 68/3 = 22\,\frac{2}{3}$.

17.7 The market demand function is $q = \begin{cases} 0 & \text{if } p \geq 16, \\ 16 - p & \text{if } 10 \leq p \leq 16, \\ 26 - 2p & \text{if } 0 \leq p \leq 10. \end{cases}$

Work first with $q = 26 - 2p$ or $p = \frac{1}{2}(26 - q)$ on the interval $0 \leq p \leq 10$.

$\Pi = \frac{1}{2}q(26 - q) - (10 + q^2) = 13q - \frac{3}{2}q^2 - 10$.

$\Pi' = 13 - 3q = 0$ implies $q = \frac{13}{3} = 4\,\frac{1}{3}$, so $p = \frac{1}{2}(21\,\frac{2}{3}) > 10$.

This answer is not compatible with $0 \leq p \leq 10$.

Work with $q = 16 - p$ or $p = 16 - q$ on the interval $10 \leq p \leq 16$.

$\Pi = q(16 - q) - 10 - q^2 = 16q - 2q^2$.

$\Pi' = 16 - 4q = 0$ implies $q = 4$, so $p = 12 \in [10, 16]$.

$\Pi = 12 \cdot 4 - 10 - 4^2 = 48 - 10 - 16 = 22$.

17.8 $\sum x_i = 15, \sum y_i = 15, \sum x_i y_i = 65, \sum x_i^2 = 71, n = 4$.

$$m^* = \frac{4 \cdot 65 - 15 \cdot 15}{4 \cdot 71 - 15^2} = \frac{35}{59}, \qquad b^* = \frac{71 \cdot 15 - 15 \cdot 65}{71 \cdot 4 - 15 \cdot 15} = \frac{90}{59}.$$

So,

$$y = \frac{35}{59}x + \frac{90}{59}.$$

17.9 *a*) $0 \leq (a - b)^2 = a^2 - 2ab + b^2$ implies $2ab \leq a^2 + b^2$.

b)
$$\begin{aligned}(x_1 + \cdots + x_n)^2 &= x_1^2 + \cdots + x_n^2 + \sum_{i<j} 2x_i x_j \\ &\geq x_1^2 + \cdots + x_n^2 + \sum_{i<j}(x_i^2 + x_j^2) \\ &\geq x_1^2 + \cdots + x_n^2 + (n-1)(x_1^2 + \cdots + x_n^2) \\ &\geq n(x_1^2 + \cdots + x_n^2).\end{aligned}$$

c) The Hessian for (12) is $\begin{pmatrix} \sum x_i^2 & \sum x_i \\ \sum x_i & n \end{pmatrix}$.

$H_1 = \sum x_i^2 \geq 0$. $H_2 = n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2 \geq 0$, by part *b*. H is positive definite, so (m^*, b^*) is a global min.

17.10 $H_2 = 0$ implies each $2x_i x_j = x_i^2 + x_j^2$, so each $x_i = x_j$. All points lie on the same vertical line.

17.11
$$S = (Ax_1 + By_1 + C - z_1)^2 + \cdots + (Ax_n + By_n + C - z_n)^2.$$
$$\frac{\partial S}{\partial A} = \sum_i 2(Ax_i + By_i + C - z_i) \cdot x_i = 0$$
$$\frac{\partial S}{\partial B} = \sum_i 2(Ax_i + By_i + C - z_i) \cdot y_i = 0$$
$$\frac{\partial S}{\partial C} = \sum_i 2(Ax_i + By_i + C - z_i) \cdot 1 = 0.$$
$$\begin{aligned}\left(\sum_i x_i^2\right) A + \left(\sum x_i y_i\right) B + \left(\sum x_i\right) C &= \sum_i x_i z_i \\ \left(\sum_i x_i y_i\right) A + \left(\sum y_i^2\right) B + \left(\sum y_i\right) C &= \sum_i y_i z_i \\ \left(\sum x_i\right) A + \left(\sum y_i\right) B + n\,C &= \sum z_i.\end{aligned}$$

17.12 $m^* = \dfrac{\overline{xy} - \overline{x} \cdot \overline{y}}{\overline{x^2} - \overline{x}^2}, \qquad b^* = \dfrac{\overline{x^2}\,\overline{y} - \overline{x}\,\overline{xy}}{\overline{x^2} - \overline{x}^2}.$

Chapter 18

18.1 $(0, +\sqrt{3})$ is a local minimum, and $(0, -\sqrt{3})$ is a local maximum. Neither is a global maximum or minimum. See figure.

18.2 The problem is

$$\begin{aligned} \max \quad & x^2 + y^2 \\ \text{subject to} \quad & x^2 + xy + y^2 = 3. \end{aligned}$$

The Lagrangian is $L = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$. The first order conditions are

$$\begin{aligned} L_x &= 2x - 2\lambda x - \lambda y = 0 \\ L_y &= 2y - \lambda x - 2\lambda y = 0 \\ L_\lambda &= -(x^2 + xy + y^2 - 3) = 0. \end{aligned}$$

There are four solutions:

$$(x, y, \lambda) = \begin{cases} (-\sqrt{3}, \sqrt{3}, 2) \\ (\sqrt{3}, -\sqrt{3}, 2) \\ (1, 1, 2/3) \\ (-1, -1, 2/3) \end{cases}$$

The NDCQ holds at all four solutions. Clearly the first two solutions are maxima and the second two are minima.

18.3

$$\begin{aligned} \max \quad & (2 - x)^2 + (1 - y)^2 \\ \text{subject to} \quad & x^2 - y = 0. \end{aligned}$$

The Lagrangian is

$$L = (2 - x)^2 + (1 - y)^2 - \lambda(y - x^2).$$

The first order conditions are

$$\begin{aligned} L_x &= -2(2 - x) + 2\lambda x = 0 \\ L_y &= -2(1 - y) - \lambda = 0 \\ L_\lambda &= y - x^2 = 0. \end{aligned}$$

Solving for λ and substituting gives $2x^3 - x - 2 = 0$. Thus $x \approx 1.165$, and so $y = x^2 \approx 1.357$. Finally $\nabla(y - x^2) = (-2x, 1) \neq (0,0)$ so NDCQ holds.

18.4 The location and type of the critical points are independent of $k > 0$, so assume without loss of generality that $k = 1$.

$$\begin{aligned} \max \quad & x_1^a x_2^{1-a} \\ \text{subject to} \quad & (p_1 x_1 + p_2 x_2 - I) = 0. \end{aligned}$$

The Lagrangian is

$$L = x_1^a x_2^{1-a} - \lambda(p_1 x_1 + p_2 x_2 - I).$$

The first order conditions are

$$\begin{aligned} L_{x_1} &= a x_1^{a-1} x_2^{1-a} - \lambda p_1 = 0 \\ L_{x_2} &= (1-a) x_1^a x_2^{-a} - \lambda p_2 = 0 \\ L_\lambda &= p_1 x_1 + p_2 x_2 - I = 0. \end{aligned}$$

The solution is

$$x_1 = \frac{aI}{p_1} \qquad x_2 = \frac{(1-a)I}{p_2}.$$

Since the constraint is linear, NDCQ holds.

18.5

$$\begin{aligned} \min \quad & x^2 + y^2 + z^2 \\ \text{subject to} \quad & 3x + y + z = 5 \\ & x + y + z = 1. \end{aligned}$$

The Lagrangian is

$$L = x^2 + y^2 + z^2 - \lambda_1(3x + y + z - 5) - \lambda_2(x + yz - 1).$$

The first order conditions are

$$\begin{aligned} L_x &= 2x - 3\lambda_1 - \lambda_2 = 0 \\ L_y &= 2y - \lambda_1 - \lambda_2 = 0 \\ L_z &= 2z - \lambda_1 - \lambda_2 = 0 \end{aligned}$$

$$L_{\lambda_1} = 3x + y + z - 5 = 0$$
$$L_{\lambda_2} = x + y + z - 1 = 0.$$

This linear system of five equations in five unknowns has a unique solution: $(2, -1/2, -1/2)$. The Jacobian of the constraints is $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which has rank 2, and so the NDCQ holds.

18.6 Substitute $y = 0$ into all the equations.

$$\begin{aligned} \max \quad (\min) \quad & x + z^2 \\ \text{subject to} \quad & x^2 + z^2 = 1. \end{aligned}$$

The Lagrangian is

$$L = x + z^2 - \lambda(x^2 + z^2 - 1)$$

and the first order conditions are

$$\begin{aligned} L_x &= 1 - 2\lambda x = 0 \\ L_z &= 2z - \lambda 2z = (1 - \lambda)2z = 0 \\ L_\lambda &= x^2 + z^2 - 1 = 0. \end{aligned}$$

There are four solutions:

$$(x, y, z, \lambda) = \begin{cases} (1/2, 0, \sqrt{3}/2, 1) \\ (1/2, 0, -\sqrt{3}/2, 1) \\ (1, 0, 0, 1/2) \\ (-1, 0, 0, -1/2) \end{cases}.$$

A check shows that the first two correspond to local maxima with a value of $5/4$, the third to a critical point with a value of 1, and the last to a local minimum with a value of -1.

18.7 Substitute the constraint $xz = 3$ into the objective function:

$$\begin{aligned} \max \quad & 3 + yz \\ \text{subject to} \quad & y^2 + z^2 = 1. \end{aligned}$$

The Lagrangian is

$$L = 3 + yz - \lambda(y^2 + z^2 - 1).$$

The first order conditions are

$$\begin{aligned} L_y &= z - 2\lambda y = 0 \\ L_z &= y - 2\lambda z = 0 \\ L_\lambda &= y^2 + z^2 - 1 = 0. \end{aligned}$$

The solutions are the four (y, z) pairs such that $y = \pm 1/\sqrt{2}$ and $z = \pm 1/\sqrt{2}$. For a maximum, y and z must have the same sign, so the solutions are $(3\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$ and $(-3\sqrt{2}, -1/\sqrt{2}, -1/\sqrt{2})$. The value of the maximand in each case is $7/2$, and NDCQ holds.

18.8 Suppose there are n variables and m constraints. The Jacobian of the constraints is

$$\begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} \end{pmatrix}.$$

This matrix can have rank m only if $m \leq n$.

18.9 A simple substitution makes this much easier. Let $X = x^2$, $Y = y^2$ and $Z = z^2$. The maximization problem is now

$$\begin{aligned} \max \quad & XYZ \\ \text{subject to} \quad & X + Y + Z = c^2 \end{aligned}$$

together with inequality constraints $X \geq 0$, $Y \geq 0$ and $Z \geq 0$, which we will ignore for the moment. This is a familiar problem. The Lagrangian is

$$L = XYZ - \lambda(X + Y + Z - c^2).$$

The first order conditions are

$$\begin{aligned} L_X &= YZ - \lambda = 0 \\ L_Y &= XZ - \lambda = 0 \\ L_Z &= XY - \lambda = 0 \\ L_\lambda &= X + Y + Z - c^2 = 0 \end{aligned}$$

The solution is

$$X = Y = Z = \frac{c^2}{3}$$

and we see that the inequality constraints are never binding. Going back to the original problem, the solution is

$$x = y = z = \frac{c}{\sqrt{3}}.$$

The value of the objective function at its maximum is $(c^2/3)^3$, so

$$(c^2/3)^3 \geq \left(\frac{x^2 + y^2 + z^2}{3}\right)^3$$

for all (x, y, z) in the constraint set. Ranging over all values of c,

$$(x^2y^2z^2)^{1/3} \leq \left(\frac{x^2 + y^2 + z^2}{3}\right).$$

18.10 The Lagrangian is

$$L = x^2 + y^2 - \lambda(2x + y - 2) + \nu_1 x + \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= 2x - 2\lambda + \nu_1 = 0 \\ L_y &= 2y - \lambda + \nu_2 = 0 \\ &\lambda(2x + y - 2) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

Solve by enumerating cases. Is there a solution with $x = 0$? If so, then $\nu_1 = 2\lambda$ and $y = 2$. If $y = 2$, then $\nu_2 = 0$, so $\lambda = 4$ and $\nu_1 = 8$, which is consistent with the FOCs. This is a solution. Is there a solution with $y = 0$? If so, then $\nu_2 = \lambda$ and $x = 1$. If $x = 1$, then $\nu_1 = 0$, so $\lambda = 1$ and $\nu_1 = 1$, which is consistent with the FOCs. This is a solution. $x = y = \nu_1 = \nu_2 = \lambda = 0$ is a solution. If neither x nor y are 0, then $\nu_1 = \nu_2 = 0$. Then $x = 4/5$ and so $y = 2/5$. This is consistent. Among these four points the global maximum occurs at $(0, 2)$ and the value of f is 4.

18.11 The Lagrangian is

$$L = 2y^2 - x - \lambda(x^2 + y^2 - 1) + \nu_1 x + \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= -1 - 2\lambda x + \nu_1 = 0 \\ L_y &= 4y - 2\lambda y + \nu_2 = 0 \\ &\lambda(x^2 + y^2 - 1) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

The only solution to the first order conditions is

$$x = 0 \quad y = 1 \quad \nu_1 = 1 \quad \nu_2 = 0 \quad \lambda = 2$$

so the optimum is $x = 0$ and $y = 1$, and the value of f is 2.

18.12 *a*) The problem is

$$\begin{aligned} \max \quad & xyz + z \\ \text{subject to} \quad & x^2 + y^2 + z \leq 6 \\ & x \geq 0, y \geq 0, z \geq 0. \end{aligned}$$

The first order conditions are

$$\begin{aligned} L_x &= yz - 2\lambda x + \nu_1 = 0 \\ L_y &= xz - 2\lambda y + \nu_2 = 0 \\ L_z &= xy + 1 - \lambda + \nu_3 = 0 \\ &\lambda(x^2 + y^2 + z - 6) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\nu_3 z = 0 \\ &\nu_1 \geq 0, \quad \nu_2 \geq 0, \quad \nu_3 \geq 0, \quad \lambda \geq 0. \end{aligned}$$

b) There is no solution to the first order conditions with $\lambda = 0$, because $\lambda = 0$ implies $xy + 1 + \nu_3 = 0$, and this equation has no nonnegative solution. Since $\lambda > 0$, the constraint must be binding at every solution to the first order conditions.

c) If $x = 0$, then $\nu_2 = 2\lambda y$. Since $\lambda > 0$, $\nu_2 y = 2\lambda y^2 = 0$ implies $y = 0$ and $\nu_2 = 0$. Therefore $z = 6$, $\nu_1 = 0$, $\nu_3 = \lambda - 1$, and $\lambda \geq 1$.

d) If $x > 0$, then $\nu_1 = 0$ so $yz = 2\lambda x$. Since $\lambda > 0$, y and z are both positive, so ν_2 and ν_3 are both 0. Thus four equations in four unknowns are the first order conditions for L_x, L_y and L_z and the constraint equality:

$$yz - 2\lambda x = 0$$
$$xz - 2\lambda y = 0$$
$$xy + 1 - \lambda = 0$$
$$x^2 + y^2 + z = 6.$$

Solving for λ and substituting gives the equation system

$$-2x - 2x^2y + yz = 0$$
$$-2y - 2xy^2 + xz = 0$$
$$x^2 + y^2 + z = 6.$$

e) This equation system has only one solution that satisfies all the nonnegativity constraints: $x = 1$, $y = 1$, and $z = 4$. Then $\lambda = 2$.

18.13 The Lagrangian is

$$L = U(x_1, x_2) - \lambda(p_1x_1 + p_2x_2 - I) + \nu_1x_1 + \nu_2x_2.$$

The first order conditions include

$$L_{x_1} = U_{x_1} - \lambda p_1 + \nu_1 = 0$$
$$L_{x_2} = U_{x_2} - \lambda p_2 + \nu_2 = 0$$
$$\lambda(p_1x_1 + p_2x_2 - I) = 0.$$

If $\lambda = 0$, then $U_{x_i} + \nu_i = 0$ has to have a nonnegative solution, and this is impossible if at every $(x_1, x_2) \geq 0$ at least one of the U_{x_i} exceeds 0. Thus $\lambda > 0$, so $p_1x_1 + p_2x_2 = I$.

At most one nonnegativity constraint can bind because the origin cannot solve the first order conditions (since $\lambda > 0$). Thus there are at most two binding constraints: The budget constraint and one of the inequality constraints. If one inequality constraint binds, the budget-constraint row of the matrix of derivatives of the binding constraints has two nonzero entries, and the row corresponding to the inequality constraint has one 0 and one 1, so the matrix is nonsingular. If only the budget constraint binds, the positivity of any one price guarantees that the matrix (now 1 by 2) has full rank. In either case NDCQ holds.

18.14 Assume that the p_i are all positive. The Lagrangian is

$$L = xyz - \lambda(p_1 x + p_2 y + p_3 z - I) + \lambda_2 x + \lambda_3 y + \lambda_4 z.$$

The first order conditions are

$$\begin{aligned}
L_x &= yz - \lambda_1 p_1 + \lambda_2 = 0\\
L_y &= xz - \lambda_1 p_2 + \lambda_3 = 0\\
L_z &= xy - \lambda_1 p_3 + \lambda_4 = 0\\
&\lambda_1(p_1 x + p_2 y + p_3 z - I) = 0\\
&\lambda_2 x = 0\\
&\lambda_3 y = 0\\
&\lambda_4 z = 0\\
&\lambda_1 \geq 0 \quad \text{for all } i.
\end{aligned}$$

One set of solutions has any two of x, y and z equal to 0 and the remaining variable taking on any value in the unit interval. For these solutions $\lambda_i = 0$ for all i and the value of the objective function is 0. These are the only solutions with $\lambda_1 = 0$.

If $\lambda_1 > 0$, then the principal constraint binds, and so at least one of x, y and z must be positive. Suppose, for instance, that $x > 0$, then $\lambda_2 = 0$ so $yz = \lambda p_1 \neq 0$, and both y and z are positive. Thus $\lambda_2 = \lambda_3 = \lambda_4 = 0$. From here it is easy to show that the only solution with $\lambda_1 > 0$ has $x = I/p_1$, $y = I/p_2$ and $z = I/p_3$. At this point the objective function is positive, and so it must be the maximum. Finally, $\lambda_1 = I^2/9p_1p_2p_3$.

18.15 The Lagrangian is

$$L = 3xy - x^3 - \mu(2x - y + 5) - \lambda(-5x - 2y + 37) + \nu_1 x + \nu_2 y.$$

The first order conditions are

$$\begin{aligned}
L_x &= 3y - 3x^2 - 2\mu + 5\lambda + \nu_1 = 0\\
L_y &= 3x + \mu + 2\lambda + \nu_2 = 0\\
&\lambda(-5x - 2y + 37) = 0\\
&\nu_1 x = 0\\
&\nu_2 y = 0\\
&2x - y + 5 = 0\\
&\lambda, \quad \nu_1, \quad \nu_2, \quad x, \quad y, \quad 5x + 2y - 37 \quad \text{all} \quad \geq 0.
\end{aligned}$$

If either x or y are 0, the equality constraint and the inequality constraint cannot simultaneously be satisfied. Thus $\nu_1 = \nu_2 = 0$.

If the inequality constraint is binding, then solving it and the inequality constraint give $x = 3$ and $y = 11$. From the first order conditions $L_x = 0$ and $L_y = 0$ it can be seen that $\lambda < 0$, so there can be no such solution to the first order conditions. Consequently $\lambda = 0$.

After substituting all the known multipliers, the remaining variables can be solved for: $x = 5$, $y = 15$ and $\mu = -15$.

18.16 The Jacobian matrix of the binding constraints at $(1, 0)$ is

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

which is non-singular.

18.17 The Lagrangian is

$$L = x^2 - 2y - \lambda(-x^2 - y^2 + 1) - \nu_1 x - \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= 2x + 2\lambda x - \nu_1 = 0 \\ L_y &= -2 + 2\lambda y - \nu_2 = 0 \\ &\lambda(x^2 + y^2 - 1) = 0 \\ &\nu_1 x = 0 \\ &\nu_2 y = 0 \\ &\lambda, \quad \nu_1, \quad \nu_2, \quad x, \quad y, \quad 1 - x^2 - y^2 \geq 0. \end{aligned}$$

If $\lambda = 0$ or $y = 0$, then $\nu_2 < 0$, so in any solution $\lambda > 0$, $y > 0$ and $\nu_2 = 0$. Multiplying the $L_x = 0$ condition by x gives $2x^2(1 + \lambda) = 0$, so $x = 0$, and consequently $y = 1$. Then $\lambda = 1$ and $\nu_1 = 0$.

18.18 The Lagrangian is

$$L = 2x^2 + 2y^2 - 2xy - 9y - \lambda_1(-4x - 3y + 10) - \lambda_2(y - 4x^2 + 2) - \nu_1 x - \nu_2 y.$$

The first order conditions are

$$\begin{aligned} L_x &= 4x - 2y + 4\lambda_1 + 8\lambda_2 x - \lambda_3 = 0 \\ L_y &= 4y - 2x - 9 + 3\lambda_1 - \lambda_2 - \lambda_4 = 0 \end{aligned}$$

$$\lambda_1(-4x - 3y + 10) = 0$$
$$\lambda_2(y - 4x^2 + 2) = 0$$
$$\lambda_3 x = 0$$
$$\lambda_4 y = 0.$$

Suppose that $x = y = 0$. The first order conditions imply $\lambda_1 = \lambda_2 = 0$, and so $\lambda_3 = 0$ and $\lambda_4 = -9$, which contradicts the requirement that $\lambda_4 \geq 0$. Next suppose that $x > y = 0$. Then $\lambda_3 = 0$, and so $x + \lambda_1 + 2\lambda_2 x = 0$. Hence at least one of λ_1 and λ_2 must be negative, which contradicts the requirement that all multipliers be nonnegative. Next suppose that $y > x = 0$. Then $\lambda_4 = \lambda_2 = 0$. Conclude from the first equation that $\lambda_1 > 0$ (else $\lambda_3 < 0$, which is a contradiction). Thus $y = 10/3$ and the second equation implies that $\lambda_1 < 0$, which is a contradiction. So the only solutions have $x > 0$ and $y > 0$.

Suppose $x, y > 0$. Then $\lambda_3 = \lambda_4 = 0$. Suppose $\lambda_1 > 0$ and $\lambda_2 > 0$. Then $4x + 3y = 10$ and $y - 4x^2 = 2$. Thus $3x^2 + x - 4 = 0$, and so $x = 1$ and $y = 2$. Then $\lambda_1 + 2\lambda_2 = 0$. Since the multipliers are nonnegative, $\lambda_1 = \lambda_2 = 0$ which is a contradiction. Suppose $\lambda_1 = 0$ and $\lambda_2 > 0$. Then the first order conditions lead to the equation $-16x^2 + 2x + 17 + \lambda_2 = 0$. This equation has no nonnegative root, which contradicts a nonnegativity constraint. Suppose that $\lambda_1 = \lambda_2 = 0$. Then $x = 3/2$ and $y = 3$ is the only solution to the first two first order conditions. But this violates the constraint $y - 4x^2 \geq -2$.

Suppose that $\lambda_1 > 0$ and $\lambda_2 = 0$. The first order conditions have a solution with $x = 28/37$, $y = 86/37$ and $\lambda_1 = 15/37$.

18.19 Let (x^*, y^*) be a minimizer of f on the set $g(x, y) \geq b$ where $g(x^*, y^*) = b$. Then $\nabla g(x^*, y^*)$ points onto the constraint set since it points into region of higher g-values. Since (x^*, y^*) minimizes f on the constraint set, f increases as one moves into the constraint set. So, $\nabla f(x^*, y^*)$ points into the constraint set, too. $\nabla g(x^*, y^*)$ and $\nabla f(x^*, y^*)$ point along the same line and both point into the constraint set. So, they point in the same directions.

More rigorously, if NDCQ holds, $\nabla g(x^*, y^*) \cdot h > 0$ implies $\nabla f(x^*, y^*) \cdot h \geq 0$. Write $\nabla f(x^*, y^*) = \nabla g(x^*, y^*) + w$, where $w \cdot \nabla g(x^*, y^*) = 0$. If $w \neq \mathbf{0}$, then the system $\nabla g(x^*, y^*) \cdot h = 0$ and $w \cdot h < 0$ has a solution, so $\nabla f(x^*, y^*) \cdot h < 0$. Perturbing a solution gives an h such that $\nabla g(x^*, y^*) \cdot h > 0$ and $\nabla f(x^*, y^*) \cdot h < 0$, which is a contradiction. Consequently $w = \mathbf{0}$. If $\lambda < 0$, again $\nabla g(x^*, y^*) \cdot h > 0$ and again implies $\nabla f(x^*, y^*) \cdot h < 0$, which is a contradiction.

18.20 For simplicity of notation, suppose that the first h inequality constraints (the g_i constraints) and the first j nonnegativity constraints are the binding constraints. Then NDCQ requires that

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_j} & \frac{\partial g_1}{\partial x_{j+1}} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & & & & \vdots \\ \frac{\partial g_h}{\partial x_1} & \cdots & \frac{\partial g_h}{\partial x_j} & \frac{\partial g_h}{\partial x_{j+1}} & \cdots & \frac{\partial g_h}{\partial x_n} \\ 1 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix}$$

has rank $h+j$. The last j rows are linearly independent. The entire collection of rows will be linearly independent if and only if the upper right submatrix has full rank. This is the condition of Theorem 18.7.

18.21

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_1(\mathbf{x}) \geq b_1, \\ & \vdots \\ & g_k(\mathbf{x}) \geq b_k \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

The Lagrangian is

$$L = f(\mathbf{x}) - \lambda_1(g_1(\mathbf{x}) - b_1) - \cdots - \lambda_k(g_k(\mathbf{x}) - b_k).$$

Equations (40) becomes for $i = 1, \ldots, n$,

$$\frac{\partial L}{\partial x_i} \geq 0, \qquad x_i \frac{\partial L}{\partial x_i} = 0$$

and for $j = 1, \ldots, k$,

$$\frac{\partial L}{\partial \lambda_j} \leq 0, \qquad \lambda_j \frac{\partial L}{\partial \lambda_j} = 0.$$

Chapter 19

19.1 *a*)

$$\begin{aligned} L &= x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3.3) \\ L_x &= 2x - \lambda(2x + y) = 0 \\ L_y &= 2y - \lambda(x + 2y) = 0 \end{aligned}$$

$$\lambda = \frac{2x}{(2x+y)} = \frac{2y}{(x+2y)}.$$

Consequently $x = \pm y$. The solutions to the first order conditions are $(x, y) = \pm(\sqrt{1.1}, -\sqrt{1.1})$ with $\lambda = 2/3$ and $(x, y) = \pm(\sqrt{3.3}, -\sqrt{3.3})$ with $\lambda = 2$. Min square of distance is $1.1 + 1.1 = 2.2$; Max square of distance is $3.3 + 3.3 = 6.6$.

b) Using Theorem 19.1 and Exercise 18.2,

$$\begin{aligned} V(3.3) &\approx V(3) + \lambda^*(0.3) \\ &= 2 + (2/3)0.3 \quad = 2.2 \\ \text{or} \quad V(3.3) &= 6 + 2(.3) \qquad = 6.6, \end{aligned}$$

the exact same answers as in *a*.

19.2 *a*) $V(.8) \approx V(1) + \lambda^*(-.2) = 5/4 + 1(-.2) = 1.05$.

b) Using the method of Exercise 18.6,

$$\begin{aligned} &L_x = 1 - 2\lambda x = 0, \\ &L_y = 1 - 2\lambda y - \mu = 0, \\ &L_z = (1-\lambda)2z = 0. \\ &\lambda = 1 \Longrightarrow x = 1/2, y = 0, z^2 = 0.8 - 0.25 = 0.55; z = \sqrt{0.55} \\ &x^2 + y + z^2 = 0.5 + 0 + 0.55 = 1.05, \quad \text{as in } a. \end{aligned}$$

19.3 *a*)

$$\begin{aligned} L &= 50x^{1/2}y^2 - \lambda(x + y - 80). \\ L_x &= 25x^{-1/2}y^2 - \lambda = 0, \\ L_y &= 100x^{1/2}y - \lambda = 0. \end{aligned}$$

Dividing,

$$\begin{aligned} 4x/y = 1 \quad &\Longrightarrow \quad y = 4x \\ &\Longrightarrow \quad x = 16, \;\; y = 64, \;\; Q = 819{,}200, \;\; \lambda = 25{,}600. \end{aligned}$$

b) $Q^*(79) \approx Q^*(80) + \lambda^*(-1) = 819{,}200 - 25{,}600 = 793{,}600$.

c) As in *a*, $y = 4x$. $5x = 79 \Longrightarrow x = 15.8$, $y = 63.2$.

$Q = 50 \cdot (15.8)^{1/2}(63.2)^2 = 793{,}839.5$.

$\Delta Q = -25{,}360.5$.

19.4 $f^*(0.9) \approx f^*(1) + \lambda^*(-0.1) = 2 + 2(-0.1) = 1.8.$

19.5 $f^*(6.2) \approx f^*(6) + \lambda^*(0.2) = 8 + 2(0.2) = 8.4.$

19.6 See equations (19.30) and (19.31), or Theorem 19.9.

19.7
$$\frac{d}{da_j} f(\mathbf{x}^*(\mathbf{a})) = \sum_i \frac{\partial f}{\partial x_i}(\mathbf{x}^*(\mathbf{a}))\frac{\partial x_i^*}{\partial a_j}(\mathbf{a})$$
$$= \sum_i \sum_k \mu_k(\mathbf{a})\frac{\partial h_k}{\partial x_i}(\mathbf{x}^*(\mathbf{a}))\frac{\partial x_i^*}{\partial a_j}(\mathbf{a})$$

(because $0 = \frac{\partial L}{\partial x_i} = \frac{\partial f}{\partial x_i} - \sum_k \mu_k \frac{\partial h_k}{\partial x_i}$)

$$= \sum_k \mu_k(\mathbf{a}) \sum_i \frac{\partial h_k}{\partial x_i}(\mathbf{x}^*(\mathbf{a}))\frac{\partial x_i}{\partial a_j}(\mathbf{a}) = \mu_j(\mathbf{a}),$$

(because $h_k(\mathbf{x}(\mathbf{a})) \equiv a_k \implies \sum_i \frac{\partial h_k}{\partial x_i}\frac{\partial x_i}{\partial a_j} = \delta_{kj}$, where $\delta_{kj} = 1$ if $k = j$ and $= 0$ if $k \neq j$).

19.8 Consider the problem of maximizing $\mathbf{x} \mapsto f(\mathbf{x})$ subject to: $g_1(\mathbf{x}) \leq b_1, \dots, g_k(\mathbf{x}) \leq b_k$, and $h_1(\mathbf{x}) = a_1, \dots, h_m(\mathbf{x}) = a_m$. Let $\mathbf{x}^*(\mathbf{b}, \mathbf{a})$ denote the solution of this problem with corresponding multipliers $\lambda_1^*(\mathbf{b}, \mathbf{a}), \dots, \lambda_k^*(\mathbf{b}, \mathbf{a})$ for the inequality constraints and $\mu_1^*(\mathbf{b}, \mathbf{a}), \dots, \mu_m^*(\mathbf{b}, \mathbf{a})$ for the inequality constraints. Suppose $\mathbf{x}^*, \lambda_i^*, \mu_j^*$ are C^1 functions of $(\mathbf{b}, \mathbf{a})$ around $(\mathbf{b}^*, \mathbf{a}^*)$ and that NDCQ holds there. Then,

$$\lambda_j^*(\mathbf{b}, \mathbf{a}) = \frac{\partial}{\partial b_j} f(\mathbf{x}^*(\mathbf{b}^*, \mathbf{a}^*)) \quad \text{and} \quad \mu_i^*(\mathbf{b}, \mathbf{a}) = \frac{\partial}{\partial a_i} f(\mathbf{x}^*(\mathbf{b}^*, \mathbf{a}^*)).$$

19.9 Consider the problem of minimizing $f(\mathbf{x})$ subject to:

$$g_1(\mathbf{x}) \geq b_1, \dots, g_k(\mathbf{x}) \geq b_k,\ h_1(\mathbf{x}) = a_1, \dots, h_m(\mathbf{x}) = a_m.$$

Let $\mathbf{x}^*(\mathbf{b}, \mathbf{a})$ denote the minimizing $\mathbf{x}$ with corresponding multipliers $\lambda_1^*(\mathbf{b}, \mathbf{a}), \dots, \lambda_k^*(\mathbf{b}, \mathbf{a})$ for the inequality constraints and $\mu_1^*(\mathbf{b}, \mathbf{a}), \dots, \mu_m^*(\mathbf{b}.\mathbf{a})$ for the equality constraints, all for the Lagrangian

$$L = f(\mathbf{x}) - \sum_i \lambda_i \big(g_i(\mathbf{x}) - b_i\big) - \sum_j \mu_j (h_j(\mathbf{x}) - a_j).$$

Then,

$$\frac{\partial}{\partial b_i} f\big(\mathbf{x}^*(\mathbf{b}, \mathbf{a})\big) = \frac{\partial L}{\partial b_i}(\mathbf{x}^*, \mathbf{b}^*, \mathbf{a}^*, \lambda^*, \mu^*)$$

and

$$\frac{\partial}{\partial a_j} f(\mathbf{x}^*(\mathbf{b}, \mathbf{a})) = \frac{\partial L}{\partial a_j}(\mathbf{x}^*, \mathbf{b}^*, \mathbf{a}^*, \lambda^*, \mu^*).$$

19.10 Form $L(\mathbf{x}, a, \mu) = f(\mathbf{x}) - \sum \mu_j h_j(\mathbf{x}, a)$. Let $\big(\mathbf{x}^*(a^*), \mu^*(a^*)\big)$ denote the maximizer and its multiplier when the parameter $a = a^*$. $a \longmapsto h_j(\mathbf{x}^*(a), a)$ is the zero function.

$$\begin{aligned} f(\mathbf{x}^*(a)) &= f(\mathbf{x}^*(a)) - \sum \mu_j^*(a) h_j(\mathbf{x}^*(a); a) \\ &= L(\mathbf{x}^*(a), \mu^*(a); a) \text{ for all } a. \\ \frac{d}{da} f(\mathbf{x}^*(a)) &= \frac{d}{da} L(\mathbf{x}^*, \mu^*(a); a) \\ &= \sum_i \frac{\partial L}{\partial x_i}(\mathbf{x}^*(a), \mu^*(a), a) \frac{dx_i^*}{da}(a) \\ &\quad + \sum_j \frac{\partial L}{\partial \mu_j}(\mathbf{x}^*(a), \mu^*(a), a) \frac{d\mu_j^*}{da}(a) + \frac{\partial L}{\partial a}(\mathbf{x}^*(a), \mu^*(a), a) \\ &= 0 + \sum_j (-h_j(\mathbf{x}^*(a), a)) \frac{d\mu_j^*}{da}(a) + \frac{\partial L}{\partial a}(\mathbf{x}^*(a), \mu^*(a), a) \\ &= \frac{\partial L}{\partial a}(\mathbf{x}^*, \mu^*(a), a). \end{aligned}$$

19.11

$$\begin{aligned} \max \quad & f(x) \\ \text{subject to} \quad & h_1(\mathbf{x}) = a_1, \dots, h_k(\mathbf{x}) = a_k. \end{aligned}$$

Suppose the constraint qualification holds and that the solution $\mathbf{x}^*$ depends on $\mathbf{a} = (a_1, \dots, a_k)$. The Lagrangian is $L(\mathbf{x}, \lambda; \mathbf{a}) = f(\mathbf{x}) - \sum \lambda_i (h_i(\mathbf{x}) - a_i)$. By Theorem 19.5,

$$\frac{\partial}{\partial a_j} f(\mathbf{x}^*(\mathbf{a})) = \frac{\partial L}{\partial a_j}(\mathbf{x}^*(\mathbf{a}), \lambda^*(\mathbf{a}); \mathbf{a}).$$

It follows from the above formula for L that $\frac{\partial L}{\partial a_j} = \lambda_j$. So, $\frac{\partial}{\partial a_j} f(\mathbf{x}^*(\mathbf{a})) = \lambda_j(\mathbf{a})$. But this is the conclusion of Theorem 19.1.

19.12

$$\begin{aligned} \max \quad & x^2 + y^2 \\ \text{subject to} \quad & x^2 + xy + .9y^2 = 3. \end{aligned}$$

Write the constraint as $x^2 + xy + by^2 = 3$.

$$L = x^2 + y^2 - \lambda(x^2 + xy + by^2 - 3).$$
$$\frac{\partial L}{\partial b} = -\lambda y^2.$$

For $b = 1$, the max is at $(x, y, \lambda) = (\pm\sqrt{3}, \mp\sqrt{3}, 2)$ with $f^* = 6$; and the min is at $(x, y, \lambda) = (\pm 1, \pm 1, 2/3)$ with $f^* = 2$.

For $b = 0.9$, $f^* \approx 6 + (-2 \cdot 3) \cdot (-.1) = 6.6$ at the max and $f^* \approx 2 + (-(2/3) \cdot 1) \cdot (-.1) = 2 + (2/30) = 2\,\frac{1}{15}$ at the min. If we want actual distance and not distance squared, we would take the square roots of these numbers and obtain 2.569 and 1.438, respectively.

19.13
$$\begin{aligned} \max \quad & x^2 + x + ay^2 \\ \text{subject to} \quad & 2x + 2y \le 1, x \ge 0, y \ge 0. \end{aligned}$$

The Lagrangian is

$$L = x^2 + x + ay^2 - \lambda_1(2x + 2y - 1) + \lambda_2 x + \lambda_3 y.$$

For $a = 4$, $x^* = 0$, $y^* = 0.5$, $\lambda_1^* = 2$, $\lambda_2^* = 3$, $\lambda_3^* = 0$, and $f^* = 1$. At these values,

$$\frac{\partial L}{\partial a} = y^2 = 0.5^2 = 0.25$$

so

$$f^*(4.1) \approx f^*(4) + \frac{\partial L}{\partial a} \cdot \Delta a = 1 + 0.25(0.1) = 1.025.$$

19.14 18.2): $H = \begin{pmatrix} 0 & 2x + y & x + 2y \\ 2x + y & 2 - 2\lambda & -\lambda \\ x + 2y & -\lambda & 2 - 2\lambda \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 \\ 3 & 2/3 & -2/3 \\ 3 & -2/3 & 2/3 \end{pmatrix}$

at the minimizer $(1, 1, 2/3)$. $\det H = -24 < 0$, the SOC for a constrained min. At the max, $(\sqrt{3}, -\sqrt{3}, 2)$,

$$\det H = \det \begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{3} & -2 & -2 \\ -\sqrt{3} & -2 & -2 \end{pmatrix} = +24 > 0,$$

the SOC for a constrained max.

18.3): $\det H = \det \begin{pmatrix} 0 & 2x & -1 \\ 2x & 2+2\lambda & 0 \\ -1 & 0 & 2 \end{pmatrix} = -(2 + 2\lambda + 8x^2) < 0,$

the SOC for a constrained min.

18.5): $\begin{pmatrix} 0 & 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 2 \end{pmatrix}$

has positive determinant, the SOC for a constrained min when there are 3 variables and 2 constraints.

19.15 If $\frac{\partial h}{\partial x}(x^*, y^*) \neq 0$, then C_h can be written as $x = \psi(y)$ around (x^*, y^*); i.e., $h\big(\psi(y), y\big) \equiv 0$ for all y near y^* and

$$\psi'(y) = -\frac{\partial h}{\partial y}(\psi(y), y) \bigg/ \frac{\partial h}{\partial x}(\psi(y), y).$$

Let $F(y) = f\big(\psi(y), y\big)$. So, $F'(y^*) = 0$ and $F''(y^*) < 0$ implies that y^* is a strict local max of F and that (x^*, y^*) is a strict local constrained max of f.

$$\begin{aligned} F'(y^*) &= \frac{\partial f}{\partial x}(\psi(y^*), y^*)\psi'(y^*) + \frac{\partial f}{\partial y}(\psi(y^*), y^*) \\ &= \frac{\partial f}{\partial x}(\psi(y^*), y^*)\psi'(y^*) + \frac{\partial f}{\partial y}(\psi(y^*), y^*) \\ &\quad - \mu^* \left[\frac{\partial h}{\partial x}(\psi(y^*), y^*)\psi'(y^*) + \frac{\partial h}{\partial y}(\psi(y^*), y^*) \right] \\ &= \frac{\partial L}{\partial x}(x^*, y^*)\psi'(y^*) + \frac{\partial L}{\partial y}(x^*, y^*) \\ &= 0, \quad \text{since } \frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0 \text{ by FOCs.} \end{aligned}$$

As in the proof of Theorem 19.7,

$$\begin{aligned} F''(y^*) &= L_{xx}(\psi')^2 + 2L_{xy}\psi' + L_{yy} + L_x\psi'' \\ &= L_{xx}\left(\frac{-h_y}{h_x}\right)^2 + 2L_{xy}\left(\frac{-h_y}{h_x}\right) + L_{yy} + 0 \\ &= \frac{1}{h_x^2}\left(h_y^2 L_{xx} - 2h_x h_y L_{xy} + L_{yy} h_x^2\right), \end{aligned}$$

which is negative by hypothesis *b* of the Theorem.

19.16 The proof is basically the same, except that hypothesis b now implies that $F''(x^*) > 0$. The conditions $F'(x^*) = 0$ and $F''(x^*) > 0$ imply that x^* is a strict local min of F and that (x^*, y^*) is a strict local constrained min of f.

19.18

$$\max \quad x_1^2 x_2$$
$$\text{subject to} \quad 2x_1^2 + x_2^2 = a.$$

The Lagrangian is $L = x_1^2 x_2 - \lambda(2x_1^2 + x_2^2 - a)$. The first order conditions are:

$$L_{x_1} = 2x_1x_2 - 4\lambda x_1 = 0$$
$$L_{x_2} = x_1^2 - 2\lambda x_2 = 0$$
$$L_\lambda = -(2x_1^2 + x_2^2 - a) = 0.$$

The Jacobian for these expressions with respect to (x_1, x_2, λ) is

$$\begin{pmatrix} 2x_2 - 4\lambda & 2x_1 & -4x_1 \\ 2x_1 & -2\lambda & -2x_2 \\ -4x_1 & -2x_2 & 0 \end{pmatrix}.$$

At $x_1 = x_2 = 1$ and $\lambda = 0.5$, its determinant is 48. The implicit function theorem concludes that we can solve $L_{x_1} = 0, L_{x_2} = 0, L_\lambda = 0$ for (x_1, x_2, λ) as C^1 functions of a near $a = 3$.

19.19 If $Dh(\mathbf{x})^*$ does not have maximal rank, we can use elementary row operations to transform one of the last k rows of $D^2L_{\mathbf{x},\mu}$ in (31) to a zero row. This would imply that $D^2L_{\mathbf{x},\mu}$ itself does not have maximal rank.

19.20 The new Lagrangian is

$$\lambda_0(x^2 + x + 4y^2) - \lambda_1(2x + 2y - 1) + \lambda_2 y + \lambda_3 z.$$

The first two first order conditions on page 446 would become:

$$\lambda_0(2x + 1) - 2\lambda_1 + \lambda_2 = 0, \quad 8\lambda_0 y_2 - 2\lambda_1 + \lambda_3 = 0.$$

The other 9 FOCs are unchanged; we would include conditions (e) and (f) from Theorem 19.11. If $\lambda_0 = 0$, the above 2 equations yield: $2\lambda_1 = \lambda_2 = \lambda_3$. By (f), this common value cannot be zero. By conditions 3, 4, 5, the constraints are binding: $x = 0, y = 0, 2x + 2y = 1$, an impossibility. So, $\lambda_0 \neq 0$. By (e), $\lambda_0 = 1$ and we proceed as in Example 18.13.

19.21 18.10: c, d, e; 18.11: c; 18.12: c; 18.17: d; 18.18: d.

19.22 *a*)

$$L = \lambda_0 x - \lambda_1(y - x^4) - \lambda_2(x^3 - y) - \lambda_3(x - 0.5).$$

$$\frac{\partial L}{\partial x} = \lambda_0 + 4\lambda_1 x^3 - 3\lambda_2 x^2 - \lambda_3 = 0$$

$$\frac{\partial L}{\partial y} = -\lambda_1 + \lambda_2 = 0 \quad \Longrightarrow \quad \lambda_1 = \lambda_2.$$

If $\lambda_1 = \lambda_2 > 0$, the first two constraints are binding and $y = x^3 = x^4$. Since $x \leq 0.5, x = 0$. In this case, $\lambda_0 = \lambda_3 = 0$.

If $\lambda_1 = \lambda_2 = 0$, then $L_x = 0$ becomes $\lambda_0 = \lambda_3$. By condition (f) of Theorem 19.11, $\lambda_0 = \lambda_3 > 0$.

Since $\lambda_3(x - 0.5) = 0, x = 0.5$.

But then, $y \geq x^3 \quad \Longrightarrow \quad y \geq 0.125$; $y \leq x^4 \quad \Longrightarrow \quad y \leq 0.0625$; a contradiction.

The only solution is: $\lambda_0 = \lambda_3, x = 0, \lambda_1 = \lambda_2 > 0$.

b) If we worked with $\lambda_0 = 1$ in part *a*, the first order conditions would be:

$$1 + 4\lambda_1 x^3 = 3\lambda_2 x + \lambda_3$$
$$\lambda_1 = \lambda_2.$$

If $\lambda_1 = \lambda_2 = 0$, then $\lambda_3 = 1$ and $x = 0.5$. As in part *a*, this contradicts the first two constraints.

If $\lambda_1 = \lambda_2 > 0$, then the first two constraints are binding: $y = x^3 = x^4$. Since $x < 0.5$, $x = 0$. Then, by the first FOC above, $\lambda_3 = 1$. On the other hand, this would contradict $\lambda_3(x - 0.5) = 0$.

We conclude that there is no solution to the FOCs with $\lambda_0 = 1$.

19.23 $C_h = \{(x, y) : x^3 + y^2 = 0\}$, a cusp in the left-half plane. Since $Dg(0, 0) = (0, 0)$, $Dg(0, 0)\mathbf{v} \leq 0$ for all $\mathbf{v}$, including $\mathbf{v} = \mathbf{e}_1 = (1, 0)$. Any curve $\alpha(t)$ with $\alpha(0) = (0, 0)$ and $\alpha'(0) = (1, 0)$ moves into the right-half plane where g is positive; i.e., $t \longmapsto g(\alpha(t))$ is increasing for t small so that for $t > 0$ and small, $g(\alpha(t)) > 0$.

19.24 In the proof of Theorem 18.4, instead of considering the curve $t \rightarrow b_1 - t$, work with the curve $t \rightarrow b_2 - t$ for $t \geq 0$. By the implicit function theorem, we can solve (47) for $x_{j_1}, \ldots, x_{j_e}$; that is, there exists a C^1 curve $t \rightarrow \mathbf{x}(t)$ for $t \in [0, \epsilon)$ such that

$$\mathbf{x}(0) = x^* \quad \text{and} \quad g_2\big(\mathbf{x}(t)\big) = b_2 - t, g_j(\mathbf{x}(t)) = b_j \text{ for } j = 1, 3, \ldots, e.$$

Let $\mathbf{v} = \mathbf{x}'(0)$. From the Chain Rule, $Dg_2(\mathbf{x}^*)\mathbf{v} = -1$ and $Dg_j(\mathbf{x}^*)\mathbf{v} = 0$ for $j \neq 2$. As in the proof of Theorem 18.4,

$$Df(\mathbf{x}^*)\mathbf{v} \leq 0 \quad \text{and} \quad 0 = D_{\mathbf{x}}L(\mathbf{x}^*)\mathbf{v} = Df(\mathbf{x}^*)\mathbf{v} + \lambda_2.$$

We conclude that $\lambda_2 \geq 0$.

19.25 If $\mathbf{x}^*$ maximizes f on $C_{g,h}$, it is also a local max of f subject to the equality constraints $g_1(\mathbf{x}) = b_1, \dots, g_{k_0}(\mathbf{x}) = b_{k_0}, h_1(\mathbf{x}) = c_1, \dots, h_m(\mathbf{x}) = c_m$. By Theorem 18.2, there are multipliers $\lambda_1, \dots, \lambda_{k_0}, \mu_1, \dots, \mu_m$ so that conditions (*a*), (*c*) and (*e*) hold. If we choose $\lambda_{k_0+1} = \cdots = \lambda_k = 0$, condition (*b*) holds. Use the argument in the proof of Theorem 18.4 (or Exercise 19.24) to show condition (*d*): each $\lambda_i \geq 0$.

Chapter 20

20.1 *a*) Degree 6, *b*) no, *c*) degree 0, *d*) degree 1, *e*) no, *f*) degree 0.

20.2 *a*)

$$\begin{aligned} f_{x_1} &= 2x_1x_2 + 3x_2^2, \quad f_{x_2} = x_1^2 + 6x_1x_2 + 3x_2^2 \\ x_1 f_{x_1} + x_2 f_{x_2} &= 2x_1^2x_2 + 3x_1x_2^2 + x_1^2x_2 + 6x_1x_2^2 + 3x_2^2 \\ &= 3x_1^2x_2 + 9x_1x_2^2 + 3x_2^3 \\ &= 3f(x_1, x_2). \end{aligned}$$

b)

$$\begin{aligned} f_{x_1} &= 7x_1^6x_2x_3^2 + 30x_1^5x_2^4, \\ f_{x_2} &= x_1^7x_3^2 + 20x_1^6x_2^3 - 5x_2x_3^5, \\ f_{x_3} &= 2x_1^7x_2x_3 - 5x_2^5x_3^4. \\ x_1 f_{x_1} + x_2 f_{x_2} + x_3 f_{x_3} &= 7x_1^7x_2x_3^2 + 30x_1^6x_3^4 + x_1^7x_2x_3^2 + 20x_1^6x_2^4 \\ &\quad - 5x_2^2x_3^5 + 2x_1^7x_2x_3^2 - 5x_2^5x_3^5 \\ &= 10x_1^7x_2x_3^2 + 50x_1^6x_2^4 - 10x_2^5x_3^5 = 10f. \end{aligned}$$

20.3 $f(t\mathbf{x}) = t^a f(\mathbf{x}), \quad g(t\mathbf{x}) = t^b g(\mathbf{x})$.

$(f \cdot g)(t\mathbf{x}) = f(t\mathbf{x}) \cdot g(t\mathbf{x}) = t^a f(\mathbf{x}) \cdot t^b g(\mathbf{x}) = t^{a+b}(f \cdot g)(\mathbf{x})$.

20.4 $F(tx_1, tx_2) = A(a_1 t^\rho x_1^\rho + a_2 t^\rho x_2^\rho)^{1/\rho} = tA(a_1 x_1^\rho + a_2 x_2^\rho)^{1/\rho} = tF(x_1, x_2)$.

20.5
$$x_1 f_{x_1} + x_2 f_{x_2} = rf$$
$$x_1 f_{x_1x_1} + x_2 f_{x_2x_1} = (r-1) f_{x_1}$$
$$x_1 f_{x_1x_2} + x_2 f_{x_2x_2} = (r-1) f_{x_2}$$

since f_{x_1} and f_{x_2} are homogeneous of degree $(r-1)$.

$$x_1(x_1 f_{x_1x_1} + x_2 f_{x_2x_1}) + x_2(x_1 f_{x_1x_2} + x_2 f_{x_2x_2}) = (r-1)x_1 f_{x_1} + (r-1)x_2 f_{x_2}.$$

So, $x_1^2 f_{x_1x_1} + 2x_1x_1 f_{x_1x_2} + x_2^2 f_{x_2x_2} = r(r-1)f.$

20.6 f is homogeneous $\Longleftrightarrow$ on any ray $\{t\mathbf{x} : t > 0\}$ from $\mathbf{0}$, $t \longrightarrow f(t\mathbf{x})$ is a homogeneous monomial αt^k. Suppose f is homogeneous of degree a and g is homogeneous of degree b. Choose $\mathbf{x}$ so that $f(\mathbf{x}) \neq \mathbf{0}$ and $g(\mathbf{x}) \neq \mathbf{0}$ and $f(\mathbf{x}) + g(\mathbf{x}) \neq \mathbf{0}$.

On the ray through $\mathbf{x}$, $t \longrightarrow f(t\mathbf{x}) + g(t\mathbf{x})$ is $t^a f(\mathbf{x}) + t^b g(\mathbf{x})$ (with $a \neq b$, say $a < b$). So, $t^a f(\mathbf{x}) + t^b g(\mathbf{x}) = t^a[f(\mathbf{x}) + t^{b-a} g(\mathbf{x})]$.

If $f+g$ is homogeneous, this equals $t^c[f(\mathbf{x}) + g(\mathbf{x})]$ for some c. In summary,

$$t^a[f(\mathbf{x}) + t^{b-a}g(\mathbf{x})] = t^c[f(\mathbf{x}) + g(\mathbf{x})] \quad \text{for all } t > 0.$$

Divide both sides by t^c:

$$t^{a-c}[f(\mathbf{x}) + t^{b-a}g(\mathbf{x})] = f(\mathbf{x}) + g(\mathbf{x}).$$

Let $t \longrightarrow 0$. If $a > c$, LHS is zero. If $a < c$, LHS is “infinite.” Therefore, $a = c$ and $f(\mathbf{x}) + g(\mathbf{x}) = f(\mathbf{x}) + t^{b-a}g(\mathbf{x})$. At $t = 0$, $f(\mathbf{x}) + g(\mathbf{x}) = f(\mathbf{x}) \Longrightarrow g(\mathbf{x}) = 0$, a contradiction.

20.7 f is homogeneous of every degree. In the previous exercise, we need to assume that neither f nor g is the zero function.

20.8 *a*) $ye^{x/y}$, *b*) $y\ln(x/y)$, *c*) $5y$, *d*) $(x_1^2/x_3) + (x_2^3/x_3^2)$,
e) $(x_1^2 + x_2^2)/x_3$.

20.9 $3xy + 2$, $\{3xy + 2 = 5\}$, $\{3xy + 2 = 14\}$; $(xy)^2$, $\{(xy)^2 = 1\}$, $\{(xy)^2 = 16\}$; $(xy)^3 + (xy)$, $\{(xy)^3 + (xy) = 2\}$; $\{(xy)^3 + (xy) = 68\}$, e^{xy}, $\{e^{xy} = e\}$, $\{e^{xy} = e^4\}$; $\ln(xy)$, $\{\ln(xy) = 0\}$, $\{\ln(xy) = \ln(4)\}$.

20.10 *a*) $\dfrac{3y}{3x} = \dfrac{y}{x} = \dfrac{1}{2}$.

b) $\dfrac{2(xy)y}{2(xy)x} = \dfrac{y}{x} = \dfrac{1}{2}$.

c) $\dfrac{e^{xy}y}{e^{xy}x} = \dfrac{y}{x} = \dfrac{1}{2}$.

d) $\dfrac{1}{x}\Big/\dfrac{1}{y} = \dfrac{y}{x} = \dfrac{1}{2}$.

20.11 $z^4 + z^2$, yes; $z^4 - z^2$, no; $z/(z+1)$, yes; $\sqrt{z}$, yes; $\sqrt{z^2+4}$, yes.

20.12 *a*) yes, $7z^2 + 2$; *b*) yes, $\ln z + 1$; *c*) no; *d*) yes, $z^{1/3}$.

20.13 Suppose f is homogeneous of degree k. Since $z \longrightarrow z^{1/k}$ is a monotonic transformation, f is equivalent to $f^{1/k}$. But $f^{1/k}$ is homogeneous of degree 1, since

$$f^{1/k}(t\mathbf{x}) = [f(t\mathbf{x})]^{1/k} = [t^k f(\mathbf{x})]^{1/k} = tf^{1/k}(\mathbf{x}).$$

20.14 For $U = x^{1/2}y^{1/2}$, $\dfrac{\partial^2 U}{\partial x^2} = -\dfrac{1}{4}x^{-3/2}y^{1/2} < 0$ for $(x, y) \in \mathbf{R^2}_{++}$. But for the equivalent $V = x^3y^3$, $\dfrac{\partial^2 V}{\partial x^2} = 6xy^3 > 0$. Not an ordinal property.

20.15 If $f' > 0$ for all x, f is invertible and its inverse g satisfies $g' > 0$ for all x in its domain. Since g is a monotonic transformation, $g(f(x)) = x$ is equivalent to f and is homogeneous of degree one.

20.16 Let g be homothetic so that $g = \varphi(h)$ where h is homogeneous of degree k. Now, $h^{1/k}$ is homogeneous of degree 1 and $\psi(z) = z^k$ is a monotone transformation. $g = \varphi(\psi(h^{1/k})) = (\varphi \circ \psi)(h^{1/k})$ and $\varphi \circ \psi$ is a monotone transformation.

20.17 *a*) Yes. This function is a monotone transformation of a homogeneous function of degree 3, $\varphi(x^2y + xy^2)$, where $\varphi(z) = e^z$.

b) Yes. This function is a monotone transformation of a homogeneous function of degree 5, $\varphi(x^2y^3)$, where $\varphi(z) = \ln z$.

c) Yes. This function is a monotone transformation of a homogeneous function of degree 3, $\varphi(xy^2)$, where $\varphi(z) = z^3 + 3z^2 + 3z + 9$. Check the derivative to see that $\varphi(z)$ is increasing.

d) No. The slope of the level set passing through (x, y) is $(y/x)(2x_1)/(x+1)$. This slope is not constant along rays emanating from the origin.

e) Yes. This function is a monotone transformation of a homogeneous function of degree 2, $\varphi(xy)$, where $\varphi(z) = z^2/(z+1)$.

20.18 *a*) $\dfrac{e^{x^2y+xy^2}\cdot(2xy+y^2)}{e^{x^2y+xy^2}\cdot(x^2+2xy)} = \dfrac{2xy+y^2}{x^2+2xy}$ is homogenous of degree zero.

b) $\dfrac{2}{x}\Big/\dfrac{3}{y} = \dfrac{2}{3}\dfrac{y}{x}$ is homogeneous of degree zero.

c) $\dfrac{3x^2y^4+4xy^2+12}{xy^5+2y^3+4y}$ is not homogenous of degree 0.

d) $\dfrac{2xy+y}{x^2+x}$ is not homogeneous of degree zero.

e) The ratio of partial derivatives is obviously homogeneous of degree 0.

f) $\dfrac{4x^3+2xy^2-3}{2x^3y+4y^3-8}$ is not homogeneous of degree zero; f is not homothetic.

20.19 Suppose u is homothetic, so that $u(x,y) = \varphi\big(h(x,y)\big)$ for some homogeneous function h and monotone φ with $\varphi' > 0$.

$$\begin{aligned}
\frac{\dfrac{\partial u}{\partial x}(tx,ty)}{\dfrac{\partial u}{\partial y}(tx,ty)} &= \frac{\varphi'(h(tx,ty))\cdot\dfrac{\partial h}{\partial x}(tx,ty)}{\varphi'(h(tx,ty))\cdot\dfrac{\partial h}{\partial y}(tx,ty)} \\
&= \frac{t^{k-1}\dfrac{\partial h}{\partial x}(x,y)}{t^{k-1}\dfrac{\partial h}{\partial y}(x,y)} = \frac{\dfrac{\partial h}{\partial x}(x,y)}{\dfrac{\partial h}{\partial y}(x,y)} \\
&= \frac{\varphi'(h(x,y))\dfrac{\partial h}{\partial x}(x,y)}{\varphi'(h(x,y))\dfrac{\partial h}{\partial y}(x,y)} \\
&= \frac{\dfrac{\partial u}{\partial x}(x,y)}{\dfrac{\partial u}{\partial y}(x,y)}.
\end{aligned}$$

20.20 Suppose $u(\mathbf{x}) = u(\mathbf{y})$, but $u(\alpha\mathbf{x}) > u(\alpha\mathbf{y})$ for some $\alpha > 0$. Since u is strictly monotone,

$$u\left(\frac{1}{\alpha}(\alpha\mathbf{x})\right) > u\left(\frac{1}{\alpha}(\alpha\mathbf{y})\right); \text{ i.e., } u(\mathbf{x}) > u(\mathbf{y}),$$

a contradiction. Therefore, $u(\alpha\mathbf{x}) = u(\alpha\mathbf{y})$.

Chapter 21

21.1

$$\begin{aligned} f(\mathbf{x}) &= \mathbf{a}\cdot\mathbf{x}. \\ f(t\mathbf{x}) &= \mathbf{a}\cdot(t\mathbf{x}) = t\mathbf{a}\cdot\mathbf{x} = tf(\mathbf{x}), \quad \text{homogeneous of degree 1.} \\ f(t\mathbf{x}+(1-t)\mathbf{y}) &= \mathbf{a}\cdot[t\mathbf{x}+(1-t)\mathbf{y}] = t\mathbf{a}\cdot\mathbf{x}+(1-t)\mathbf{a}\cdot\mathbf{y} \\ &= tf(\mathbf{x})+(1-t)f(\mathbf{y}). \end{aligned}$$

So, f is concave and convex.

21.2 *a*) $f'' = 3e^x + 60x^2 + \dfrac{1}{x^2} > 0 \implies$ convex.

b) $D^2 f = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}$ negative definite $\implies$ concave.

c) $\begin{pmatrix} 3e^x & 0 & 0 \\ 0 & 60y^2 & 0 \\ 0 & 0 & 1/z^2 \end{pmatrix}$ positive definite $\implies$ convex.

d) $D^2 f = A\begin{pmatrix} a(a-1)x^{a-2}y^bz^c & abx^{a-1}y^{b-1}z^c & acx^{a-1}y^bz^{c-1} \\ abx^{a-1}y^{b-1}z^c & b(b-1)x^ay^{b-2}z^c & bcx^ay^{b-1}z^{c-1} \\ acx^{a-1}y^bz^{c-1} & bcx^ay^{b-1}z^{c-1} & c(c-1)x^ay^bz^{c-2} \end{pmatrix}$

First LPM: $a(a-1)x^{a-2}y^bz^c$.
Second LPM: $ab(1-a-b)x^{2a-2}y^{2b-2}z^{2c}$.
Third LPM: $abc(a+b+c-1)x^{3a-2}y^{3b-2}z^{3c-2}$.
If $0 < a < 1$, $0 < b < 1$, $0 < c < 1$ and $0 < a+b+c < 1$, then f is concave.

21.3 Let $f(\mathbf{x}) = \sum_{i,j} a_{ij}x_ix_j$.

$$\text{Hessian } D^2 f = \begin{pmatrix} 2a_{11} & 2a_{12} & \cdots & 2a_{1n} \\ 2a_{12} & 2a_{22} & \cdots & 2a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 2a_{1n} & 2a_{2n} & \cdots & 2a_{nn} \end{pmatrix} = 2\cdot A.$$

So, f is concave $\Longleftrightarrow$ $D^2 f$ is negative semidefinite $\Longleftrightarrow$ A is negative semidefinite.

21.4 $f(t) = at^k$, $f' = kat^{k-1}$, $f'' = k(k-1)at^{k-2}$.
f is convex if $k(k-1)a \geq 0$.
f is concave if $k(k-1)a \leq 0$.

21.5 Show that $f : \mathbf{R^1} \to \mathbf{R^1}$ is concave if and only if the set on or below its graph in $\mathbf{R^2}$ is a convex set. Suppose f is concave and that (x_1, y_1) and (x_2, y_2) lie below its graph G; i.e., $f(x_1) \geq y_1$, $f(x_2) \geq y_2$. A point on the line L joining these 2 points is $(tx_2 + (1-t)x_1, ty_2 + (1-t)y_1)$.

$$f(tx_2 + (1-t)x_1) \geq tf(x_2) + (1-t)f(x_1) \geq ty_2 + (1-t)y_1.$$

So, segment L lies below G, and the set below G is convex. Conversely, suppose the set on or below G is convex and that $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are in this set. Then, so is $(tx_1 + (1-t)x_2, tf(x_1) + (1-t)f(x_2))$ for $t \in [0, 1]$. The statement that these points lie on or below G means: $tf(x_1) + (1-t)f(x_2) \leq f(tx_1 + (1-t)x_2)$. f is concave.

21.6 In the figure, ℓ_1 is tangent to the graph at $(x, f(x))$ and ℓ_2 is tangent at $(y, f(y))$; ℓ_0 is the "secant" line joining these 2 points. (4) and (5) state: slope $\ell_2 \leq$ slope $\ell_0 \leq$ slope ℓ_1. See figure.

21.7
$$\begin{aligned} g(f(t\mathbf{x} + (1-t)\mathbf{y}) &= af(t\mathbf{x} + (1-t)\mathbf{y}) + b \\ &\geq a[tf(\mathbf{x}) + (1-t)f(\mathbf{y})] + b \\ &= t[af(\mathbf{x}) + b] + (1-t)[af(\mathbf{y}) + b] \\ &= tg(f(\mathbf{x})) + (1-t)g(f(\mathbf{y})); \end{aligned}$$

i.e., $g \circ f$ is concave.

21.8 Suppose f and g are C^2. Then,

$$(f \circ g)''(x) = f''(g(x))g'(x)^2 + f'(g(x))g''(x).$$

If f is concave so that $f'' \leq 0$, if g is concave so that $g'' \leq 0$ and if f is monotone increasing so that $f' \geq 0$, then $(f \circ g)'' \leq 0$ and $f \circ g$ is concave. This can be proved directly using the definitions of concave and monotone without the C^2 hypothesis.

21.9 $\left(\frac{1}{f}\right)'' = \frac{2f'^2}{f^3} - \frac{f''}{f^2}$. If $f'' \leq 0$ (f concave) and f and $f' > 0$, then $1/f$ is convex.

21.10 $(f \cdot g)'' = (f''g + 2f'g' + fg'')$. If $g \geq 0$, $f \geq 0$ and $f' \cdot g' < 0$, then $f \cdot g$ is concave.

21.11 $\Pi(\mathbf{x}) = p\mathbf{g}(\mathbf{x}) - \mathbf{w} \cdot \mathbf{x}$. $\mathbf{x} \longmapsto -\mathbf{w} \cdot \mathbf{x}$ is concave. If g is concave, so is Π by Theorem 21.8.

21.12 *a*) $\Pi(q) = q \cdot F(q) - C(q)$.

b) $\Pi'' = 2F'(q) + qF''(q) - C''(q)$. If F is decreasing and concave and C is convex, Π is concave. (See Exercise 21.10.)

21.13 Since f is C^1 and concave, $f(\mathbf{y}) - f(\mathbf{x}_0) \leq Df(\mathbf{x}_0) \cdot (\mathbf{y} - \mathbf{x}_0)$ for all $\mathbf{y}$ in U. If $Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0$ for all $\mathbf{y}$ in U, then $f(\mathbf{y}) - f(\mathbf{x}_0) \leq 0$ for all $\mathbf{y}$ in U; i.e., $f(\mathbf{y}) \leq f(\mathbf{x}_0)$ for all $\mathbf{y}$ in U. In particular, if $Df(\mathbf{x}_0) = 0$, $Df(\mathbf{x}_0)(\mathbf{y} - \mathbf{x}_0) \leq 0$ for all $\mathbf{y}$ in U, and $\mathbf{x}_0$ is a global max.

21.14 If f is convex, $g = -f$ is concave. Apply Theorem 21.2 to g:

$$\begin{aligned} g(y) - g(x) &\leq g'(x)(y - x) && \text{for all } x, y \text{ in } I. \\ -f(y) + f(x) &\leq -f'(x)(y - x) && \text{for all } x, y \text{ in } I. \\ f(y) - f(x) &\geq f'(x)(y - x) && \text{for all } x, y \text{ in } I. \end{aligned}$$

21.15 For each i, $f_i(t\mathbf{x} + (1 - t)\mathbf{y}) \geq tf_i(\mathbf{x}) + (1 - t)f_i(y)$.
Multiply through by a_i and sum over i:

$$\begin{aligned} \sum_i a_i f_i(t\mathbf{x} + (1 - t)\mathbf{y}) &\geq \sum_i ta_i f_i(\mathbf{x}) + (1 - t)a_i f_i(\mathbf{y}) \\ &= t \sum_i a_i f_i(\mathbf{x}) + (1 - t) \sum_i a_i f_i(\mathbf{y}). \end{aligned}$$

So, $\sum a_i f_i$ is concave.

21.16 $\pi(p, \mathbf{w}) = \max_{y,\mathbf{x}} \{py - \mathbf{w} \cdot \mathbf{x} : y \leq g(\mathbf{x})\}$.

Let $(p, \mathbf{w})$ and $(p', \mathbf{w}')$ be two price-wage combinations.
Let $(p'', \mathbf{w}'') = t(p', \mathbf{w}') + (1 - t)(p, \mathbf{w})$ with corresponding optimal y'', $\mathbf{x}''$.

$$\begin{aligned}\pi(p'', \mathbf{w}'') &= p''y'' - \mathbf{w}'' \cdot \mathbf{x}'' \\ &= t(p'y'' - \mathbf{w}' \cdot \mathbf{x}'') + (1-t)(py'' - \mathbf{w} \cdot \mathbf{x}'') \\ &\leq t\pi(p', \mathbf{w}') + (1-t)\pi(p, \mathbf{w}),\end{aligned}$$

since $(y'', \mathbf{x}'')$ is not necessarily optimal for $(p', \mathbf{w}')$ or $(p, \mathbf{w})$. So, π is convex.

$(y', \mathbf{x}')$ maximizes $p\mathbf{y} - \mathbf{w} \cdot \mathbf{x}$ if and only if it maximizes $tpy - t\mathbf{w} \cdot \mathbf{x}$ for any $t > 0$. The corresponding optimal profit is $t\pi(p, \mathbf{w})$, i.e., $t\pi(p, \mathbf{w}) = \pi(tp, t\mathbf{w})$.

21.17 Let $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$. Let $V(\mathbf{a}) = \max\{\mathbf{a} \cdot \mathbf{x} : \mathbf{x} \in U\}$. Show V is convex and homogeneous of degree 1.

$\mathbf{x}_0$ maximizes $\mathbf{x} \longmapsto \mathbf{a} \cdot \mathbf{x}$ for $\mathbf{x} \in U$ iff it maximizes $\mathbf{x} \longmapsto t\mathbf{a} \cdot \mathbf{x}$ for $\mathbf{x} \in U$ for any fixed $t > 0$.

$V(t\mathbf{a}) = t\mathbf{a} \cdot \mathbf{x}_0 = t(\mathbf{a} \cdot \mathbf{x}_0) = tV(\mathbf{a})$, and V is homogeneous of degree 1.

Let $\mathbf{a}'' = t\mathbf{a}' + (1-t)\mathbf{a}$ and let $\mathbf{x}''$ be the maximizer of $\mathbf{x} \longmapsto \mathbf{a}'' \cdot \mathbf{x}$ for $\mathbf{x} \in U$.

$$\begin{aligned}V(\mathbf{a}'') = \mathbf{a}'' \cdot \mathbf{x}'' &= t\mathbf{a}'\mathbf{x}'' + (1-t)\mathbf{a} \cdot \mathbf{x}'' \\ &\leq tV(\mathbf{a}') + (1-t)V(\mathbf{a}),\end{aligned}$$

since $V(\mathbf{a}') \geq \mathbf{a}' \cdot \mathbf{x}$ and $V(\mathbf{a}) \geq \mathbf{a} \cdot \mathbf{x}$ for any $\mathbf{x} \in U$.

21.18 *a*) Both, *b*) both, *c*) both, *d*) neither, *e*) neither,

f) quasiconvex, *g*) both, *h*) neither.

21.19 *a*) $\Longrightarrow$ *b*): Suppose $f(\mathbf{x}) \geq f(\mathbf{y})$. Then, $\mathbf{x}$ and $\mathbf{y}$ are both in $C_{f(\mathbf{y})}$. By *a* $t\mathbf{x} + (1-t)\mathbf{y}$ is in $C_{f(\mathbf{y})}$, too. That is, $f(t\mathbf{x} + (1-t)\mathbf{y}) \geq f(\mathbf{y})$.

b) $\Longrightarrow$ *c*): Suppose $f(\mathbf{x}) \leq f(\mathbf{y})$; i.e., $f(\mathbf{x}) = \min\{f(\mathbf{x}), f(\mathbf{y})\}$. By *b*, $f(t\mathbf{x} + (1-t)\mathbf{y}) \geq f(\mathbf{x}) = \min\{f(\mathbf{x}), f(\mathbf{y})\}$. Similarly, if $f(\mathbf{x}) \geq f(\mathbf{y})$, *b*) implies $f(t\mathbf{x} + (1-t)\mathbf{y}) \geq f(\mathbf{y}) = \min\{f(\mathbf{x}), f(\mathbf{y})\}$.

c) $\Longrightarrow$ *a*): Suppose $f(\mathbf{x})$ and $f(\mathbf{y}) \geq a$; that is, $\mathbf{x}, \mathbf{y} \in C_a$. Then $f(t\mathbf{x} + (1-t)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}$ (by *c*) $\geq a$ since $f(\mathbf{x}) \geq a$ and $f(\mathbf{y}) \geq a$. Thus $t\mathbf{x} + (1-t)\mathbf{y} \in C_a$, so C_a is a convex set.

21.20 The following are equivalent:

a) $C_a^- = \{\mathbf{x} \in U : f(\mathbf{x}) \leq a\}$ is a convex set for all $x \in U$.

b) For all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0, 1]$, $f(\mathbf{x}) \leq f(\mathbf{y})$ implies $f(t\mathbf{x} + (1-t)\mathbf{y}) \leq f(\mathbf{y})$.

c) For all $\mathbf{x}, \mathbf{y} \in U$ and $t \in [0, 1]$, $f(t\mathbf{x} + (1-t)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}$.

21.21 We show that a quasiconcave function cannot have a strict interior minimum. Suppose $\mathbf{x}^*$ is such a minimum. Write $\mathbf{x}^* = t\mathbf{y}_1 + (1-t)\mathbf{y}_2$ where $f(\mathbf{y}_1) > f(\mathbf{x}^*)$ and $f(\mathbf{y}_2) > f(\mathbf{x}^*)$. By Theorem 21.12*c*, $f(\mathbf{x}^*) \geq \min\{f(\mathbf{y}_1), f(\mathbf{y}_2)\} > f(\mathbf{x}^*)$, a contradiction.

21.22 Suppose $F(\mathbf{x}) \geq F(\mathbf{y})$; i.e., $g(f_1(x_1) + \cdots + f_k(x_k)) \geq g(f_1(y_1) + \cdots + f_k(y_k))$.

Since g is monotone, $f_1(x_1) + \cdots + f_k(x_k) \geq f_1(y_1) + \cdots + f_k(y_k)$.

Since $\varphi(x_1, \ldots, x_k) = f_1(x_1) + \cdots + f_k(x_k)$ is concave and therefore quasiconcave, $\varphi(t\mathbf{x} + (1-t)\mathbf{y}) \geq \varphi(\mathbf{y})$.

Then $g(\varphi(t\mathbf{x} + (1-t)\mathbf{y})) \geq g(\varphi(\mathbf{y}))$. So, $F(t\mathbf{x} + (1-t)\mathbf{y}) \geq F(\mathbf{y})$, and F is quasiconcave.

21.23 *a, b*) The bordered Hessian has determinant ye^{-3x}. For $(x,y) \in \mathbf{R^2}$, it takes on both signs. For $(x,y) \in \mathbf{R^2}_+$, it is only positive. *a*) Neither, *b*) quasiconcave.

c) Monotone transformation $z \longmapsto z^3$ of the linear (quasiconcave and quasiconvex) function $2x + 3y$. Both quasiconcave and quasiconvex.

d) $(x,y,z) \longmapsto e^x + 5y^4 + |z|$ is a convex function. So $(x,y,z) \longmapsto (e^x + 5y^4 + |z|)^{1/2}$ is quasiconvex.

e) A monotone transformation $(z \longmapsto z^{1/3})$ of the concave function $(y - x^4)$. Therefore, quasiconcave.

f) Each level set $\{f = a\}$ is the graph of $y = a(x^2 + 1)$; for $a > 0$, the set above this graph is a convex set. So f is quasiconcave.

g) By *f*, quasiconcave for $y > 0$ and quasiconvex for $y < 0$. It is neither on all $\mathbf{R^2}$.

h) Level set $\{f = a\}$ is the graph of $y = ax^2$ for $a > 0$. Set above level set is convex set, so f is quasiconcave. Also, the determinant of the bordered Hessian is $2x^{-8}y > 0$.

i) Monotone transformation $x \longmapsto ke^x$ of convex function $\mathbf{x} \longmapsto \mathbf{x}^T A\mathbf{x}$. Therefore, quasiconvex.

21.24 Suppose F is pseudoconcave on U. Let $C_{\mathbf{x}^*} = \{\mathbf{y} \in U : DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0\}$; $\mathbf{x} \in C_{\mathbf{x}^*}$.

By definition of pseudoconcave, if $\mathbf{y} \in C_{\mathbf{x}^*}$, $F(\mathbf{y}) \leq F(\mathbf{x}^*)$. Therefore, $\mathbf{x}^*$ maximizes F on $C_{\mathbf{x}^*}$. Conversely, suppose $\mathbf{x}^*$ maximizes F on $C_{\mathbf{x}^*}$. Let $\mathbf{y} \in U$. Suppose $DF(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$. Then, $\mathbf{y} \in C_{\mathbf{x}^*}$ and $F(\mathbf{y}) \leq F(\mathbf{x}^*)$; that is, F is pseudoconcave.

21.25 *a*) Suppose $f(L(\mathbf{x})) \geq f(L(\mathbf{y}))$. Since f is quasiconcave, $f(tL\mathbf{x} + (1-t)L\mathbf{y}) \geq f(L(\mathbf{y}))$. Since L is linear, this inequality can be written as $f(L(t\mathbf{x} + (1-t)\mathbf{y})) \geq f\big(L(y)\big)$. So $f \circ L$ is quasiconcave.

b) Suppose $D(f \circ L)(\mathbf{x}^*)(\mathbf{y} - \mathbf{x}^*) \leq 0$. By the Chain Rule, $D(f \circ L)(\mathbf{x}^*) = Df(L\mathbf{x}^*) \circ L$. Then, f pseudoconcave and $Df(L\mathbf{x}^*)(L\mathbf{y} - L\mathbf{x}^*) \leq 0$ imply $f(L\mathbf{y}) \leq f(L\mathbf{x}^*)$; that is, $(f \circ L)(\mathbf{y}) \leq (f \circ L)(\mathbf{x}^*)$.

21.26 The solution of $\max xy$ subject to $2x + 2y \leq 8$ is $x^* = 2$ and $y^* = 2$, and the value of the multiplier is $\lambda = 1$. The saddle point inequality is $xy - (2x + 2y - 8) \leq 4$. But at $x = y = 1$, the left-hand term is 5. xy is quasiconcave and pseudoconcave on $\mathbf{R^2}_{++}$, but not concave.

21.27 f is concave in $(\mathbf{x}, \mathbf{a})$ and g_i convex in $(\mathbf{x}, \mathbf{a})$.

Thus $L(\mathbf{x}, \mathbf{a}, \lambda) = f(\mathbf{x}, \mathbf{a}) - \sum \lambda_i g_i(\mathbf{x}, \mathbf{a})$ is a concave function.

$Z(\mathbf{a}) =$ maximizers of $\mathbf{x} \longmapsto f(\mathbf{x}, \mathbf{a})$ such that $g_i(\mathbf{x}, \mathbf{a}) \leq 0$ for $i = 1, \ldots, k$.
$V(\mathbf{a}) = f\big(Z(\mathbf{a}), \mathbf{a}\big)$.

Let $\mathbf{x}_1 \in Z(\mathbf{a}_1)$ and $\mathbf{x}_2 \in Z(\mathbf{a}_2)$. $V(\mathbf{a}_1) = f(\mathbf{x}_1, \mathbf{a}_1)$ and $V(\mathbf{a}_2) = f(\mathbf{x}_2, \mathbf{a}_2)$.

Let $(\mathbf{x}_3, \mathbf{a}_3) = t(\mathbf{x}_1, \mathbf{a}_1) + (1-t)(\mathbf{x}_2, \mathbf{a}_2)$.

$$\begin{aligned}
V(\mathbf{a}_3) &= V(t\mathbf{a}_1 + (1-t)\mathbf{a}_2)\\
&\leq f(t\mathbf{x}_1 + (1-t)\mathbf{x}_2, t\mathbf{a}_1 + (1-t)\mathbf{a}_2)\\
&\leq tf(\mathbf{x}_1, \mathbf{a}_1) + (1-t)f(\mathbf{x}_2, \mathbf{a}_2)\\
&= tV(\mathbf{a}_1) + (1-t)V(\mathbf{a}_2).
\end{aligned}$$

The first inequality requires that $\mathbf{x}_3$ is in the constraint set for $f(\cdot, \mathbf{a}_3)$; i.e., each $g_i(\mathbf{x}_3, \mathbf{a}_3) \leq 0$. But since each g_i is convex and each $g_i(\mathbf{x}_j, \mathbf{a}_j) \leq 0$ for $j = 1, 2$,

$$g_i(\mathbf{x}_3, \mathbf{a}_3) \leq tg_i(\mathbf{x}_1, \mathbf{a}_1) + (1-t)g_i(\mathbf{x}_2, \mathbf{a}_2) \leq 0.$$

21.28 Let $\mathbf{a}_3 = t\mathbf{a}_1 + (1-t)\mathbf{a}_2$. Let $\mathbf{x}_i \in Z(\mathbf{a}_i)$ denote a constrained max of $f(\cdot, \mathbf{a}_i)$ on the constraint set C_{a_i} for each $i = 1, 2, 3$. Each $\mathbf{x}_i$ satisfies the constraints $\mathbf{g} \leq \mathbf{0}$. Therefore, $f(\mathbf{x}_3, \mathbf{a}_i) \leq V(\mathbf{a}_i)$ for $i = 1, 2$.

$$\begin{aligned}
V(\mathbf{a}_3) &= f(\mathbf{x}_3, t\mathbf{a}_1 + (1-t)\mathbf{a}_2)\\
&\leq tf(\mathbf{x}_3, \mathbf{a}_1) + (1-t)f(\mathbf{x}_3, \mathbf{a}_2)\\
&\leq tV(\mathbf{a}_1) + (1-t)V(\mathbf{a}_2).
\end{aligned}$$

21.29 Theorem 15.1.

Chapter 22

22.1 $x_1 = \dfrac{Ia}{p_1(a+b)}, \quad x_2 = \dfrac{Ib}{p_2(a+b)}, \quad \lambda = \dfrac{I^{a+b-1}a^a b^b}{p_1^a p_2^b}(a+b)^{1-a-b},$

$$V = \frac{I^{a+b}a^a b^b}{p_1^a p_2^b}(a+b)^{-a-b}, \quad \frac{\partial V}{\partial I} = \lambda.$$

22.2 By FOCs (5), $a_i p_j x_j = a_j p_i x_i$ for all $i, j = 1, \ldots, n$. Substituting into (6) yields

$$x_i = \frac{a_i I}{p_i} \quad \text{for } i = 1, \ldots, n.$$

$$\lambda = \frac{1}{p_1} a_1 x_1^{a-1} x_2^{a_2} \cdots x_n^{a_n} = \frac{a_1}{p_1}\left(\frac{a_1 I}{p_1}\right)^{a_1 - 1} \cdots \left(\frac{a_i I}{p_i}\right)^{a_i} \cdots \left(\frac{a_n I}{p_n}\right)^{a_n}$$
$$= \frac{a_1^{a_1} \cdots a_n^{a_n}}{p_1^{a_1} \cdots p_n^{a_n}}.$$

22.3

$$\sum s_i \eta_i = \sum_i \left(\frac{p_i \xi_i}{I}\right) \cdot \frac{I}{\xi_i}\frac{\partial \xi_i}{\partial I} = \sum_i p_i \frac{\partial \xi_i}{\partial I}$$
$$= \frac{\partial}{\partial I} \sum_i p_i \xi_i(\mathbf{p}, I) = \frac{\partial}{\partial I} \cdot I = 1.$$
$$\sum_k \epsilon_{jk} + \eta_j = \sum_k \frac{p_k}{\xi_j}\frac{\partial \xi_j}{\partial p_k} + \frac{I}{\xi_j}\frac{\partial \xi_j}{\partial I}$$
$$= \frac{1}{\xi_j}\left(\sum_k p_k \frac{\partial \zeta_j}{\partial p_k} + I\frac{\partial \zeta_j}{\partial I}\right) = 0,$$

by Euler's theorem, since ξ_j is homogeneous of degree zero.

22.4 $\dfrac{\partial \tilde{L}}{\partial x_i} = p_i - \nu \dfrac{\partial U}{\partial x_i}(\mathbf{x}) - \nu_i = 0 \quad$ for all i.

For i as in (33), $x_i > 0$; so $\nu_i = 0$. Then, $p_i > 0$ implies $\nu > 0$. Let $\lambda = 1/\nu > 0$. The above equation becomes:

$$\lambda p_i = \frac{\partial U}{\partial x_i}(\mathbf{x}) + \lambda \nu_i \geq \frac{\partial U}{\partial x_i}(\mathbf{x})$$

with equality if $x_i > 0$ (when $\nu_i = 0$). Condition a holds.
Since $\nu > 0$, $U(x^*) = u$. That is, $U(Z(\mathbf{p}, u)) = u$ and b holds.

The equations for $x^* = Z(\mathbf{p}, u)$ where $\mathbf{x}_i^* > 0$ are

$$\begin{aligned} U_{x_i}(\mathbf{x}) - \lambda p_i &= 0, \quad i = 1, \cdots, n, \\ U(\mathbf{x}) - u &= 0. \end{aligned}$$

The Jacobian for this system is (13); and c follows.
That $(\partial E/\partial u) = \lambda$ is exactly Theorem 19.1. By the Envelope Theorem,

$$\frac{\partial E}{\partial p_i} = \frac{\partial \tilde{L}}{\partial p_i} = x_i = Z_i(\mathbf{p}, u).$$

22.5 Let $\mathbf{x}^* = \xi(\mathbf{p}, I)$, the maximizer of U subject to $\mathbf{p} \cdot \mathbf{x} \le I$ and $\mathbf{x} \ge \mathbf{0}$. Let $V(\mathbf{p}, I) = U(\xi(\mathbf{p}, I)) = u^*$. By (6), $I = \mathbf{p} \cdot \xi(\mathbf{p}, I)$.
We want to show that $\mathbf{x}^*$ minimizes $\mathbf{p} \cdot \mathbf{x}$ subject to $U(\mathbf{x}) \ge u^*$ and $\mathbf{x} \ge \mathbf{0}$. Suppose not, i.e., that there is $\mathbf{z}' \ge \mathbf{0}$ with $U(\mathbf{z}') \ge u^*$ and $\mathbf{p} \cdot \mathbf{z}' < \mathbf{p} \cdot \mathbf{x}^*$. Since U is continuous and monotone, we can perturb $\mathbf{z}'$ to $\mathbf{z}''$ so that $\mathbf{p} \cdot \mathbf{z}'' < \mathbf{p} \cdot \mathbf{x}^*$ and $U(\mathbf{z}'') > U(\mathbf{x}^*) = u^*$. This contradicts $\mathbf{x}^*$ as max of U on $\mathbf{p} \cdot \mathbf{x} \le I$. Therefore, $\mathbf{x}^* = Z(\mathbf{p}, u^*) = Z(\mathbf{p}, V(\mathbf{p}, I))$, and b holds. The equation $\mathbf{p} \cdot \xi(\mathbf{p}, I) = I$ implies d.

22.6 $u = V(\mathbf{p}, E(\mathbf{p}, u))$ by Theorem 22.7c. By Theorem 22.2 and 22.6, V and E are C^1 functions under the hypothesis of Theorem 22.5. Taking the p_i partial derivative via the Chain Rule yields:

$$\begin{aligned} 0 &= \frac{\partial V}{\partial p_i}(\mathbf{p}, E(\mathbf{p}, u)) + \frac{\partial V}{\partial I}(\mathbf{p}, E(\mathbf{p}, u)) \cdot \frac{\partial E}{\partial p_i}(\mathbf{p}, u) \\ &= \frac{\partial V}{\partial p_i}(\mathbf{p}, E(\mathbf{p}, u) + \frac{\partial V}{\partial I}(\mathbf{p}, E(\mathbf{p}, u)) \cdot Z_i(\mathbf{p}, u) \quad \text{(by (34))} \\ &= \frac{\partial V}{\partial p_i}(\mathbf{p}, I) + \frac{\partial V}{\partial I}(\mathbf{p}, I) \cdot \xi_i(\mathbf{p}, I) \end{aligned}$$

where $I = E(\mathbf{p}, u)$, using Theorem 22.7 again.

22.7 One computes that for $U = x_1^{a_1} \cdots x_n^{a_n}$, with $a_1 + \cdots + a_n = 1$,

$$\begin{aligned} \xi_i(\mathbf{p}, I) &= a_i I / p_i, \\ z_i(\mathbf{p}, u) &= u \left(\frac{p_1}{a_1}\right)^{a_1} \cdots \left(\frac{p_n}{an}\right)^{a_n} \cdot \frac{a_i}{p_i}. \end{aligned}$$

For $\quad i \ne j, \partial \xi_j / \partial p_i = 0 \quad$ and

$$\frac{\partial Z_j}{\partial p_i}(\mathbf{p}, u) = u \left(\frac{p_i}{a_i}\right)^{a_i - 1} \left(\frac{p_j}{a_j}\right)^{a_j - 1} \prod_{k \ne i,j} \left(\frac{p_k}{a_k}\right)^{a_k}$$

$$u = \left(\frac{a_1}{p_1}\right)^{a_1} \cdots \left(\frac{a_n}{p_n}\right)^{a_n} \cdot I = V(\mathbf{p}, I).$$

So, $\dfrac{\partial Z_j}{\partial p_i}(\mathbf{p}, V(\mathbf{p}, I)) = \dfrac{a_i}{p_i} \cdot \dfrac{a_j}{p_j} \cdot I.$

Also, $\dfrac{\partial \xi_j}{\partial I}(p, I) \cdot \xi_i = \dfrac{a_j}{p_j} \cdot \dfrac{a_i}{p_i} I.$

So, $0 = \dfrac{\partial \xi_j}{\partial p_i}(\mathbf{p}, I) = \dfrac{\partial Z_j}{\partial p_i}(\mathbf{p}, V(\mathbf{p}, I)) - \dfrac{\partial \xi_j}{\partial I}(\mathbf{p}, I) \cdot \xi_i, \quad \text{for} \quad i \neq j.$

22.8 The linearized system is

$$\begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & U_{11} & U_{12} \\ -p_2 & U_{12} & U_{22} \end{pmatrix} \begin{pmatrix} d\lambda \\ dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} x_1 dp_1 + x_2 dp_2 - dI \\ \lambda dp_1 \\ \lambda dp_2 \end{pmatrix}.$$

Let $D =$ the determinant of the coefficient matrix.

$$\frac{\partial x_2}{\partial p_1} = \frac{\begin{vmatrix} 0 & -p_1 & x_1 \\ -p_1 & U_{11} & \lambda \\ -p_2 & U_{12} & 0 \end{vmatrix}}{D} = x_1(U_{11}p_2 - U_{12}p_1) + \lambda p_1 p_2.$$

$$x_1 \frac{\partial x_2}{\partial p_1} = x_1 \begin{vmatrix} 0 & -p_1 & -1 \\ -p_1 & U_{11} & 0 \\ -p_2 & U_{12} & 0 \end{vmatrix} = x_1(-U_{11}p_2 + U_{12}p_1).$$

$$\frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial I} = \lambda p_1 p_2.$$

Similarly,

$$\frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial I} = \lambda p_1 p_2.$$

22.9 $x_k = p_k^{1/(r-1)} I \Big/ \left(p_1^{r/(r-1)} + p_2^{r/(r-1)}\right), \quad \text{for } k = 1, 2,$

$$V = I\left(p_1^{r/(r-1)} + p_2^{r/(r-1)}\right)^{(r-1)/r},$$

$$e = U\left(p_1^{r/(r-1)} + p_2^{r/(r-1)}\right)^{(r-1)/r}.$$

22.10 Multiplying p and $\mathbf{w}$ by $r > 0$, multiplies the profit function by r. The $\mathbf{x}^*$ that maximizes $\Pi(\mathbf{x})$ is exactly the $\mathbf{x}^*$ that maximizes $r\Pi(\mathbf{x})$; i.e., $\mathbf{x}^*(rp, r\mathbf{w}) = \mathbf{x}^*(p, \mathbf{w})$.

22.11 Continuing the previous exercise,

$$\begin{aligned}\Pi^*(rp, r\mathbf{w}) &= \Pi(\mathbf{x}^*(rp, r\mathbf{w}))\\ &= rpf(\mathbf{x}^*(rp, r\mathbf{w})) - r\mathbf{w}\cdot\mathbf{x}^*(rp, r\mathbf{w})\\ &= rpf(\mathbf{x}^*(p, \mathbf{w})) - r\mathbf{w}\cdot\mathbf{x}^*(p.\mathbf{w})\\ &= r\Pi^*(p, \mathbf{w}).\end{aligned}$$

22.12 The Lagrangian for this problem is $L = pf(x) - \mathbf{w}\cdot\mathbf{x} + \sum \nu_i x_i$. The first order conditions

$$L_{x_i} = p\frac{\partial f}{\partial x_i} - w_i + \nu_i = 0$$

along with $\nu_i \geq 0$ translate to (40):

$$p\frac{\partial f}{\partial x_i}(\mathbf{x}) - w_i \leq 0.$$

If $x_i^* > 0$, $\nu_i x_i^* = 0$ implies $\nu_i = 0$ and (40) becomes (41). In the case where each $x_i^* > 0$, we can treat this problem as an unconstrained max problem. If $\mathbf{x}^*$ is a critical point, i.e., satisfies (41), and if the Hessian is negative definite, $\mathbf{x}^*$ is a strict local max by Theorem 17.2. Conversely, if $\mathbf{x}^*$ is an interior local max, it satisfies (41) and the Hessian is negative semidefinite by Theorems 17.6 and 17.7.

22.13 The monotonicity assumption implies NDCQ. The Lagrangian is

$$L = \mathbf{w}\cdot\mathbf{x} - \lambda(f(\mathbf{x})) - y) - \sum_i \nu_i x_i.$$

The FOCs are:

$$L_{x_i} = w_i - \lambda\frac{\partial f}{\partial x_i}(\mathbf{x}) - \nu_i = 0 \quad \text{for } i = 1, \dots, n$$

or

$$w_i = \lambda\frac{\partial f}{\partial x_i}(\mathbf{x}) + \nu_i \geq \lambda\frac{\partial f}{\partial x_i}(\mathbf{x}). \qquad (*)$$

((49) in the text has the wrong sign.) If $x_i > 0$, $\nu_i x_i = 0$ implies $\nu_i = 0$ and $(*)$ becomes

$$w_i = \lambda\frac{\partial f}{\partial x_i}(\mathbf{x}). \qquad (**)_i$$

By the monotonicity condition, there is an i such that $x_i^* > 0$. For such an i, $(**)_i$ holds and $w_i > 0$ implies $\lambda > 0$. For i, j with $x_i > 0$ and $x_j > 0$, divide $(**)_i$ by $(**)_j$ to obtain (50).
The Jacobian of the FOCs is the matrix in (51). So, (51) follows from the discussion below (19.23). The converse follows directly from Theorems 19.6 and 19.9 applied to this problem.

22.14 By Theorem 19.9, the solution $\mathbf{z}^*(y, \mathbf{w})$ of Problem (48) is a C^1 function of y and $\mathbf{w}$. So is $C(y, \mathbf{w}) = \mathbf{z}^*(y, \mathbf{w}) \cdot \mathbf{w}$, the optimal value of the objective function of (48). Suppose we are at an interior solution where $\mathbf{z}^* > \mathbf{0}$. The Lagrangian for Problem (48) is $L = \mathbf{w} \cdot \mathbf{x} - \lambda(f(\mathbf{x}) - y)$. By Theorem 19.5,

$$\frac{\partial C}{\partial w_i} = \frac{\partial L}{\partial w_i} = z_i^*.$$

Then,

$$\frac{\partial z_i^*}{\partial w_j} = \frac{\partial^2 C}{\partial w_j \partial w_i} = \frac{\partial^2 C}{\partial w_i \partial w_j} = \frac{\partial}{\partial w_i}\left(\frac{\partial C}{\partial w_j}\right) = \frac{\partial z_j^*}{\partial w_i}.$$

22.15 $x_1^*(y, w_1, w_2)$, $x_2^*(y, w_1, w_2)$, and $\lambda(y, w_1, w_2)$ satisfy

$$\lambda \frac{\partial f}{\partial x_1}(x_1, x_2) = w_1$$

$$\lambda \frac{\partial f}{\partial x_2}(x_1, x_2) = w_2$$

$$f(x_1, x_2) = y.$$

Taking total differentials yields

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = dy$$

$$\frac{\partial f}{\partial x_1} d\lambda + \lambda \frac{\partial^2 f}{\partial x_1^2} dx_1 + \lambda \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_2 = dw_1$$

$$\frac{\partial f}{\partial x_2} d\lambda + \lambda \frac{\partial^2 f}{\partial x_1 \partial x_2} dx_1 + \lambda \frac{\partial^2 f}{\partial x_2^2} dx_2 = dw_2.$$

$$\frac{\partial x_1}{\partial w_2} = \frac{\det \begin{pmatrix} 0 & 0 & f_{x_2} \\ f_{x_1} & 0 & \lambda f_{x_1 x_2} \\ f_{x_2} & 1 & \lambda f_{x_2 x_2} \end{pmatrix}}{D} = \frac{f_{x_1} f_{x_2}}{D}$$

$$\frac{\partial x_2}{\partial w_1} = \frac{\det\begin{pmatrix} 0 & f_{x_1} & 0 \\ f_{x_1} & \lambda f_{x_1 x_1} & 1 \\ f_{x_2} & \lambda f_{x_1 x_2} & 0 \end{pmatrix}}{D} = \frac{f_{x_1} f_{x_2}}{D},$$

where D is the determinant of the coefficient matrix. So, $\partial x_1/\partial w_2 = \partial x_2/\partial w_1$.

22.16 By Shepard's lemma,

$$x_1 = \frac{\partial C}{\partial w_1} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_2}{w_1}}\right) \quad \text{and} \quad x_2 = \frac{\partial C}{\partial w_2} = y\left(1 + \frac{1}{2}\sqrt{\frac{w_1}{w_2}}\right).$$

$$\sqrt{\frac{w_1}{w_2}} = 2\left(\frac{x_2}{y} - 1\right) = \frac{y}{2(x_1 - y)}.$$

So, $y^2 = 4(x_1 - y)(x_2 - y)$ or $3y^2 - 4(x_1 + x_2)y + 4x_1x_2 = 0$ or $y = \frac{2}{3}\left[(x_1 + x_2) + \sqrt{x_1^2 - x_1x_2 + x_2^2}\right]$.

22.17 $\mathbf{x}^*$ is a local Pareto optimum for $u_1, \ldots, u_A$ on C if there is an open set U about $\mathbf{x}^*$ so that if $\mathbf{x} \in C \cap U$, $\mathbf{x} \neq \mathbf{x}^*$, and $u_j(\mathbf{x}) > u_j(\mathbf{x}^*)$ for some j, then $u_i(\mathbf{x}) < u_i(\mathbf{x}^*)$ for some i.

$\mathbf{x}^*$ is a strict Pareto optimum for $u_1, \ldots, u_A$ on C if $\mathbf{x} \in C$ and $u_j(\mathbf{x}) \geq u_j(\mathbf{x}^*)$ for some j; then $u_i(\mathbf{x}) < u_i(\mathbf{x}^*)$ for some $i = 1, \ldots, A$.

$\mathbf{x}^*$ is a strict local Pareto optimum for $u_1 \ldots, u_A$ on C if there is an open set U about $\mathbf{x}^*$ so that if $\mathbf{x} \in C \cap U$ and $u_j(\mathbf{x}) \geq u_j(\mathbf{x}^*)$ for some j, then $u_i(\mathbf{x}) < u_i(\mathbf{x}^*)$ for some i.

22.18 By the proof of Theorem 22.17, if $\mathbf{x}^*$ satisfies the hypotheses of the theorem, $\mathbf{x}^*$ is a strict local max of u_i on

$$C_i = \{\mathbf{x} \in C : u_j(\mathbf{x}) \geq u_j(\mathbf{x}^*) \quad \text{for} \quad j \neq i\},$$

for each $i = 1, \ldots, A$. Suppose there is an index i and an $\mathbf{x}$ in C such that $u_i(\mathbf{x}) \geq u_i(\mathbf{x}^*)$. Then, $\mathbf{x}$ cannot be in C_i. That is, there is an index j such that $u_j(\mathbf{x}) < u_j(\mathbf{x}^*)$. But this is just the definition of a strict Pareto optimum.

22.19 For each i consider the problem of maximizing u_i on the constraint set C_i as in the previous exercise. Its Lagrangian is

$$u_i(\mathbf{x}) - \sum_{j \neq i} \alpha_j(u_j(\mathbf{x}) - u_j(\mathbf{x}^*)) - \sum_{k=1}^{M} \lambda_k(g_k(\mathbf{x}) - b_k) - \sum_{p=1}^{N} \mu_p(h_p(\mathbf{x}) - c_p).$$

By hypothesis, there exist $\alpha_1, \dots, \alpha_A, \lambda_1, \dots, \lambda_M, \mu_1, \dots, \mu_N$ such that for $s = 1, \dots, n$

$$\frac{\partial u_i}{\partial x_s} - \sum_{j \neq i} \frac{\alpha_j}{\alpha_i} \frac{\partial u_j}{\partial x_s} - \sum_k \frac{\lambda_k}{\alpha_i} \frac{\partial g_k}{\partial x_s} - \sum_p \frac{\mu_p}{\alpha_i} \frac{\partial h_p}{\partial x_s} = 0$$

at $\mathbf{x} = \mathbf{x}^*$. This is the FOC for our original problem. By hypothesis, u_i is pseudoconcave, u_j for $j \neq i$ and all g_ks are quasiconcave, and the h_p's are linear. By Theorem 21.22, $\mathbf{x}^*$ is a global max of u_i on C_i. Since this is true for each i, there cannot exist a point with $u_j(\mathbf{x}) \geq u_j(\mathbf{x}^*)$ for all j with strict inequality for some i; that is, $\mathbf{x}^*$ is a global Pareto optimum.

22.20 $U^k(\mathbf{z}^1, \dots, \mathbf{z}^A) = u^k(\mathbf{z}^k)$.

$$D_{\mathbf{z}^j} U^k(\mathbf{z}^1, \dots, \mathbf{z}^A)(\mathbf{v}^1, \dots, \mathbf{v}^A) = \begin{cases} 0 & \text{if } j \neq k \\ Du^k(\mathbf{z}^k)\mathbf{v}^k & \text{if } j = k \end{cases}$$

since U^k is independent of $\mathbf{z}^j$ for $j \neq k$ and $U^k(\mathbf{z}) = u^k(\mathbf{z}^k)$ for $k = j$. Then

$$\sum_{k=1}^{A} \alpha^k DU^k(\mathbf{Y})\mathbf{V} = \sum_{k=1}^{A} \alpha^k \sum_j D_{\mathbf{z}_j} U^k(\mathbf{Y})\mathbf{v}_j = \sum_{k=1}^{A} \alpha^k Du^k(\mathbf{y}^k)\mathbf{v}^k.$$

22.21 Consider the problem of maximizing

$$\mathbf{Y} \longmapsto W(\mathbf{Y}) = \sum_{k=0}^{A} \frac{1}{\lambda^k} u^k(\mathbf{y}^k) \quad \text{subject to} \quad \sum \mathbf{y}^k = \mathbf{b} \text{ and } \mathbf{y}^k \geq \mathbf{0} \text{ in } \mathbf{R^n}.$$

The Lagrangian for this problem is

$$L = \sum (1/\lambda^k) u^k(\mathbf{y}^k) - \mathbf{p} \cdot \left(\sum_k \mathbf{y}^k - \mathbf{b} \right) + \sum \mathbf{r}^k \cdot \mathbf{y}^k,$$

where $\mathbf{p}$ and r^k are n-vectors, $\mathbf{r}^k \geq \mathbf{0}$. The FOCs are:

$$D_{\mathbf{y}^k} L = (1/\lambda^k) D\mathbf{u}^k(\mathbf{y}^k) - \mathbf{p} + \mathbf{r}^k = \mathbf{0}.$$

Since $\mathbf{r}^k \geq 0$, these FOCs can be written as

$$\frac{1}{\lambda^k} D\mathbf{u}^k(\mathbf{y}^k) - \mathbf{p} \leq \mathbf{0}, \quad \text{and} \quad \frac{1}{\lambda^k} \frac{\partial u^k}{\partial x_j^k}(\mathbf{y}^k) = p_j \text{ if } x_j^k > 0.$$

But these are exactly conditions (66) and (67) with $\mathbf{p} = \mathbf{c}$ and $1/\lambda^k = \alpha^k$.

22.22 *a*) (Compare with Section 15.4.) Let $\gamma_1 \equiv \beta_1/(1-\beta_1)$ and $\gamma_2 \equiv \beta_2/(1-\beta_2)$. The six equations for a competitive equilibrium are

$$\begin{aligned} p_1x_1 &= \gamma_1 p_2 x_2 \\ p_1y_1 &= \gamma_2 p_2 y_2 \\ p_1x_1 + p_2x_2 &= p_1 \\ p_1y_1 + p_2y_2 &= p_2 \\ x_1 + y_1 &= 1 \\ x_2 + y_2 &= 1. \end{aligned}$$

The six variables are prices p_1, p_2, consumer 1's bundle (x_1, x_2) and consumer 2's bundle (y_1, y_2). The first two equations are the consumers' first order conditions, the next two are their budget constraints, and the last two are the pure exchange equations. Since prices are relative, we set $p_2 = 1$. Since the system has a redundant equation, we will drop the fourth equation. (See Section 15.4.) Solve for y_1 and y_2 in the last two equations and substitute into the second; this yields

$$p_1(1-x_1) = \gamma_2(1-x_2) \quad \text{or} \quad p_1 - p_1x_1 = \gamma_2 - \gamma_2x_2.$$

Substitute γ_1x_2 for p_1x_1:

$$p_1 = (\gamma_1 - \gamma_2)x_2 + \gamma_2.$$

Combining the first and third equations yields

$$\gamma_1x_2 + x_2 = p_1.$$

Solve these two equations for

$$x_2 = \frac{\gamma_2}{1+\gamma_2}, \qquad p_1 = \frac{(1+\gamma_1)\gamma_2}{1+\gamma_2}.$$

Then, $y_2 = \dfrac{1}{1+\gamma_2}$ by equation (6). Use the first two equations to compute

$$x_1 = \frac{\gamma_1}{1+\gamma_1}, \qquad y_1 = \frac{1}{1+\gamma_1}.$$

In terms of β_1 and β_2, these solutions are

$$x_1 = \beta_1, \quad x_2 = \beta_2, \quad y_1 = 1 - \beta_1, \quad y_2 = 1 - \beta_2,$$
$$p_1 = \beta_2/(1 - \beta_1), \quad p_2 = 1.$$

b) By Theorem 22.19, the equation (68) for a Pareto optimum are

$$\frac{\dfrac{\partial u^1}{\partial x_1}}{\dfrac{\partial u^1}{\partial x_2}}(x_1, x_2) = \frac{\dfrac{\partial u^2}{\partial y_1}}{\dfrac{\partial u^2}{\partial y^2}}(y_1, y_2) = \frac{\dfrac{\partial u^2}{\partial y_1}}{\dfrac{\partial u^2}{\partial y^2}}(1 - x_1, 1 - x_2)$$

since $x_1 + y_1 = 1$ and $x_2 + y_2 = 1$. These become:

$$\frac{\beta_1}{1 - \beta_1} \cdot \frac{x_2}{x_1} = \frac{\beta_2}{1 - \beta_2} \cdot \frac{1 - x_2}{1 - x_1},$$

which gives the curve of Pareto optima

$$x_2 = \frac{\beta_1 x_1}{\beta_2} + (\beta_1 - \beta_2)x_1,$$

a curve that goes through both $(0, 0)$ and $(1, 1)$. All are Pareto superior to the initial allocation.

22.23 *a*) Solid curves are indifference curves of u^2.
Dashed curves are indifference curves of u^1.
Dark line is set of Pareto optima.
Interior Pareto optima are solutions of

$$\frac{\dfrac{\partial u^1}{\partial x_1}(x_1, x_2)}{\dfrac{\partial u^1}{\partial x_2}(x_1, x_2)} = \frac{\dfrac{\partial u^2}{\partial x_1}(2 - x_1, 2 - x_2)}{\dfrac{\partial u^2}{\partial x_2}(2 - x_1.2 - x_2)}$$

or

$$\frac{x_2 + 1}{x_1 + 1} = -2[(2 - x_1) - 1] = 2(x_1 - 1)$$

or

$$x_2 = 2x_1^2 - 3,$$

a curve in the Edgeworth Box $[0,2]\times[0,2]$ from $(\sqrt{3/2},0)$ to $(\sqrt{5/2},2)$. Checking how the indifference curves cross on the boundary of $[0,2]\times[0,2]$, one notes that the segment from 1 to $\sqrt{3/2}$ on $x_2 = 0$ and the segment from $\sqrt{5/2}$ to 2 on $x_2 = 2$ are Pareto optima. See figure.

b) Is the point A in the figure realizable as a competitive equilibrium? A is the global max of u_2 in the Edgeworth Box, and therefore the max of u_2 on any budget line. Choose prices so that A is the max of u_1 on the corresponding budget line. A will be a competitive equilibrium for these prices.

Chapter 23

23.1 $$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = (-1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

23.2 *a*) $$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \implies (3-r)(5-r) = 0 \implies r_1 = 3, r_2 = 5.$$

$$\begin{pmatrix} 3-3 & 0 \\ 4 & 5-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0v_1 + 0v_2 \\ 4v_1 + 2v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$\begin{pmatrix} 3-5 & 0 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + 0v_2 \\ 4v_1 + 0v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) $$\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix} \implies (-1-r)(4-r) + 6 = 0 \implies r^2 - 3r + 2 = 0$$
$$\implies r_1 = 2, r_2 = 1.$$

$$\begin{pmatrix} -1-2 & 3 \\ -2 & 4-2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3v_1 + 3v_2 \\ -2v_1 + 2v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} -1-1 & 3 \\ -2 & 4-1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2v_1 + 3v_2 \\ -2v_1 + 3v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

c) $$\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix} \implies (-r)(-3-r) + 2 = 0 \implies r^2 + 3r + 2 = 0$$
$$\implies r_1 = -1, r_2 = -2.$$

$$\begin{pmatrix} 0-(-1) & -2 \\ 1 & -3-(-1) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 - 2v_2 \\ v_1 - 2v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 0+2 & -2 \\ 1 & -3+2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 - 2v_2 \\ v_1 - v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$d)$ $$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{pmatrix} \Longrightarrow (-r)(7-r)(-3-r) - (-2)(7-r)(1) = 0$$

$$\Longrightarrow (7-r)[(-r)(3-r) - (-2)(1)] = 0$$

$$\Longrightarrow (7-r)(r^2 - 3r + 2)$$

$$\Longrightarrow r_1 = 7, r_2 = -1, r_3 = -2.$$

$$\begin{pmatrix} -7 & 0 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & -10 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -7v_1 + 0v_2 - 2v_3 \\ 0v_1 + 0v_2 + 0v_3 \\ v_1 + 0v_2 - 10v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 6 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - 2v_3 \\ 6v_2 \\ v_1 - 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 0 & -2 \\ 0 & 5 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2v_1 - 2v_3 \\ 5v_2 \\ v_1 - v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

23.3 The eigenvalues of D are the values of r that make $|D - rI| = 0$.
$|D - rI| = (d_{11} - r)(d_{22} - r)\dots(d_{nn} - r) = 0$.

The only solutions to this equation are $r \in \{d_{11}, d_{22}, \dots, d_{nn}\}$, the diagonal entries of D.

23.4 The eigenvalues of A, an upper triangular matrix, are the solutions to $\det(A - rI) = 0$,

$$\det(A - rI) = \begin{vmatrix} a_{11} - r & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} - r & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} - r & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} - r \end{vmatrix}$$

$$= (a_{11} - r) \begin{vmatrix} a_{22} - r & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} - r & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} - r \end{vmatrix}$$

$$= (a_{11} - r)(a_{22} - r)\cdots(a_{nn} - r) = 0,$$

by the inductive hypothesis. The only solutions to this equation are $r \in \{a_{11}, a_{22}, \ldots, a_{nn}\}$, which are the diagonal entries. The same proof works for a lower-triangular matrix, by expanding across the top row each time (rather than down the first column as above).

23.5 $(A - rI)\mathbf{v} = 0 \iff A\mathbf{v} = rI\mathbf{v} = r\mathbf{v}$

$$\iff A^{-1}A\mathbf{v} = A^{-1}r\mathbf{v} = rA^{-1}\mathbf{v} \iff \mathbf{v} = rA^{-1}\mathbf{v}$$

$$\iff \frac{1}{r}\mathbf{v} = A^{-1}\mathbf{v} \iff \left(A^{-1}\mathbf{v} - \frac{1}{r}\mathbf{v}\right) = 0 \iff \left(A^{-1} - \frac{1}{r}\right)\mathbf{v} = 0.$$

So, if r is an eigenvalue of A, then $1/r$ is an eigenvalue of A^{-1}. Similarly, if $1/r$ is an eigenvalue of A^{-1}, $1/(1/r)$ is an eigenvalue of $(A^{-1})^{-1} = A$. Therefore, r is an eigenvalue of $A \iff 1/r$ is an eigenvalue of A^{-1}.

23.6 $\begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and

$$\begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = -1\begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

23.7 *a*) $A = \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} \Longrightarrow \lambda = 3, \lambda = 2 \Longrightarrow v_1 - v_2 = 0,$

$$v_1 = 0 \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},\ P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

$$D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

b) $A = \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix}$; $\quad (-\lambda)(5 - \lambda) + 1 = \lambda^2 - 5\lambda + 1 = 0$

$$\Longrightarrow \lambda = \frac{5 \pm \sqrt{25 - 4}}{2} = \frac{5 + \sqrt{21}}{2}, \frac{5 - \sqrt{21}}{2}.$$

$$-\left(\frac{5 + \sqrt{21}}{2}\right)v_1 + v_2 = 0 \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ \dfrac{5 + \sqrt{21}}{2} \end{pmatrix},$$

$$-\left(\frac{5 - \sqrt{21}}{2}\right)v_1 + v_2 = 0 \Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ \dfrac{5 - \sqrt{21}}{2} \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 1 \\ \dfrac{5+\sqrt{21}}{2} & \dfrac{5-\sqrt{21}}{2} \end{pmatrix} \text{ and } D = \begin{pmatrix} \dfrac{5+\sqrt{21}}{2} & 0 \\ 0 & \dfrac{5-\sqrt{21}}{2} \end{pmatrix}.$$

c) $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$,

$$(1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 3)(\lambda - 2), \quad \lambda_i = 3, 2.$$

$-2v_1 - v_2 = 0 \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$; and

$-v_1 - v_2 = 0 \Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

d) $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix} : \Longrightarrow$

$$\begin{aligned} &(3-r)[(2-r)(3-r) - 1 - 1] \\ &= (3-r)(r^2 - 5r + 4) \\ &\Longrightarrow r_1 = 3, r_2 = 4, r_3 = 1. \end{aligned}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_2 = 0, -v_1 - v_3 = 0$$

$$\Longrightarrow v_1 = -v_3 \Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} -1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_1 = -v_2, -v_2 = v_3$$

$$\Longrightarrow -(-v_2) - 2v_2 - (-v_2) = 0$$

$$\Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_2 = 2v_1, v_2 = 2v_3$$

$$\Longrightarrow v_1 = v_3, -v_1 + 2v_1 - v_1 = 0$$

$$\Longrightarrow \mathbf{v}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$e)\quad A = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \Longrightarrow \begin{aligned} &(1-r)\begin{vmatrix} 4-r & -2 \\ 1 & 1-r \end{vmatrix} \\ &= (1-r)[(4-r)(1-r)+2] \\ &= (1-r)(r-3)(r-2) = 0 \end{aligned}$$

$$\Longrightarrow r_1 = 1, r_2 = 2, r_3 = 3.$$

$$\begin{pmatrix} 3 & -2 & -2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_1 = 0, -2v_2 - 2v_3 = 0$$

$$\Longrightarrow \mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & -2 & -2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_2 = 0, v_1 = v_3$$

$$\Longrightarrow \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & -2 & -2 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \Longrightarrow v_2 = 0, v_1 = 2v_3$$

$$\Longrightarrow \mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

$$P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

23.8 Using the solutions for 23.7:

$$a)\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 3^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 2^n \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$b)\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 \left(\tfrac{5+\sqrt{21}}{2}\right)^n \begin{pmatrix} 1 \\ \frac{5+\sqrt{21}}{2} \end{pmatrix} + c_2 \left(\tfrac{5-\sqrt{21}}{2}\right)^n \begin{pmatrix} 1 \\ \frac{5-\sqrt{21}}{2} \end{pmatrix}.$$

$$c)\quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = c_1 2^n \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 3^n \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

$$d)\quad \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1 3^n \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_2 4^n \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + c_3 1^n \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

e) $$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1 1^n \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_2 2^n \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 3^n \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

23.9 $$\begin{pmatrix} b_1 & b_2 & \cdots & b_{m-1} & b_m \\ 1-d_1 & 0 & \cdots & 0 & 0 \\ 0 & 1-d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1-d_{m-1} & 0 \end{pmatrix}$$

23.10 The general solution is given in (18). As $n \to \infty$, the first term in (18) dominates; it goes to ∞ while the second term stays bounded. As $n \to \infty$, the ratio of the size of first population to that of the second approaches 4-to-1.

23.11 $$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 4 & 0 \\ .5 & 0 & 0 \\ 0 & .2 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}.$$

The eigenvalues are $0, -1, 2$ with corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -10 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 40 \\ 10 \\ 1 \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = c_1 0^n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_2 (-1)^n \begin{pmatrix} -10 \\ 5 \\ -1 \end{pmatrix} + c_3 2^n \begin{pmatrix} 40 \\ 10 \\ 1 \end{pmatrix}.$$

23.12 *a*)
$$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}^{1/3} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3^{1/3} & 0 \\ 0 & 5^{1/3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3^{1/3} & 0 \\ 2 \cdot 5^{1/3} - 2 \cdot 3^{1/3} & 5^{1/3} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}^{1/2} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3^{1/2} & 0 \\ 0 & 5^{1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 3^{1/2} & 0 \\ 2 \cdot 5^{1/2} - 2 \cdot 3^{1/2} & 5^{1/2} \end{pmatrix}.$$

b)
$$\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}^{1/3} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1^{1/3} & 0 \\ 0 & 2^{1/3} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 - 2^{4/3} & 3 \cdot 2^{1/3} - 3 \\ 2 - 2^{4/3} & 3 \cdot 2^{1/3} - 2 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}^{1/2} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1^{1/2} & 0 \\ 0 & 2^{1/2} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 - 2^{3/2} & 3 \cdot 2^{1/2} - 3 \\ 2 - 2^{3/2} & 3 \cdot 2^{1/2} - 2 \end{pmatrix}.$$

c)
$$\begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}^{1/3} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1^{1/3} & 0 \\ 0 & -2^{1/3} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 2^{1/3} - 2 & 2 - 2^{4/3} \\ 2^{1/3} - 1 & 1 - 2^{4/3} \end{pmatrix}.$$

$A^{1/2}$ is not defined since A has negative eigenvalues.

d)
$$\begin{pmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{pmatrix}^{1/3}$$
$$= \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 7^{1/3} & 0 & 0 \\ 0 & -1^{1/3} & 0 \\ 0 & 0 & -2^{1/3} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 0 & 2 \end{pmatrix}.$$

$A^{1/2}$ is not defined since A has negative eigenvalues.

23.13 Suppose $P = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n)$ is invertible. Then, $A\mathbf{v}_i = r_i\mathbf{v}_i$ for $i = 1, \ldots, n$

$$\iff A(\mathbf{v}_1 \ \cdots \ \mathbf{v}_n) = (\mathbf{v}_1 \ \cdots \ \mathbf{v}_n)\begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix}$$
$$\iff AP = PD \iff P^{-1}AP = D.$$

23.14 *a*) $3 + 5 = 3 = 5$ and $3 \cdot 5 = 15$.

b) $4 - 1 = 2 + 1$ and $(-1)4 - (-2)3 = 2 = 2 \cdot 1$.

c) $0 - 3 = -1 - 2$ and $-(-2 \cdot 1) = 2 = (-1) \cdot (-2)$.

d) $0 + 7 - 3 = 7 - 1 - 2$ and $14 = 7 \cdot (-1) \cdot (-2)$.

23.15 *a*) Columns sum to 1, so $r_1 = 1$ is an eigenvalue. By Theorem 23.9*a*, $r_1 + r_2 = 0.8$; so $r_2 = -0.2$.

b) Columns are equal, so $r_1 = 0$ is an eigenvalue. By Theorem 23.9*a*, $r_1 + r_2 = 2$; so $r_2 = 2$.

c) Columns sum to 2, so $r_1 = 2$ is an eigenvalue. By inspection, $r_2 = -1$ is a second eigenvalue and it has multiplicity two. Eigenvalues are $2, -1, -1$.

d) By inspection, $r_1 = 3$ is an eigenvalue and so is $r_2 = 0$. By Theorem 23.9*a*, $\sum r_i = 6$; so $r_3 = 3$, too.

23.16 *a*) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$: $(3-r)(1-r) + 1 = r^2 - 4r + 4 = (r-2)^2 \Longrightarrow r = 2, 2 \Longrightarrow v_1 + v_2 = 0 \Longrightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Longrightarrow \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow P = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

b) $\begin{pmatrix} -5 & 2 \\ -2 & -1 \end{pmatrix}$: $(-5-r)(-1-r) + 4 = r^2 + 6r + 9 \Longrightarrow r = -3, -3 \Longrightarrow -2v_1 + 2v_2 = 0 \Longrightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\begin{pmatrix} -2 & 2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \mathbf{w} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Longrightarrow P = \begin{pmatrix} 1 & 0 \\ 1 & 1/2 \end{pmatrix}.$$

c) $\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}$ $(3-r)(-3-r) + 9 = r^2 - 9 + 9 = r^2 \Longrightarrow r = 0, 0 \Longrightarrow 3v_1 + 3v_2 = 0 \Longrightarrow \mathbf{v} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$.

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} \Longrightarrow \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Longrightarrow P = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}.$$

23.17 *a*) $\mathbf{z}_{n+1} = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}\mathbf{z}_n$;

$$\mathbf{z}_n = (c_0 2^n + nc_1 2^{n-1})\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_1 2^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

b) $\mathbf{z}_{n+1} = \begin{pmatrix} -5 & 2 \\ -2 & -1 \end{pmatrix}\mathbf{z}_n$;

$$\mathbf{z}_n = (c_0(-3)^n + nc_1(-3)^{n-1})\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1(-3)^n \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}.$$

c) $\mathbf{z}_{n+1} = \begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix}\mathbf{z}_n$;

$$\mathbf{z}_n = P\mathbf{y}_n = \begin{pmatrix} 3 & 1 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} k_0 t + k_1 \\ k_0 \end{pmatrix} = \begin{pmatrix} 3k_0 t + (3k_1 + k_0) \\ -3k_0 t - 3k_1 \end{pmatrix}.$$

23.18 Verify $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ where $P = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 1 & 4 & -3 \\ 1 & 1 & 0 \end{pmatrix}.$$

Find P^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right).$$

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 & -4 \\ 1 & 4 & -3 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

23.19 $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \qquad r_1 = r_2 = 3, \qquad v_1 + v_2 = 0.$

a) Let $\mathbf{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Longrightarrow w_1 + w_2$

$= -1, -w_1 - w_2 = 1.$

Let $\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Longrightarrow P = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \Longrightarrow P^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$

$$P^{-1}AP = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

b) Let $\mathbf{v} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$. $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} \Longrightarrow w_1 + w_2 = 4,$
$-w_1 - w_2 = -4.$

Let $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Longrightarrow P = \begin{pmatrix} 4 & 2 \\ -4 & 2 \end{pmatrix} \Longrightarrow P^{-1} = \frac{1}{16}\begin{pmatrix} 2 & -2 \\ 4 & 4 \end{pmatrix}$.

$$P^{-1}AP = \begin{pmatrix} 1/8 & -1/8 \\ 1/4 & 1/4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} 5/8 & -1/8 \\ 3/4 & 3/4 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

23.20 The general solution is $\mathbf{z}_n = c_0 r_1^n \mathbf{v}_1 + nc_1 r_1^{n-1}\mathbf{v}_1 + c_1 r_1^n \mathbf{v}_2$. Since $|r_1| < 1$, $r_1^n \longrightarrow 0$ and the first and last terms in $\mathbf{z}_n$ tend to $\mathbf{0}$. What happens to nr_1^{n-1} as $n \longrightarrow \infty$? The first factor $\longrightarrow \infty$, but second $\longrightarrow 0$. Using L'Hôpital's rule,

$$\lim_{n\to\infty} nr_1^{n-1} = \lim_{n\to\infty} \frac{n}{r_1^{1-n}} = \lim_{n\to\infty} \frac{1}{(1-n)r_1^{-n}} = \lim_{n\to\infty} \frac{r_1^n}{(1-n)} \longrightarrow 0.$$

23.21 The characteristic equation is $\begin{vmatrix} 3-r & 1 & 1 \\ 1 & 2-r & 1 \\ -1 & -1 & 1-r \end{vmatrix} = -(r-2)^3 = 0.$

To find eigenvectors for $r = 2$, solve $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Since the rank of this matrix is two, the solution set is generated by $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix} \Longrightarrow P^{-1} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

$$P^{-1}CP = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

23.22 $\begin{vmatrix} 4-r & 0 & 0 \\ -1 & 4-r & 2 \\ 0 & 0 & 4-r \end{vmatrix} = (4-r)^3 = 0.$

To find the corresponding eigenvalue(s),

$$(C - 4I)\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So, $2v_3 = v_1$ and v_2 is anything, e.g., $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

$$(C - 4I)^2\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

this implies that $\mathbf{v}_3$ can be anything, e.g., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}.$$

23.23 If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are independent eigenvectors for eigenvalue r^* and 3×3 matrix A, let $P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3)$ and let $D = r^*I$. Since $A\mathbf{v}_i = r^*\mathbf{v}_i$ for $i = 1, 2, 3$, $AP = PD$. Since P is invertible (by the independence of the $\mathbf{v}_i$s), $P^{-1}AP = D = r^*I$. Then, $A = P^{-1}(r^*I)P = r^*(P^{-1}IP) = r^*I$.

23.24 $\begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} r_1 & 1 & 0 \\ 0 & r_1 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \begin{pmatrix} r_3 & 0 & 0 \\ 0 & r_1 & 1 \\ 0 & 0 & r_1 \end{pmatrix}, \begin{pmatrix} r_1 & 1 & 0 \\ 0 & r_1 & 1 \\ 0 & 0 & r_1 \end{pmatrix}$, where any of r_1, r_2, r_3 may be equal.

23.25 *a*) Example 14: $\mathbf{z}_n = c_1 2^n \mathbf{v}_1 + (c_2 3^n + nc_3 3^{n-1})\mathbf{v}_2 + c_3 3^n \mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

b) Exercise 21:

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \left[c_1 2^n + nc_2 2^{n-1} + \frac{n(n-1)}{2} c_3 2^{n-2} \right] \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + (c_2 2^n + nc_3 2^{n-1}) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + c_3 2^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

c) Exercise 22: $\mathbf{z}_n = c_1 4^n \mathbf{v}_1 + (c_2 4^n + nc_3 4^{n-1})\mathbf{v}_2 + c_3 4^n \mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

23.26 *a*) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$. *b*) $\begin{pmatrix} -3 & 1 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. *c*) $\begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$.

d) $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 5 & 0 & 1 \end{pmatrix}$. *e*) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. *f*) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

23.28 *a*)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \sqrt{13}^n \left[(c_1 \cos n\theta - c_2 \sin n\theta) \begin{pmatrix} 1 \\ -2 \end{pmatrix} - (c_2 \cos n\theta + c_1 \sin n\theta) \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right]$$

where $\cos\theta = 2/\sqrt{13}$.

b)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 2^n \left\{ [c_1 \cos(n\pi/2) - c_2 \sin(n\pi/2)] \begin{pmatrix} -2 \\ 1 \end{pmatrix} - [c_1 \sin(n\pi/2) + c_2 \cos(n\pi/2)] \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}.$$

$c)$ $$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = 5^{n/2}\left[(c_1 \cos n\theta - c_2 \sin n\theta)\begin{pmatrix} -5 \\ 1 \end{pmatrix} - (c_2 \cos n\theta + c_1 \sin n\theta)\begin{pmatrix} 0 \\ -2 \end{pmatrix}\right]$$

where $\cos\theta = 1/\sqrt{5}$.

$d)$ $$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = 2^n\left\{[c_1\cos(n\pi/2) - c_2\sin(n\pi/2)]\begin{pmatrix} 2 \\ -5 \\ -2 \end{pmatrix} - [c_1\sin(n\pi/2) + c_2\cos(n\pi/2)]\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}\right\} + c_3 2^n\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}.$$

$e)$ $$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = 2^n\left\{\left[c_1\cos\frac{n\pi}{2} - c_2\sin\frac{n\pi}{2}\right]\begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} - \left[c_1\sin\frac{n\pi}{2} + c_2\cos\frac{n\pi}{2}\right]\begin{pmatrix} 0 \\ -\frac{2}{3} \\ 0 \end{pmatrix}\right\} + c_3 2^n\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}.$$

23.29 Show $P^{-1}AP = \begin{pmatrix} 1+3i & 0 \\ 0 & 1-3i \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}$,

$P = \begin{pmatrix} 1 & 1 \\ 3i & -3i \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6}i \\ \frac{1}{2} & \frac{1}{6}i \end{pmatrix}$.

$$\begin{aligned} P^{-1}AP &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{6}i \\ \frac{1}{2} & \frac{1}{6}i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3i & -3i \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}+\frac{3}{2}i & \frac{1}{2}-\frac{1}{6}i \\ \frac{1}{2}-\frac{3}{2}i & \frac{1}{2}+\frac{1}{6}i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3i & -3i \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}+\frac{3}{2}i+\frac{3}{2}i+\frac{1}{2} & \frac{1}{2}+\frac{3}{2}i-\frac{3}{2}i-\frac{1}{2} \\ \frac{1}{2}-\frac{3}{2}i+\frac{3}{2}i-\frac{1}{2} & \frac{1}{2}-\frac{3}{2}i-\frac{3}{2}i+\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} 1+3i & 0 \\ 0 & 1-3i \end{pmatrix}. \end{aligned}$$

23.30 $a)$ $c_1\begin{pmatrix} 5 \\ 3 \end{pmatrix} + c_2(.2)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$b)$ $c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2(.5)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$$c)\ c_1\begin{pmatrix}2\\2\\1\end{pmatrix}+2^{-n/2}\left(\left[c_1\cos(7n\pi/4)-c_2\sin(7n\pi/4)\right]\begin{pmatrix}-1\\1\\0\end{pmatrix}\right.$$

$$\left.-\left[c_2\cos(7n\pi/4)+c_1\sin(7n\pi/4)\right]\begin{pmatrix}0\\-1\\1\end{pmatrix}\right).$$

23.31 White males: $y = .002x + .864y = .002(1-y) + .864y \Longrightarrow y = .0145$ or 1.45%.

Black males: $y = .004(1-y) + .898y$ or $y = .0377$ or 3.77%.

23.32 Columns of M add to 1; that is, $\sum_j m_{ij} = 1$.
Let $\mathbf{p}$ be a probability vector, so $p_i \geq 0$ for all i and $\sum_i p_i = 1$. $(M\mathbf{p})_i = \sum_j m_{ij}p_j$. So,

$$\sum_i (M\mathbf{p})_i = \sum_i\sum_j m_{ij}p_j = \sum_j p_j \sum_i m_{ij} = \sum_j p_j\cdot 1 = \sum p_j = 1.$$

23.33 *a*) x_n = probability nth day is sunny.
y_n = probability nth day is cloudy without rain.
z_n = probability nth day is rainy.

$$\begin{pmatrix}x_{n+1}\\y_{n+1}\\z_{n+1}\end{pmatrix}=\begin{pmatrix}0.5&0.25&0\\0.5&0.25&0.5\\0&0.5&0.5\end{pmatrix}\begin{pmatrix}x_n\\y_n\\z_n\end{pmatrix}.$$

$$b)\ M^2=\begin{pmatrix}0.5&0.25&0\\0.5&0.25&0.5\\0&0.5&0.5\end{pmatrix}^2=\begin{pmatrix}3/8&3/16&1/8\\3/8&7/16&3/8\\1/4&3/8&1/2\end{pmatrix}.$$

c) Eigenvector $\mathbf{v}$ for $r = 1$:

$$\begin{pmatrix}-0.5&0.25&0\\0.5&-0.75&0.5\\0&0.5&-0.5\end{pmatrix}\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Longrightarrow \mathbf{v}=\begin{pmatrix}1\\2\\2\end{pmatrix}.$$

Long-run distribution vector is $\begin{pmatrix}0.2\\0.4\\0.4\end{pmatrix}$.

23.34 Stock A: $\begin{pmatrix}0.75&0.1\\0.25&0.9\end{pmatrix}\Longrightarrow\begin{pmatrix}-0.25&0.1\\0.25&-0.1\end{pmatrix}\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$

Stock A will sell for \$5 $\frac{2}{7}$ of the time and for \$10 $\frac{5}{7}$ of the time.
Its expected value is $\$5 \cdot (2/7) + \$10 \cdot (5/7) = \$8\frac{4}{7}$.

Stock B: $\begin{pmatrix} 0.9 & 0.3 \\ 0.1 & 0.7 \end{pmatrix} \Longrightarrow \begin{pmatrix} -0.1 & 0.3 \\ 0.1 & -0.3 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Stock B will sell for \$6 $\frac{3}{4}$ of the time and for \$12 $\frac{1}{4}$ of the time.
Its expected value is $\$6 \cdot (3/4) + \$12 \cdot (1/4) = \$7.5$.

23.35 $\begin{pmatrix} 1 & 4 \\ .5 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ .5 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ .5 & 2 \end{pmatrix}$, strictly positive. Eigenvalues are 2 and -1. An eigenvector for $r = 2$ is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

23.36 $\begin{pmatrix} 0 & 0 & 1.6 \\ .8 & 0 & 0 \\ 0 & .4 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1.6 \\ .8 & 0 & 0 \\ 0 & .4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0.64 & 0 \\ 0 & 0 & 1.28 \\ .32 & 0 & 0 \end{pmatrix}$.

Every multiple of the matrix will have only three nonzero entries; so it's not regular. However, it still satisfies all the properties listed under Theorem 23.15.

23.37 *a*) Eigenvalues by inspection are $-2, 6$. Corresponding normalized eigenvectors are:

$$\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix},$$

an orthonormal set. Take $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

b) Eigenvalues are 0, 5. Corresponding normalized eigenvectors are:

$$\begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix},$$

an orthonormal set. Take $Q = \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$.

c) $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$.

d) Eigenvalues are $1, 0, 3$. $Q = \begin{pmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$.

e) Eigenvalues are $0, 3, 3$. $Q = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$.

f) Eigenvalues are $1, 4, 3$. $Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}$.

23.38 *a*) Eigenvalues are $3, 1$. $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

b) Eigenvalues are $4, -2$. $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

c) Eigenvalues are $0, 3, 3$. One Q is $\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix}$.

23.39 Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ are mutually orthogonal. Suppose $c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$. Take the dot product with $\mathbf{v}_i$ on each side: $c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + \cdots + c_i(\mathbf{v}_i \cdot \mathbf{v}_i) + \cdots + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) = 0$.

Since the $\mathbf{v}_i$'s are orthogonal, $\mathbf{v}_j \cdot \mathbf{v}_i = 0$ for $j \neq i$. Therefore, $c_i(\mathbf{v}_i \cdot \mathbf{v}_i) = 0$.

Since $\mathbf{v}_i \cdot \mathbf{v}_i \neq 0$, c_i must equal zero.

Since all c_i's are zero, $\mathbf{v}_i$s are linearly independent.

23.40 $A^T = A^{-1} \Longrightarrow \det A = 1/\det A \Longrightarrow (\det A)^2 = 1$.

23.41 Suppose $\mathbf{w}_1, \dots, \mathbf{w}_k$ are mutually orthogonal eigenvectors of symmetric matrix A. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be the corresponding vectors of length 1; that is, $\mathbf{v}_i = \mathbf{w}_i/\|\mathbf{w}_i\|$.
Let P be the $k \times k$ matrix $P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k)$. Then,

$$P^TP = \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_k \end{pmatrix} (\mathbf{v}_1 \quad \cdots \quad \mathbf{v}_k) = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_1 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_1 \cdot \mathbf{v}_k \\ \mathbf{v}_2 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_2 \cdot \mathbf{v}_k \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_k \cdot \mathbf{v}_1 & \mathbf{v}_k \cdot \mathbf{v}_2 & \cdots & \mathbf{v}_k \cdot \mathbf{v}_k \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

23.42 In (49),

$$Q^TQ = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

(53) is a similar calculation.

23.43 Straightforward.

23.44
$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} \cdot \mathbf{w}_1$$
$$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \left[\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \Bigg/ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right] \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \cdot \mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \cdot \mathbf{w}_2$$
$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1/2}{3/2} \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}.$$

23.45
$$\mathbf{w}_1 \cdot \mathbf{w}_2 = \mathbf{w}_1 \cdot \left(\mathbf{v}_2 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 \right)$$
$$= \mathbf{w}_1 \cdot \mathbf{v}_2 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_2}{\mathbf{w}_1 \cdot \mathbf{w}_1} (\mathbf{w}_1 \cdot \mathbf{w}_1)$$
$$= \mathbf{w}_1 \cdot \mathbf{v}_2 - \mathbf{w}_1 \cdot \mathbf{v}_2$$
$$= 0.$$

$$\begin{aligned}\mathbf{w}_1 \cdot \mathbf{w}_3 &= \mathbf{w}_1 \cdot \left(\mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \cdot \mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \cdot \mathbf{w}_2\right) \\ &= \mathbf{w}_1 \cdot \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}(\mathbf{w}_1 \cdot \mathbf{w}_2) \\ &= \mathbf{w}_1 \cdot \mathbf{v}_3 - \mathbf{w}_1\mathbf{v}_3 - 0 \\ &= 0. \\ \mathbf{w}_2 \cdot \mathbf{w}_3 &= \mathbf{w}_2 \cdot \left(\mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1} \cdot \mathbf{w}_1 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2} \cdot \mathbf{w}_2\right) \\ &= \mathbf{w}_2 \cdot \mathbf{v}_3 - \frac{\mathbf{w}_1 \cdot \mathbf{v}_3}{\mathbf{w}_1 \cdot \mathbf{w}_1}(\mathbf{w}_2 \cdot \mathbf{w}_1) - \frac{\mathbf{w}_2 \cdot \mathbf{v}_3}{\mathbf{w}_2 \cdot \mathbf{w}_2}(\mathbf{w}_2 \cdot \mathbf{w}_2) \\ &= \mathbf{w}_2 \cdot \mathbf{v}_3 - 0 - \mathbf{w}_2 \cdot \mathbf{v}_3 = 0.\end{aligned}$$

23.46 Suppose $P^TAP = D_1$ and $P^TBP = D_2$ with D_1, D_2 diagonal. Then,

$$A = (P^T)^{-1}D_1P^{-1} = PD_1P^{-1} \quad \text{and} \quad B = (P^T)^{-1}D_2P^{-1} = PD_2P^{-1},$$

since $P^T = P^{-1}$. So,

$$\begin{aligned}AB &= PD_1P^{-1}PD_2P^{-1} = PD_1D_2P^{-1} = PD_2D_1P^{-1} \\ &= PD_2P^{-1}PD_1P^{-1} = BA,\end{aligned}$$

since diagonal matrices commute.

23.48 If $\mathbf{x}^TA\mathbf{x} > 0$ and $\mathbf{x}^TB\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^TA\mathbf{x} + \mathbf{x}^TB\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

23.49 *a*) Let A be a symmetric matrix. Then, the following are equivalent:

i) A is negative definite,

ii) There is a nonsingular B such that $A = -B^TB$,

iii) There is a nonsingular Q such that $Q^TAQ = -I$.

Proof: A is negative definite if and only if $-A$ is positive definite. Apply Theorem 23.18 to find a nonsingular B such that $B^TB = -A$ and a nonsingular Q such that $Q^T(-A)Q = I$; i.e., $Q^TAQ = -I$. The converse arguments work the same way.

b) Let A be a symmetric matrix. Then, the following are equivalent:

i) A is positive semidefinite,

ii) There is a matrix B such that $A = B^TB$,

iii) There is a matrix Q such that $Q^TAQ = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$.

Proof: There is a matrix P of eigenvectors such that (56) and (57) hold, where $r_1, \ldots, r_k$ are the nonnegative eigenvalues of A. The proof that $a \Longleftrightarrow b$ is similar to the proof on page 628, except that $\mathbf{x}^T A \mathbf{x}$ may be zero for nonzero $\mathbf{x}$.

The proof that $a \Longrightarrow c$ is similar to that in Theorem 18, but with

$$Q = \left(\frac{1}{\sqrt{r_1}}\mathbf{v}_1 \quad \cdots \quad \frac{1}{\sqrt{r_h}}\mathbf{v}_h \quad \mathbf{0} \quad \cdots \quad \mathbf{0} \right)$$
$$= P \begin{pmatrix} \frac{1}{\sqrt{r_1}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\sqrt{r_h}} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix},$$

where we order the k eigenvalues of A so that $r_1, \ldots, r_h$ are the strictly positive eigenvalues of A and $r_{h+1} = r_k = 0$.

23.50 For example, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = B^T B$ for $B = \begin{pmatrix} \sqrt{\frac{1}{2}} & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}$ and for $B = \begin{pmatrix} -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{3}{2}} & \sqrt{\frac{3}{2}} \end{pmatrix}$.

23.51 Let $L = \text{diag}\,\{r_1, \ldots, r_k\}$ and let $R = \text{diag}\,\{1/\sqrt{r_1}, \ldots, 1/\sqrt{r_k}\}$, so that $Q = PR$. Then,

$$Q^T A Q = R^T P^T A P R = R^T L R = I.$$

In order for R to be defined, all the eigenvalues r_i must be positive, a condition equivalent to A's being positive definite.

23.52 Let $T_0 = \min \mathbf{v}^T A \mathbf{v}$ for $\|\mathbf{v}\| = 1$. If $T_0 > 0$, A is positive definite and we can take $t = 0$. Suppose that $T_0 \leq 0$. Choose $t > -T_0$ so that $T_0 + t > 0$. For any vector $\mathbf{v}$ with $\|\mathbf{v}\| = 1$,

$$\begin{aligned} \mathbf{v}^T (A + tI)\mathbf{v} &= \mathbf{v}^T A \mathbf{v} + t\,\mathbf{v}^T I \mathbf{v} \\ &= \mathbf{v}^T A \mathbf{v} + t\|\mathbf{v}\|^2 \\ &> T_0 + t \\ &> 0. \end{aligned}$$

For any vector $\mathbf{w}$ write $\mathbf{w} = r\mathbf{v}$ where $r = \|\mathbf{w}\|$ and $\mathbf{v} = \mathbf{w}/(\|\mathbf{w}\|)$. Then, $\mathbf{v}^T(A + tI)\mathbf{v} > 0$ since $\|\mathbf{v}\| = 1$ and

$$\mathbf{w}^T(A + tI)\mathbf{w} = (r\mathbf{v})^T(A + tI)(r\mathbf{v}) = r^2(\mathbf{v}^T(A + tI)\mathbf{v}) > 0.$$

23.53 Proof of Theorem 23.5: Section 23.9 contains the proof for $h = 1, 2$. Suppose the theorem is true for $h - 1$ distinct eigenvalues. Suppose that $r_1, \ldots, r_h$ are h distinct real eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \ldots \mathbf{v}_h$. $A\mathbf{v}_i = r_i\mathbf{v}_i$, for $i = 1, \ldots, h$. Suppose

$$c_1\mathbf{v}_1 + \cdots + c_h\mathbf{v}_h = \mathbf{0}. \qquad (*)$$

Then,

$$c_1A\mathbf{v}_1 + \cdots + c_hA\mathbf{v}_h = A\mathbf{0},$$
$$c_1r_1\mathbf{v}_1 + \cdots c_hr_h\mathbf{v}_h = \mathbf{0}.$$

Multiply (∗) through by r_1:

$$c_1r_1\mathbf{v}_1 + \cdots + c_hr_1\mathbf{v}_h = \mathbf{0},$$

and subtract the previous two equations:

$$c_2(r_2 - r_1)\mathbf{v}_2 + \cdots + c_h(r_h - r_1)\mathbf{v}_h = \mathbf{0}.$$

By the inductive hypothesis, $\mathbf{v}_2, \ldots, \mathbf{v}_h$ are linearly independent vectors. Therefore, $c_2 = \cdots = c_h = 0$. Then, substituting into (∗), we find that $c_1 = 0$ too, and conclude that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_h$ are linearly independent.

23.54 Let $B = A - rI$ be the matrix in (66). One proves easily by induction that each term of det B is a product of entries with exactly one entry from each row of B and one entry from each column of B and that every such combination occurs in the expansion of det B. See Exercise 26.34. One such term in the expansion of det B is the product of its diagonal entries: $(a_{11} - r)\cdots(a_{kk} - r)$. Any other term can contain at most $k - 2$ diagonal entries $a_{jj} - r$, since if a term contains $k - 1$ diagonal entries in its product, the pattern dictates that it contain all k diagonal entries. A term that contains exactly j diagonal entries $(a_{ii} - r)$ of B will be a jth order polynomial in r.

23.55 By the previous exercise, the only term with r^{k-1} in it arises from the product of the diagonal entries of B:

$$P_k(r) = (a_{11} - r)\cdots(a_{kk} - r).$$

We prove by induction on the number of factors in $P_k(r)$ that

$$P_k(r) = \pm(r^k - (a_{11} + \cdots + a_{kk})r^{k-1}) + \text{ terms of order lower than } k-1 \text{ in } r.$$

This statement is easily seen to be true for $k = 1, 2$. Assume it is true for $P_{k-1}(r)$. Then,

$$\begin{aligned} P_k(r) &= (a_{kk} - r) \cdot P_{k-1}(r) \\ &= (a_{kk} - r)\Big[\pm(r^{k-1} - (a_{11} + \cdots + a_{k-1,k-1})r^{k-2}) + g(r)\Big] \end{aligned}$$

by the inductive hypothesis, where g is of order $\leq k - 3$ in r. Multiplying out the above expression yields:

$$\begin{aligned} P_k(r) &= \mp r(r^{k-1} - (a_{11} + \cdots + a_{k-1,k-1})r^{k-2} + g(r)) \\ &\quad \pm a_{kk}(r^{k-1} - (a_{11} + \cdots + a_{k-1,k-1})r^{k-2} + g(r)) \\ &= \mp r^k \pm (a_{11} + \cdots + a_{k-1,k-1} + a_{kk})r^{k-1} \\ &\quad \mp [rg(r) + (a_{11} + \cdots + a_{k-1,k-1})a_{kk}r^{k-2} - a_{kk}g(r)], \end{aligned}$$

where we are careful with the order of g but not its sign. Since g is of order at most $k - 3$ in r, the expression in square brackets has order at most $k - 2$ in r. The coefficient of r^{k-1} is $\pm(a_{11} + \cdots + a_{kk})$.

23.57 $\det(A - xI) = \pm\Pi_i(x - r_i) \equiv P_k(x)$, where $r_1, \ldots, r_k$ are the eigenvalues of A. We want to prove that the coefficient of x^{k-j} in $P_k(x)$ is the sum of all j-fold products of $\{r_1, \ldots, r_k\}$. We use induction on k. It is easily seen to be true for $k = 1, 2$. We assume it true for $k - 1$:

$$P_{k-1}(x) = (x - r_1)\cdots(x - r_{k-1}) = \sum_{j=0}^{k-1} b_j^{k-1} x^{k-j},$$

where $b_j^{k-1} = (-1)^j$ times the sum of all j–fold products of $r_1, \cdots, r_{k-1}$. Then,

$$\begin{aligned} P_k(x) &= (x - r_k) \cdot P_{k-1}(x) = (x - r_k) \sum_{j=0}^{k-1} b_j^{k-1} x^{k-j} \\ &= \sum_{j=0}^{k-1} b_j^{k-1} x^{k-j+1} - \sum_{j=0}^{k-1} r_k b_j^{k-1} x^{k-j} \\ &= \sum_{j=1}^{k} b_{j-1}^{k-1} x^{k-j} - \sum_{j=0}^{k-1} (r_k b_j^{k-1}) x^{k-j}. \end{aligned}$$

The coefficient b_j^k of x^{k-j} in $P_k(x)$ is 1 times the coefficient of x^{k-j-1} in $P_{k-1}(x)$ minus r_k times the coefficient of x^{k-j} in $P_{k-1}(x)$. The first term gives all j-fold products of $r_1, \dots, r_{k-1}$; the second term gives all j-fold products of $r_1, \dots, r_k$, that include r_k. Together, this gives all j-fold products of $r_1, \dots, r_k$.

23.58 Simply carry out the multiplication to find

$$\det\begin{pmatrix} a_{11}-x & a_{12} & a_{13} \\ a_{21} & a_{22}-x & a_{23} \\ a_{31} & a_{32} & a_{33}-x \end{pmatrix} = -x^3 + (a_{11}+a_{22}+a_{33})x^2$$

$$-\left(\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}\right)x + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

and

$$(r_1-x)(r_2-x)(r_3-x)$$
$$= -x^3 + (r_1+r_2+r_3)x^2 - (r_1r_2 + r_1r_3 + r_2r_3)x + r_1r_2r_3.$$

Chapter 24

24.1 *a)* $\dot{y} = \dfrac{2(c-t^2)+2t\cdot 2t}{(c-t^2)^2} = \dfrac{2}{c-t^2} + \dfrac{4t^2}{(c-t^2)^2}$ and

$$y^2 + \frac{y}{t} = \frac{4t^2}{(c-t^2)^2} + \frac{2}{c-t^2}.$$

b) $\dot{y} = \dfrac{(ct^2-1)^{1/2} - ct^2(ct^2-1)^{-1/2}}{ct^2-1} = \dfrac{-1}{(ct^2-1)^{3/2}} - \dfrac{y^3}{t^3}$

$$= -\frac{1}{(ct^2-1)^{3/2}}.$$

c) $\dot{y} = ce^t + 2t + 2$ and $y - t^2 = ce^t + 2t + 2$.

d) $\dot{y} = \frac{1}{4}(4t+6) + c_1e^t + 2c_2e^{2t}$ and $\ddot{y} = 1 + c_1e^t + 4c_2e^{4t}$.

$$3\dot{y} - 2y + t^2 = 3t + \frac{9}{2} + 3c_1e^t + 6c_2e^{2t} - t^2 - 3t - \frac{14}{4}$$
$$- 2c_1e^t - 2c_2e^{2t} + t^2$$
$$= 1 + c_1e^t + 4c_2e^{2t} = \ddot{y}.$$

24.2 *a*)

$$
\begin{aligned}
y &= k_1 e^{2t} + k_2 e^t + 1 \\
\dot{y} &= 2k_1 e^{2t} + k_2 e^t \\
\ddot{y} &= 4k_1 e^{2t} + k_2 e^t.
\end{aligned}
$$

$$
\begin{aligned}
3\dot{y} - 2y + 2 &= (6k_1 e^{2t} + 3k_2 e^t) - (2k_1 e^{2t} + 2k_2 e^t - 2) + 2 \\
&= 4k_1 e^{2t} + k_2 e^t = \ddot{y}.
\end{aligned}
$$

b)

$$
\begin{aligned}
y &= k_1 \cos t + k_2 \sin t + k_3 e^t + k_4 e^{-t} \\
\ddot{y} &= -k_1 \cos t - k_2 \sin t + k_3 e^t + k_4 e^{-t} \\
y^{[4]} &= k_1 \cos t + k_2 \sin t + k_3 e^t + k_4 e^{-t} = y.
\end{aligned}
$$

24.3

$$
\begin{aligned}
y &= k_1 \cos \sqrt{a}t + k_2 \sin \sqrt{a}t \\
\dot{y} &= -\sqrt{a}k_1 \sin \sqrt{a}t + k_2\sqrt{a} \cos \sqrt{a}t \\
\ddot{y} &= -ak_1 \cos \sqrt{a}t - ak_2 \sin \sqrt{a}t = -ay.
\end{aligned}
$$

24.4 Initial stretching and initial velocity of the spring.

24.5 *a*) By (8), $y = ke^{-t} + 5$. $y(1) = ke^{-1} + 5 = 1$ implies $k = -4e$ and $y = -4e^{1-t} + 5$.

b) $\dot{y} = -y + t$; $a = -1$ and $b = t$ in (10).

$$
\begin{aligned}
y &= \left(k + \int^t se^s\,ds\right) e^{-t} \\
&= \left(k + te^t - e^t\right) e^{-t} \quad \text{by table or integration by parts} \\
&= ke^{-t} + t - 1.
\end{aligned}
$$

$y(1) = ke^{-1} = 1$ implies $k = e$ and $y = e^{1-t} + t - 1$.

c) $\dot{y} = y - t^2$; $a = 1$ and $b = -t^2$ in (10).

$$
\begin{aligned}
y &= \left(k + \int^t (-s^2)e^{-s}\,ds\right) e^t \\
&= \left(k + t^2e^{-t} + 2te^{-t} + 2e^{-t}\right) e^t \quad \text{by table or integration by parts} \\
&= ke^t + (t^2 + 2t + 2).
\end{aligned}
$$

$y(1) = ke + 5 = 1$ implies $k = -4e^{-1}$ and $y = -4e^{t-1} + t^2 + 2t + 2$.

d) $\frac{dy}{dt} = \frac{y^3}{t^3} \Longrightarrow \int y^{-3} dy = \int t^{-3} dt,$

$$-\frac{1}{2y^2} = -\frac{1}{2t^2} + c_0, \quad \frac{1}{y^2} = \frac{1}{t^2} + c_1 = \frac{1 + c_1 t^2}{t^2}, \quad y = \frac{\pm t}{\sqrt{1 + c_1 t^2}}.$$

$y(1) = \frac{1}{\sqrt{1 + c_1}} = 1$ implies $c_1 = 0$. So $y = t$.

e) $\frac{dy}{dt} = \frac{t^3}{y^3} \Longrightarrow \int y^3\, dy = \int t^3\, dt$. So

$$\frac{1}{4} y^4 = \frac{1}{4} t^4 + c_0, \qquad y^4 = t^4 + c_1, \qquad y = \pm (t^4 + c_1)^{1/4}.$$

$y(1) = (1 + c_1)^{1/4} = 1 \Longrightarrow c_1 = 0$. So $y = t$.

f) $\dot{y} = \left(1 - t^{-1}\right) y + t^{-1} e^{2t}$. So $a = 1 - t^{-1}$ and $b = t^{-1} e^{2t}$ in (10).

$$\begin{aligned} y &= \left(k + \int^t s^{-1} e^{2s} e^{-\int^s (1-u^{-1})\, du} ds\right) e^{\int^t (1-s^{-1})\, ds} \\ &= \left(k + \int^t s^{-1} e^{2s} e^{-s + \ln s}\, ds\right) e^{t - \ln t} \\ &= \left(k + \int^t s^{-1} e^{2s} e^{-s} \cdot s\, ds\right) t^{-1} e^t \\ &= (k + e^t) t^{-1} e^t = k t^{-1} e^t + t^{-2} e^{2t}. \end{aligned}$$

$y(1) = ke + e^2 = 1 \Longrightarrow k = \frac{1 - e^2}{e} = e^{-1} - e.$

So $y = t^{-1}(e^{t-1} - e^{1+t} + e^{2t})$.

24.6

$$\begin{aligned} y &= \left(k + \int^t b(s) e^{-\int^s a}\, ds\right) e^{\int^t a}, \\ \dot{y} &= \left(k + \int^t b(s) e^{-\int a}\, ds\right) a e^{\int^t a} + b(t) e^{-\int^t a} \cdot e^{\int^t a} \\ &= a\left(k + \int^t b(s) e^{-\int a}\, ds\right) e^{\int^t a} + b(t) \\ &= ay + b. \end{aligned}$$

24.7 *a*) If a and b are constants, (10) becomes

$$y = \left(k + b \int^t e^{-as}\, ds\right) e^{at} = \left(k - \frac{b}{a} e^{-at}\right) e^{at} = k e^{at} - \frac{b}{a},$$

as in (8). If $b = 0$ and $a = a(t)$, (10) becomes $y = (k + \int 0\, ds)e^{\int^t a(s)ds} = ke^{\int^t a(s)ds}$, as in (9).

b) *i)* $0 = e^{-at}\dot{y} - e^{-at}ay = \dfrac{d}{dt}(e^{-at}y)$. So $e^{-at}y = k$, or $y = ke^{at}$.

ii) $e^{-at}(\dot{y} - ay) = e^{-at}b$. So $\dfrac{d}{dt}(ye^{-at}) = be^{-at}$.

Then $ye^{-at} = b\int^t e^{-as}ds + k = -\dfrac{b}{a}e^{-at} + k$, or $y = -\dfrac{b}{a} + ke^{at}$.

iii) $e^{-\int a}(\dot{y} - ay) = 0$. So $\dfrac{d}{dt}\left(e^{-\int^t a(s)\,ds}y(t)\right) = 0$.

$e^{-\int^t a(s)\,ds}y(t) = k$, or $y = ke^{\int^t a(s)\,ds}$.

24.8 $\displaystyle\int^y \frac{dy}{y} = \int^t a(s)\,ds$. So $\ln y = \int^t a(s)ds + k$.

Therefore $y = e^{\int^t a(s)\,ds} \cdot e^k = ce^{\int^t a(s)\,ds}$.

24.9 $y = \dfrac{a}{b} \Longrightarrow \dot{y} = 0$ and $y(a - by) = \dfrac{a}{b}\left(a - \dfrac{ba}{b}\right) = 0$.

$$y = \frac{a}{b + ke^{-at}}, \qquad y(0) = \frac{a}{b + k} = y_0.$$

$a = y_0b + y_0k$ implies $k = \dfrac{a - y_0b}{y_0} = \dfrac{a}{y_0} - b$.

So $y(t) = \dfrac{a}{b + \left(\dfrac{a}{y_0} - b\right)e^{-at}}$.

24.10 *a)* $\ddot{y} - y = 0; \quad r^2 - 1 = (r - 1)(r + 1) = 0, \quad r = \pm 1$.

$y = k_1e^t + k_2e^{-t}; \qquad \dot{y} = k_1e^t - k_2e^{-t}$.

$1 = y(0) = k_1 + k_2$, and $1 = \dot{y}(0) = k_1 - k_2$.

So $k_1 = 1$ and $k_2 = 0$; $\qquad y = e^t$.

b) $\ddot{y} - 5\dot{y} + 6y = 0; \quad r^2 - 5r + 6 = (r - 3)(r - 2) = 0$.

$y = k_1e^{3t} + k_2e^{2t}; \qquad \dot{y} = 3k_1e^{3t} + 2k_2e^{2t}$.

$3 = y(0) = k_1 + k_2; \qquad 7 = \dot{y}(0) = 3k_1 + 2k_2$.

$k_1 = 1, k_2 = 2; \qquad y = e^{3t} + 2e^{2t}$.

c) $2\ddot{y} + 3\dot{y} - 2y = 0; \qquad 2r^2 + 3r - 2 = (2r - 1)(r + 2) = 0$.

$y = c_1e^{t/2} + c_2e^{-2t}; \qquad \dot{y} = .5c_1e^{t/2} - 2c_2e^{-2t}$.

$3 = y(0) = c_1 + c_2; \qquad -1 = \dot{y}(0) = .5c_1 - 2c_2$.

$c_1 = 2, c_2 = 1; \qquad y = 2e^{t/2} + e^{-2t}$.

24.11 $\dot{y} = 4e^{2t} + e^{2t} + 2te^{2t} = 5e^{2t} + 2te^{2t}$.

$\ddot{y} = 10e^{2t} + 2e^{2t} + 4te^{2t} = 12e^{2t} + 4te^{2t}$.

$\ddot{y} - 4\dot{y} + 4y = 12e^{2t} + 4te^{2t} - (20e^{2t} + 8te^{2t}) + (8e^{2t} + 4te^{2t}) = 0$.

24.12

$$\begin{aligned} y(t) &= k_1 e^{rt} + k_2 t e^{rt}, \\ \dot{y}(t) &= r k_1 e^{rt} + k_2 e^{rt} + r k_2 t e^{rt} \\ &= (r k_1 + k_2) e^{rt} + r k_2 t e^{rt}. \end{aligned}$$

$y_0 = k_1 e^{rt_0} + k_2 t_0 e^{rt_0}$ and $z_0 = (rk_1 + k_2)e^{rt_0} + rk_2 t_0 e^{rt_0}$.

$\begin{pmatrix} y_0 e^{-rt_0} \\ z_0 e^{-rt_0} \end{pmatrix} = \begin{pmatrix} 1 & t_0 \\ r & 1 + rt_0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

The determinant of the coefficient matrix is 1, so the matrix is invertible and one can always solve for k_1 and k_2. In fact,

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 1 + rt_0 & -t_0 \\ -r & 1 \end{pmatrix} \begin{pmatrix} y_0 e^{-rt_0} \\ z_0 e^{-rt_0} \end{pmatrix}.$$

24.13 *a*) $\ddot{y} + 6\dot{y} + 9y = 0, \qquad r^2 + 6r + = (r + 3)^2 = 0.$

$y = k_1 e^{-3t} + k_2 t e^{-3t}, \qquad \dot{y} = -3k_1 e^{-3t} + k_2 e^{-3t} - 3tk_2 e^{-3t}.$

$$\begin{aligned} 0 &= y(0) = k_1, \\ 1 &= \dot{y}(0) = -3k_1 + k_2. \end{aligned}$$

So $k_1 = 0, k_2 = 1$ and $y = te^{-3t}$.

b) $4\ddot{y} + 4\dot{y} + y = 0, \qquad 4r^2 + 4r + 1 = (2r + 1)^2 = 0.$

$y = k_1 e^{-t/2} + k_2 t e^{-t/2}, \qquad \dot{y} = -\frac{1}{2}k_1 e^{-t/2} + k_2 e^{-t/2} - \frac{1}{2}k_2 t e^{-t/2}.$

$$\begin{aligned} 1 &= y(0) = k_1, \\ 1 &= \dot{y}(0) = -\frac{1}{2}k_1 + k_2. \end{aligned}$$

So $k_1 = 1, k_2 = 1.5$; and $y = e^{-t/2} + 1.5te^{-t/2}$.

24.14 *a*) $\ddot{y} + 2\dot{y} + 10y = 0, r^2 + 2r + 10 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i.$$

$$\begin{aligned} y &= e^{-t}(c_1 \cos 3t + c_2 \sin 3t), \\ \dot{y} &= e^{-t}[(-c_1 + 3c_2)\cos 3t + (-c_2 - 3c_1)\sin 3t]. \end{aligned}$$

$2 = y(0) = c_1; \qquad 1 = \dot{y}(0) = -c_1 + 3c_2.$

$c_1 = 2, c_2 = 1$, so $y = e^{-t}(2\cos 3t + \sin 3t)$.

b) $\ddot{y} + 9y = 0; \qquad r^2 + 9 = 0, r = \pm 3i.$

$y = k_1 \cos 3t + k_2 \sin 3t, \qquad \dot{y} = -3k_1 \sin 3t + 3k_2 \cos 3t.$

$2 = y(0) = k_1, \qquad 1 = \dot{y}(0) = 3k_2.$

$k_1 = 2, k_2 = \frac{1}{3}. \qquad y = 2\cos 3t + \frac{1}{3}\sin 3t.$

24.15

$$y = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t),$$
$$\dot{y} = e^{\alpha t}[(\alpha c_1 + \beta c_2)\cos\beta t + (\alpha c_2 - \beta c_1)\sin\beta t],$$
$$\ddot{y} = e^{\alpha t}[(\alpha^2 - \beta^2)c_1 + 2\alpha\beta c_2]\cos\beta t + e^{\alpha t}[(\alpha^2 - \beta^2)c_2 - 2\alpha\beta c_1]\sin\beta t.$$

$$\begin{aligned} a\ddot{y} + b\dot{y} + cy &= e^{\alpha t}\cos\beta t[a(\alpha^2 - \beta^2)c_1 + 2a\alpha\beta c_2 + b\alpha c_1 + b\beta c_2 + cc_1] \\ &\quad + e^{\alpha t}\sin\beta t[a(\alpha^2 - \beta^2)c_2 - 2a\alpha\beta c_1 + b\alpha c_2 \\ &\quad - b\beta c_1 + cc_2] \\ &= e^{\alpha t}\cos\beta t[c_1(a(\alpha^2 - \beta^2) + b\alpha + c) + c_2(2a\alpha\beta + b\beta)] \\ &\quad + e^{\alpha t}\sin\beta t[-c_1(2a\alpha\beta + b\beta) + c_2(a(\alpha^2 - \beta^2) \\ &\quad + b\alpha + c)]. \end{aligned}$$

On the other hand, since $r = \alpha + i\beta$ is a root of $ar^2 + br + c = 0$,

$$0 = a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = [a(\alpha^2 - \beta^2) + b\alpha + c] + (2a\alpha\beta + b\beta).$$

So $a(\alpha^2 - \beta^2) + b\alpha + c = 0$ and $2a\alpha\beta + b\beta = 0$.

But these are precisely the coefficients of c_1 and c_2 in the previous expression, and so that expression equals zero.

24.16 *a*) $6\ddot{y} - \dot{y} - y = 0, \qquad 6r^2 - r - 1 = (3r + 1)(2r - 1) = 0.$

$y = k_1 e^{-t/3} + k_2 e^{t/2}, \qquad \dot{y} = -\frac{1}{3}k_1 e^{-t/3} + \frac{1}{2}k_2 e^{t/2}.$

$1 = y(0) = k_1 + k_2, \qquad 0 = \dot{y}(0) = -\frac{1}{3}k_1 + \frac{1}{2}k_2.$

So $k_1 = \frac{3}{5}, k_2 = \frac{2}{5}$; and $y = \frac{3}{5}e^{-t/3} + \frac{2}{5}e^{t/2}$.

b) $\ddot{y} + 2\dot{y} + 2y = 0, \qquad r^2 + 2r + 2 = 0, \qquad r = -1 \pm i.$

$y = e^{-t}(k_1 \cos t + k_2 \sin t), \qquad \dot{y} = e^{-t}[(k_2 - k_1)\cos t - (k_1 + k_2)\sin t].$

$1 = y(0) = k_1, \qquad 0 = \dot{y}(0) = k_2 - k_1. \qquad$ So $k_1 = k_2 = 1$.

$y = e^{-t}(\cos t + \sin t).$

c) $4\ddot{y} - 4\dot{y} + y = 0, \qquad 4r^2 - 4r + 1 = (2r-1)^2 = 0.$

$y = k_1 e^{t/2} + k_2 t e^{t/2}, \qquad \dot{y} = \frac{1}{2}k_1 e^{t/2} + k_2 e^{t/2} + \frac{1}{2}k_2 t e^{t/2}.$

$1 = y(0) = k_1, \qquad 0 = \dot{y}(0) = \frac{1}{2}k_1 + k_2 \Longrightarrow k_1 = 1, k_2 = -\frac{1}{2}.$

$y(t) = e^{t/2} - \frac{1}{2}te^{t/2}.$

d) $\ddot{y} + 5\dot{y} + 6y = 0, \qquad r^2 + 5r + 6 = (r+3)(r+2) = 0.$

$y = k_1 e^{-3t} + k_2 e^{-2t}, \qquad \dot{y} = -3k_1 e^{-3t} - 2k_2 e^{-2t}.$

$1 = y(0) = k_1 + k_2, \quad 0 = \dot{y}(0) = -3k_1 - 2k_2 \Longrightarrow k_1 = -2, k_2 = 3.$

$y = -2e^{-3t} + 3e^{-2t}.$

e) $\ddot{y} - 6\dot{y} + 9y = 0, \qquad r^2 - 6r + 9 = (r-3)^2 = 0.$

$y = k_1 e^{3t} + k_2 t e^{3t}, \qquad \dot{y} = 3k_1 e^{3t} + k_2 e^{3t} + 3k_2 t e^{3t}.$

$1 = y(0) = k_1, \quad 0 = \dot{y}(0) = 3k_1 + k_2 \Longrightarrow k_1 = 1, k_2 = -3.$

$y = e^{3t} - 3te^{3t}.$

f) $\ddot{y} + \dot{y} + y = 0, \quad r^2 + r + 1 = 0, \quad r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$

$$y = e^{-t/2}\left(k_1 \cos\frac{\sqrt{3}}{2}t + k_2 \sin\frac{\sqrt{3}}{2}t\right)$$

$$\dot{y} = e^{-t/2}\left(-\frac{k_1}{2}\cos\frac{\sqrt{3}}{2}t + \frac{k_2\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t - \frac{k_2}{2}\sin\frac{\sqrt{3}}{2}t - \frac{k_1\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t\right).$$

$1 = y(0) = k_1, \quad 0 = \dot{y}(0) = -\frac{k_1}{2} + \frac{k_2\sqrt{3}}{2} \Longrightarrow k_1 = 1, k_2 = 1/\sqrt{3}.$

$y = e^{-t/2}\left(\cos\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t\right).$

24.17 $y^{[3]} - 2\ddot{y} - \dot{y} + 2y = 0$. Look for the solution $y = e^{rt}$.

$$e^{rt}(r^3 - 2r^2 - r + 2) = 0.$$

$$(r-1)(r+1)(r-2) = 0 \Longrightarrow r = 1, -1, 2.$$

$$y = k_1 e^t + k_2 e^{-t} + k_3 e^{2t}.$$

24.18 $y = Ae^{-t}, \quad \dot{y} = -Ae^{-t}, \quad \ddot{y} = Ae^{-t}.$

$\ddot{y} - 2\dot{y} - 3y = Ae^{-t} + 2Ae^{-t} - 3Ae^{-t} \equiv 0$, but $\ddot{y} - 2\dot{y} - 3y$ must equal $8e^{-t}$.

24.19 *a*) $\ddot{y} - 2\dot{y} - y = 7$.

For the general solution of the homogeneous equation: $r^2 - 2r - 1 = 0 \Longrightarrow r = 1 \pm \sqrt{2}$.

A particular solution of the nonhomogeneous equation is $y_p = -7$.

So, $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t} - 7$.

b) $\ddot{y} + \dot{y} - 2y = 6t$.

General solution of the homogeneous equation:

$$r^2 + r - 2 = (r+2)(r-1) = 0,$$
$$y = k_1 e^{-2t} + k_2 e^t.$$

Particular solution of the nonhomogeneous equation: $y = At + B, \dot{y} = A$

$\Longrightarrow A - 2At - 2B = 6t, -2A = 6, A - 2B = 0$

$\Longrightarrow A = -3, B = -3/2$.

So, $y = k_1 e^{-2t} + k_2 e^t - 3t - 3/2$.

c) $\ddot{y} - \dot{y} - 2y = 4e^{-t}$.

General solution of the homogeneous equation:

$$r^2 - r - 2 = (r-2)(r+1) = 0,$$
$$y = k_1 e^{2t} + k_2 e^{-t}.$$

Particular solution of the nonhomogeneous equation: not Ae^{-t}, but $y_p = Ate^{-t}$.

$\dot{y}_p = Ae^{-t} - Ate^{-t}, \quad \ddot{y}_p = (-2A)e^{-t} + Ate^{-t}$.

$\ddot{y}_p - \dot{y}_p - 2y_p = e^{-t}[-2A + At - (A - At) - 2At] = 4e^{-t}$.

$-3A = 4, \qquad A = -4/3, \qquad y_p = (-4/3)te^{-t}$.

So $y = k_1 e^{2t} + k_2 e^{-t} - (4/3)te^{-t}$.

d) $\ddot{y} + 2\dot{y} = \sin 2t$.

General solution of the homogeneous equation: $r^2 + 2r = 0 \Longrightarrow y = k_1 e^{0t} + k_2 e^{-2t}$

Particular solution of the nonhomogeneous equation: $y = A \sin 2t + B \cos 2t$.

$$\dot{y} = +2A \cos 2t - 2B \sin 2t,$$
$$\ddot{y} = -4A \sin 2t - 4B \cos 2t.$$
$$\ddot{y} + 2\dot{y} = (-4A - 4B) \sin 2t + (-4B + 4A) \cos 2t = \sin 2t.$$

$-4A - 4B = 1, \qquad -4B + 4A = 0 \Longrightarrow A = B = -\frac{1}{8}.$

$y = k_1 + k_2 e^{-2t} - \frac{1}{8}\sin 2t - \frac{1}{8}\cos 2t.$

e) $\ddot{y} + 4y = \sin 2t$. The general solution of the homogeneous equation is: $y = k_1 \cos 2t + k_2 \sin 2t$.

A particular solution of the nonhomogeneous equation is: $y_p = t(A\sin 2t + B\cos 2t)$.

$$\begin{aligned}
\dot{y}_p &= A\sin 2t + B\cos 2t + 2tA\cos 2t - 2tB\sin 2t,\\
\ddot{y}_p &= 4A\cos 2t - 4B\sin 2t - 4tA\sin 2t - 4tB\cos 2t,\\
\ddot{y}_p + 4y_p &= 4A\cos 2t - 4B\sin 2t = \sin 2t.
\end{aligned}$$

$A = 0, B = -1/4$. So $y = k_1\cos 2t + k_2 \sin 2t - (t/4)\cos 2t$.

f) $\ddot{y} - y = e^t$.

General solution of the homogeneous equation: $r^2 - 1 = 0 \Longrightarrow y = k_1 e^t + k_2 e^{-t}$.

Particular solution of the nonhomogeneous equation: $y_p = Ate^t$.

$\dot{y}_p = Ae^t + Ate^t, \qquad \ddot{y}_p = 2Ae^t + Ate^t.$

$\ddot{y}_p - y_p = 2Ae^t = e^t \Longrightarrow A = \frac{1}{2}.$

$y = k_1 e^t + k_2 e^{-t} + \frac{1}{2}te^t.$

24.20

$$\begin{aligned}
&\frac{d^2}{dt^2}a(y_{p_1} + y_{p_2}) + \frac{d}{dt}b(y_{p_1} + y_{p_2}) + c(y_{p_1} + y_{p_2})\\
&\quad = a\ddot{y}_{p_1} + b\dot{y}_{p_1} + cy_{p_1} + a\ddot{y}_{p_2} + b\dot{y}_{p_2} + cy_{p_2}\\
&\quad = g_1(t) + g_2(t).
\end{aligned}$$

24.21 General solution of the homogeneous equation:

$$r^2 - r - 2 = (r-2)(r+1) = 0$$
$$y = k_1 e^{2t} + k_2 e^{-t}.$$

Particular solution #1:

$$\begin{aligned}
&y_{p_1} = At + B, \dot{y}_{p_1} = A, \ddot{y}_{p_1} = 0,\\
&\ddot{y}_{p_1} - \dot{y}_{p_1} - 2y_{p_1} = -A - 2(At + B) = 6t.\\
&-2A = 6 \quad \text{and} \quad -A - 2B = 0 \Longrightarrow A = -3, B = +3/2.
\end{aligned}$$

Particular solution #2:

$$y_{p_2} = Cte^{-t},$$
$$\dot{y}_{p_2} = Ce^{-t} - Cte^{-t}, \quad \ddot{y}_{p_2} = -2Ce^{-t} + Cte^{-t}.$$
$$\ddot{y}_{p_2} - \dot{y}_{p_2} - 2y_{p_2} = [(-2C + Ct) - (C - Ct) - 2(Ct)]e^{-t} = 4e^{-t}.$$
$$-3C = 4, \qquad C = -4/3.$$

So $y = k_1 e^{3t} + k_2 e^{-2t} - 3t + (3/2) - (4/3)te^{-t}$.

24.22 See figure.

24.23 See figure.

24.24 See figure.

24.25 See figure.

24.26 See figure.

24.27 *a*) $y = 0, y = 1$ are the steady states.

$$f'(y) = \frac{y^2 + 2y - 1}{(y^2 + 1)^2}.$$

$f'(0) = -1 < 0 \Longrightarrow 0$ is asymptotically stable.

$f'(1) = 2/4 > 0 \Longrightarrow 1$ is unstable.

b) $y = -2\pi, -\pi, 0, \pi, 2\pi, \ldots,$

$f'(y) = e^y(\sin y + \cos y)$,

$f'(2n\pi) = e^{2n\pi} \cdot \cos 2n\pi > 0$,

$f'((2n + 1)\pi) = e^{(2n+1)\pi} \cdot \cos(2n + 1)\pi < 0$.

So $y = 2n\pi$ are unstable, and $y = (2n + 1)\pi$ are asymptotically stable.

c) $y = 0$ is the only steady state.

$f'(0) = 0$, so we can't use the derivative test.

The phase diagram is: ——>——·——>——

So $y = 0$ is unstable (on the right).

24.28 By (48), $p \longmapsto \mu(p)$ satisfies the differential equation:

$$\frac{d\mu}{dp} = \xi(p, \mu) = p^a \mu^b e^c.$$

Using separation of variables, we find:

$$\int \mu^{-b}\, d\mu = \int e^c\, p^a\, dp, \qquad \text{or} \qquad \frac{\mu^{1-b}}{1-b} = \frac{e^c p^{a+1}}{a+1} + k.$$

Choose k to satisfy initial condition $\mu(q) = y$:

$$k = \frac{y^{1-b}}{1-b} - \frac{e^c q^{a-1}}{a+1}.$$

Then

$$\frac{\mu^{1-b}}{1-b} = \frac{e^c}{a+1}(p^{a+1} - q^{a+1}) + \frac{y^{1-b}}{1-b},$$

or

$$\mu = \left[\frac{e^c(1-b)}{a+1}(p^{a+1} - q^{a+1}) + y^{1-b}\right]^{1/(1-b)}.$$

Chapter 25

25.1 *a*)

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -12 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

$$r^2 + 3r + 2 = (r+2)(r+1) = 0.$$

For $r = -2$,

$$\begin{pmatrix} 4 & 1 \\ -12 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $r = -1$,

$$\begin{pmatrix} 3 & 1 \\ -12 & -4 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -4 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

b)
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}6 & -3\\ -2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}.$$

$$r^2 - 7r = r(r-7) = 0 \Longrightarrow r = 0, 7.$$

For $r = 0$,

$$\begin{pmatrix}6 & -3\\ -2 & 1\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

For $r = 7$,

$$\begin{pmatrix}-1 & -3\\ -2 & -6\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

So

$$\begin{pmatrix}x\\ y\end{pmatrix} = c_1e^{0t}\begin{pmatrix}1\\ 2\end{pmatrix} + c_2e^{7t}\begin{pmatrix}3\\ -1\end{pmatrix}.$$

c)
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 4\\ 3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}. \qquad r^2 - 3r - 10 = (r-5)(r+2) = 0.$$

For $r = 5$,

$$\begin{pmatrix}-4 & 4\\ 3 & -3\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

For $r = -2$,

$$\begin{pmatrix}3 & 4\\ 3 & 4\end{pmatrix}\begin{pmatrix}4\\ -3\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}.$$

So

$$\begin{pmatrix}x\\ y\end{pmatrix} = c_1e^{5t}\begin{pmatrix}1\\ 1\end{pmatrix} + c_2e^{-2t}\begin{pmatrix}4\\ -3\end{pmatrix}.$$

d)
$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 4\\ -1 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}. \qquad r^2 - 4r + 4 = (r-2)^2 = 0.$$

For $r = 2$,

$$\begin{pmatrix}-2 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}-2 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}2\\ 1\end{pmatrix}.$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{2t}\left[c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix} + tc_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right].$$

e) $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \qquad r^2 - 2r + 2 = 0, r = 1 \pm i.$$

For $r = 1 + i$,

$$\begin{pmatrix} -3 - i & 5 \\ -2 & 3 - i \end{pmatrix}\begin{pmatrix} 5 \\ 3 + i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \mathbf{u} + i\mathbf{v} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t\left[(k_1 \cos t - k_t \sin t)\begin{pmatrix} 5 \\ 3 \end{pmatrix} - (k_1 \sin t + k_2 \cos t)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right].$$

f) $$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1 & -2 & -6 \\ 2 & 5 & 6 \\ -2 & -2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Then $(r - 3)^2(r + 3) = 0$, so $r = 3, 3, -3$. For $r = 3$,

$$\begin{pmatrix} -2 & -2 & -6 \\ 2 & 2 & 6 \\ -2 & -2 & -6 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -2 & -2 & -6 \\ 2 & 2 & 6 \\ -2 & -2 & -6 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For $r = -3$,

$$\begin{pmatrix} 4 & -2 & -6 \\ 2 & 8 & 6 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1e^{3t}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2e^{3t}\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} + c_3e^{-3t}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

25.2 $\ddot{y} = -ay$; let $z = \dot{y}$. Then, $\dot{z} = \ddot{y} = -ay$. So, $\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$.

25.3 If $z = \dot{y}$, then $\dot{z} = \ddot{y} = -\frac{a}{m}\dot{y} - mg \sin y$. So

$$\begin{aligned} \dot{y} &= z \\ \dot{z} &= -\frac{a}{m} z - mg \sin y. \end{aligned}$$

25.4 By equation (24.10), the solution of $\dot{y}_1 = ry_1 + c_2 e^{rt}$ is:

$$\begin{aligned} y_1(t) &= \left(c_1 + \int^t c_2 e^{rs} e^{-\int^s r\,du}\, ds \right) e^{rt} \\ &= \left(c_1 + \int^t c_2 e^{rs} \cdot e^{-rs}\, ds \right) e^{rt} \\ &= (c_1 + c_2 t)\, e^{rt} \end{aligned}$$

25.5 According to Chapter 23, there exists an invertible $P = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3)$ such that

$$P^{-1}AP = L \quad \text{where} \quad L = \begin{pmatrix} r & 1 & 0 \\ 0 & r & 1 \\ 0 & 0 & r \end{pmatrix}.$$

Let $y = P^{-1}\mathbf{x}$. Then, $\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}} = P^{-1}A\mathbf{x} = P^{-1}AP\mathbf{y} = L\mathbf{y}$.

Solving $\dot{\mathbf{y}} = L\mathbf{y}$,

$$\begin{aligned} \dot{y}_1 &= r_1 y_1 + y_2 \\ \dot{y}_2 &= r_1 y_2 + y_3 \\ \dot{y}_3 &= r y_3. \end{aligned}$$

So $y_3 = c_2 e^{rt}$.

From the previous exercise, conclude that $y_2 = (c_1 + c_2 t)e^{rt}$.

$\dot{y}_1 = ry_1 + y_2 = ry_1 + (c_1 + c_2 t)e^{rt}$, so $y_1 = ke^{rt} + z(t)$.

Using the method of undetermined coefficients, look for $z(t) = (A + Bt + Ct^2)e^{rt}$.

$$\dot{z} = rz + (c_1 + c_2 t)e^{rt}. \qquad (*)$$

But, $\dot{z} = (B + 2Ct)e^{rt} + (rA + rBt + rCt^2)e^{rt}$.

RHS of $(*) = (rA + rBt + rCt^2)e^{rt} + (c_1 + c_2 t)e^{rt}$.

Set these two equal, cancel e^{rt}s and set similar the coefficients of t equal:

$$B = c_1 \quad \text{and} \quad 2C = c_2.$$

So, $z(t) = (A + c_1 t + \frac{1}{2}c_2 t^2)e^{rt}$.

Let $c_0 = A + k$. Then, $y_1(t) = (c_0 + c_1 t + \frac{1}{2}c_2 t^2)e^{rt}$

The solution is $\mathbf{x}(t) = P\mathbf{y}(t)$. That is,

$$\mathbf{x}(t) = (\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3\,)\begin{pmatrix} (c_0 + c_1 t + \frac{1}{2}c_2 t^2) \\ c_1 + c_2 t \\ c_2 \end{pmatrix} e^{rt}$$

$$= e^{rt}[(c_0 + c_1 t + \tfrac{1}{2}c_2 t^2)\mathbf{v}_1 + (c_1 + c_2 t)\mathbf{v}_2 + c_2\mathbf{v}_3].$$

25.6 *a*) $y_2 = \dot{y}_1 - y_1, \qquad \dot{y}_2 = \ddot{y}_1 - \dot{y}_1$.

$\dot{y}_2 = 4y_1 - 2y_2$ becomes $\ddot{y}_1 - \dot{y}_1 = 4y_1 - 2(\dot{y}_1 - y_1)$ or $\ddot{y}_1 + \dot{y}_1 - 6y_1 = 0$.

$r^2 + r - 6 = (r + 3)(r - 2) = 0$. So, $r = -3, 2$.

$y_1 = c_1 e^{-3t} + c_2 e^{2t}$,

$y_2 = \dot{y}_1 - y_1 = -3c_1 e^{-3t} + 2c_2 e^{2t} - c_1 e^{-3t} - c_2 e^{2t} = -4c_1 e^{-3t} + c_2 e^{2t}$.

$y_1(0) = c_1 + c_2 = 1$ and $y_2(0) = -4c_1 + c_2 = 0 \implies c_1 = .2$, and $c_2 = .8$.

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} .2e^{-3t} + .8e^{2t} \\ -.8e^{-3t} + .8e^{2t} \end{pmatrix}$.

b) $y_2 = 0.2y_1 - 0.2\dot{y}_1$ and $\dot{y}_2 = 0.2\dot{y}_1 - 0.2\ddot{y}_1$.

$\dot{y}_2 = 2y_1 - 5y_2$ implies $.2\dot{y}_1 - 0.2\ddot{y}_1 = 2y_1 - y_1 + \dot{y}_1$ or $0.2\ddot{y}_1 + 0.8\dot{y}_1 + y_1 = 0 \implies .2r^2 + .8r + 1 = 0$ or $r^2 + 4r + 5 = 0$; so $r = -2 \pm i$

$$y_1 = e^{-2t}(c_1 \cos t - c_2 \sin t)$$

$$\dot{y}_1 = e^{-2t}(-2c_1 \cos t + 2c_2 \sin t - c_1 \sin t - c_2 \cos t)$$

$$= e^{-2t}[-(2c_1 + c_2)\cos t + (2c_2 - c_1)\sin t]$$

$$y_2 = e^{-2t}[0.2c_1 \cos t - 0.2c_2 \sin t + (0.4c_1 + 0.2c_2)\cos t$$

$$- (0.4c_2 - 0.2c_1)\sin t]$$

$$= e^{-2t}[(0.6c_1 + 0.2c_2)\cos t + (0.2c_1 - 0.6c_2)\sin t].$$

$1 = y_1(0) = c_1, 0 = y_2(0) = 0.6c_1 + 0.2c_2$. So $c_1 = 1, c_2 = -3$.

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{-2t} \begin{pmatrix} \cos t + 3\sin t \\ 2\sin t \end{pmatrix}.$

c) $y_2 = .2\dot{y}_1 + .6y_1, \qquad \dot{y}_2 = .2\ddot{y}_1 + .6\dot{y}_1.$

$\dot{y}_2 = -y_1 + y_2$ implies $.2\ddot{y}_1 + .6\dot{y}_1 = -y_1 + 0.2\dot{y}_1 + 0.6y_1$.

$.2\ddot{y}_1 + 0.4\dot{y}_1 + 0.4y_1 = 0$ or $\ddot{y}_1 + 2\dot{y}_1 + 2y_1 = 0$.

$r^2 + 2r + 2 = 0$ implies $r = -1 \pm i$.

$$\begin{aligned}
y_1 &= e^{-t}(c_1 \cos t - c_2 \sin t) \\
\dot{y}_1 &= e^{-t}(-c_1 \cos t + c_2 \sin t - c_1 \sin t - c_2 \cos t) \\
&= e^{-t}[-(c_1 + c_2)\cos t + (c_2 - c_1)\sin t] \\
y_2 &= 0.2\dot{y}_1 + 0.6y_1 \\
&= e^{-t}[-(0.2c_1 + 0.2c_2)\cos t + (0.2c_2 - 0.2c_1)\sin t] \\
&\quad + e^{-t}(0.6c_1 \cos t - 0.6c_2 \sin t) \\
&= e^{-t}[(0.4c_1 - 0.2c_2)\cos t + (-0.4c_2 - 0.2c_1)\sin t].
\end{aligned}$$

Then $1 = y_1(0) = c_1$ and $0 = y_2(0) = 0.4c_1 - 0.2c_2$. So, $c_1 = 1$ and $c_2 = 2$.

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{-t} \begin{pmatrix} \cos t - 2\sin t \\ -\sin t \end{pmatrix}.$

25.7 A fourth steady state is the solution of the system:

$$b_1y_1 + c_1y_2 = a_1, \qquad c_2y_1 + b_2y_2 = a_2,$$

namely

$$y_1 = \frac{a_1b_2 - a_2c_2}{b_1b_2 - c_1c_2} \quad \text{and} \quad y_2 = \frac{b_1a_2 - a_1c_2}{b_1b_2 - c_1c_2}.$$

This is the steady state in which both species survive at positive levels. It only makes sense if both y_1 and y_2 are positive.

25.8 $x = 0, y = 0$ and $x = C/D; y = A/B$.

25.9 *a*) $2y(x - y) = 0$ and $x = 2 - y^2$.

$y = 0, x = 2$ and $x = y; y^2 + y - 2 = 0$ or $y = 1, -2$.

Steady states are $(x, y) = (2, 0), (1, 1)$ and $(-2, -2)$.

b) $x + y = 0$ and $x + 4y = 0$ imply $(x, y) = (0, 0)$.

c) $xy = 0$ and $2x + 4y = 4$.

$x = 0, y = 1$ and $y = 0, x = 2$.

d) $2x = 0$ and $y^2 - x^2 = 1$ imply $(x, y) = (0, 1)$ or $(0, -1)$.

25.10 Eigenvalues of $\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ are solutions of $r^2 - 2r - 3 = (r-3)(r+1) = 0$; i.e., $r = 3, -1$.
Since $3 > 0$, $(0, 0)$ is an unstable steady state of (18). The general solution of (18) is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{3t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Whenever $c_1 \neq 0$, the solution grows like e^{3t}.

25.11 The characteristic polynomial of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is $r^2 - (a_{11} + a_{22})r + \det A = 0$. The sum of the eigenvalues equals the trace, $(a_{11} + a_{22})$. The product of the eigenvalues equals the constant term, $\det A$. If the eigenvalues are real and negative, their sum is < 0 and their product is > 0. If the eigenvalues are complex $a \pm ib$ with $a < 0$, their sum $2a$ is negative and their product $a^2 + b^2$ is positive. Conversely, if their product is positive and their sum is negative, the eigenvalues are complex numbers with negative real part or they are negative real numbers. If $\det A < 0$, the eigenvalues have opposite signs; so one must be positive. If trace $A > 0$, at least one eigenvalue must be positive or both are complex with positive real part. Finally, if $\det A > 0$ and trace $A = 0$, the eigenvalues must be complex numbers with zero real part, i.e., pure imaginary numbers $\pm ib$. The general solution in this case is $\mathbf{x} = (\cos bt)\mathbf{u} + (\sin bt)\mathbf{v}$; in this case, all orbits are periodic and $(0, 0)$ is neutrally stable.

25.12 *a*) trace $= 4 > 0$; unstable.

b) trace $= 5 > 0$; unstable.

c) trace $= 2 > 0$; unstable.

d) trace $= 8 =$ sum of roots; unstable.

25.13 Just replace A in the above solution of Exercise 25.11 by the Jacobian matrix of the system evaluated at the steady state.

25.14 The Jacobian of (1) at (0, 0), $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, has two positive eigenvalues. When one population is very small, the other behaves as in the logisitic equation with exponential growth near zero. The Jacobian of (2) at (0, 0), $\begin{pmatrix} A & 0 \\ 0 & -C \end{pmatrix}$ has one positive eigenvalue. In the absence or near absence of the predator, the prey has exponential growth.

25.15 The Jacobian of (1) at $(0, a_2/b_2)$ is $J = \begin{pmatrix} a_1 - c_1a_2/b_2 & 0 \\ -c_2a_2/b_2 & -a_2 \end{pmatrix}$. Its eigenvalues are both negative if and only if

$$a_1 - \frac{c_1a_2}{b_2} = \frac{a_1b_2 - c_1a_2}{b_2} < 0;$$

that is, $a_1b_2 - c_1a_2 < 0$ or $\frac{a_1}{c_1} < \frac{a_2}{b_2}$, or $\frac{c_1}{a_1} > \frac{b_2}{a_2}$. The left side $\left(\frac{c_1}{a_1}\right)$ measures the negative impact of the size of population 2 on the growth rate of population 1, while the right side $\left(\frac{b_2}{a_2}\right)$ measures the negative impact of the size of population 2 on its own growth rate. So, the inequality states that: if a large population 2 has a more negative impact on the growth of population 1 than it does on its own growth, then there is a stable steady state with $y_1^* = 0$ and $y_2^* > 0$.

25.16 The Jacobian of (2) at the interior equilibrium $(C/D, A/B)$ is

$$\begin{pmatrix} 0 & -BC/D \\ DA/B & 0 \end{pmatrix}.$$

Its eigenvalues are the pure imaginary numbers $\pm i\sqrt{AC}$. This is the case in which the stability of the linearization does not determine the stability of the nonlinear system.

25.17 *a*) trace $= -3 < 0$, det $= 2 > 0$; asymptotically stable.

b) trace $= 7 > 0$; unstable.

c) trace $= 3 > 0$; unstable.

d) trace $= 4 > 0$; unstable.

e) trace $= 2 > 0$; unstable.

f) trace $= 3 > 0$; unstable.

25.18 Write $\mathbf{x}(t; \mathbf{y}_0)$ for the solution $\mathbf{x}(t)$ of the autonomous system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$ that satisfies the initial condition $\mathbf{x}(0) = \mathbf{y}_0$. A steady state $\mathbf{x}^*$ of this system is asymptotically stable if there is an $\varepsilon > 0$ such that for any $\mathbf{y}_0$ with $\|\mathbf{y}_0 - \mathbf{x}_0\| < \varepsilon$, $\mathbf{x}(t; \mathbf{y}_0) \to \mathbf{x}^*$ as $t \to \infty$.

25.19 See figure.

25.20 *a*) Steady states at $(0, 0)$, $(0, 4)$, $(2, 0)$ and $(1, 3)$.

The Jacobian of the RHS is $J = \begin{pmatrix} 6 - 6x - y & -x \\ -y & 4 - x - 2y \end{pmatrix}$.

At $(0, 0)$, $J = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$ with eigenvalues 6 and 4. So, $(0, 0)$ is an unstable source.

At $(0, 4)$, $J = \begin{pmatrix} 2 & 0 \\ -4 & -4 \end{pmatrix}$ with eigenvalues 2 and -4. So, $(0, 4)$ is an unstable saddle.

At $(2, 0)$, $J = \begin{pmatrix} -6 & -2 \\ 0 & 2 \end{pmatrix}$ with eigenvalues -6 and 2. So, $(2, 0)$ is an unstable saddle.

At $(1, 3)$, $J = \begin{pmatrix} -3 & -1 \\ -3 & -3 \end{pmatrix}$ with negative eigenvalues $-3 \pm \sqrt{3}$. So, $(1, 3)$ is a sink.

All orbits starting inside the positive orthant tend to $(1, 3)$. Species reach equilibrium with both surviving. See figure.

b) Steady states in the positive quadrant are: $(0, 0)$, $(0, 6)$ and $(2, 0)$.

The Jacobian of the RHS is $J = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 6 - 2y - 2x \end{pmatrix}$.

At $(0, 0)$, $J = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \Longrightarrow$ unstable.

At $(2, 0)$, $J = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix} \Longrightarrow$ unstable.

At $(0, 6)$, $J = \begin{pmatrix} -4 & 0 \\ -12 & -6 \end{pmatrix} \Longrightarrow$ asymptotically stable.

All orbits starting in the interior of the positive orthant tend to $(0, 6)$; so the first species eventually dies out. See figure.

c) Steady states in the positive quadrant are $(0, 0)$, $(3, 0)$, $(0, 2)$.

$(0, 0)$ is unstable source.

$(3, 0)$ is asymptotically stable steady state, attracting all orbits in the interior of the positive orthant.

$(0, 2)$ is an unstable saddle.

See figure.

d) Steady states are $(0,0)$, $(2,0)$, $(0,2)$ and $(4/3, 4/3)$.

$(0,0)$ is an unstable source. $(2,0)$ and $(0,2)$ are unstable saddles.

$(4/3, 4/3)$ is an asymptotically stable steady state which attracts all orbits that start inside the positive orthant. See figure.

25.21 The fact that the lines were straight in Figure 25.12 plays almost no role in the analysis. The same argument would show that the vector field points west on the curve corresponding to segment *d* in Figure 25.12 and points south on the curve corresponding to segment *b* in Figure 25.12.See figure.

25.22 The interior equilibrium is

$$(y_1^*, y_2^*) = \left(\frac{a_1 b_2 - a_2 c_1}{b_1 b_2 - c_1 c_2}, \frac{a_2 b_1 - a_1 c_2}{b_1 b_2 - c_1 c_2} \right).$$

The Jacobian of (1) at this point is: $\begin{pmatrix} -b_1 y_1^* & -c_1 y_1^* \\ -c_2 y_2^* & -b_2 y_2^* \end{pmatrix}$, with trace $-(b_1 y_1^* + b_2 y_2^*) < 0$ and determinant $y_1^* y_2^* (b_1 b_2 - c_1 c_2)$. In order for $y_1^* > 0$, $y_2^* > 0$ and the determinant > 0, we need:

$$a_1 b_2 > a_2 c_1 \quad \text{or} \quad \frac{b_2}{a_2} > \frac{c_1}{a_1}$$

$$a_2 b_1 > a_1 c_2 \quad \text{or} \quad \frac{b_1}{a_1} > \frac{c_2}{a_2}$$

$$b_1 b_2 > c_1 c_2.$$

As we saw in Exercise 25.15, the first two inequalities mean that the negative effect of the size of each species is greater on itself than it is on the other species. The third inequality follows from the first two and emphasizes the requirement that the growth of each species has a greater impact on its own growth than on the growth of the other.

25.23 See figure.

25.24 If $V(x, y) = xy$, $\dot{V} = \frac{\partial V}{\partial x} \cdot x + \frac{\partial V}{\partial y} \cdot (-y) = y \cdot x + x(-y) = 0.$

25.25 If V were constant on some large open set, then $\dot{V}$ would have to be zero on that set and we do not know in which direction the orbits are moving, relative to the level sets of V.

25.27 *a*) See figure.

b) See figure.

c) See figure.

d) See figure.

Chapter 26

26.1 *a*) 1, *b*) 0, *c*) 0, *d*) 16, *e*) 80.

26.3 If $a_{11}a_{22} - a_{12}a_{21} = 0$ in (5), (5) becomes

$$-a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \qquad (*)$$

On the other hand,

$$\begin{aligned}
&-(a_{23}a_{11} - a_{21}a_{13})(a_{11}a_{32} - a_{31}a_{21})/a_{11} \\
&\quad = (-a_{11}^2a_{23}a_{32} + a_{11}a_{23}a_{12}a_{31} - a_{12}a_{21}a_{13}a_{31} + a_{11}a_{21}a_{13}a_{32})/a_{11} \\
&\quad = (-a_{11}^2a_{23}a_{32} + a_{11}a_{12}a_{23}a_{31} - a_{11}a_{22}a_{13}a_{31} + a_{11}a_{21}a_{13}a_{32})/a_{11} \\
&\quad = -a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{22}a_{13}a_{31} + a_{13}a_{21}a_{32} \\
&\quad = (*).
\end{aligned}$$

26.5 Use induction. Easily seen for $n = 1, 2$. Assume true for $(n-1) \times (n-1)$ matrices and let A be $n \times n$.
Let A_{ij} be as in Theorem 26.1. Let $a_{ij}^T = a_{ji}$ and $A_{ij}^T = A_{ji}$.
Expanding across row 1 of A^T yields

$$\begin{aligned}
\det A^T &= \sum_j (-1)^{1+j} a_{1j}^T \det A_{1j}^T \\
&= \sum_j (-1)^{1+j} a_{j1} \det A_{j1}, \qquad \text{by inductive hypothesis,} \\
&= \det A, \qquad \text{expanding down column 1.}
\end{aligned}$$

26.6 *c*)
$$\begin{aligned}
\det &= 1\cdot5\cdot9 + 4\cdot8\cdot3 + 7\cdot2\cdot6 - 3\cdot5\cdot7 - 2\cdot4\cdot9 - 1\cdot8\cdot6 \\
&= 45 + 96 + 84 - 105 - 72 - 48 \\
&= 225 - 225 = 0.
\end{aligned}$$

26.7 *a*) 2×2: 2 terms, 3×3: $3 \cdot 2 = 3!$ terms, $4 \times 4 = 4 \cdot 3! = 4!$ terms, $n \times n$: $n!$ terms.

Generally, let $\phi(n)$ denote the number of terms in the determinant of an $n \times n$ matrix. From Theorem 26.1 it is clear that $\phi(n) = n\phi(n-1)$, so $\phi(n) = n!\phi(1)$. Furthermore, $\phi(1) = 1$, so $\phi(n) = n!$.

b) Each of the $n!$ terms multiplies together n elements; this requires $n-1$ multiplications. Thus, in toto there are $n!(n-1)$ multiplications. There are $n!$ terms to be added or subtracted, and this involves $n!-1$ operations, so there are $n!(n-1) + n! - 1 = n \cdot n! - 1$ operations.

26.8 8 terms vs. $4! = 24$ terms.

26.9 $\det A = 1$, $\det B = 1$, $\det(A+B) = 4$. Matrix C differs from each of A and B only in the first row; its first row is the sum of their first rows. So it follows from Fact 2 that $\det(A+B) = \det C$. Calculation shows that this is the case: $\det C = 2$.

26.10 Let k be the number of elementary row operations in a string of such operations. Let A be the original matrix and R the result of applying the row operations on A in the prescribed order. Each operation leaves the determinant invariant or changes its sign. If it takes one row operation to change A to its row echelon form A_r, $\det A = \pm \det A_r$. Assume that for a string of length $k-1$, $\det A = \pm \det R$. Now suppose it takes k row operations to go from A to R. Let R' be the end result of the first $(k-1)$ operations. By the inductive hypothesis, $\det A = \pm \det R'$. Furthermore, going from R' to R either leaves the determinant invariant or changes its sign. So, $\det R = \pm \det A$.

26.11 Clearly true for 1×1 and 2×2 matrices. Suppose true for $(k-1) \times (k-1)$ upper-triangular matrices. Expand the determinant of upper-triangular matrix A along its first column:

$$\det A = a_{11} \cdot \det A_{11} + 0 \cdot \det A_{12} + \cdots + 0 \cdot \det A_{1n} = a_{11} \det A_{11}.$$

But, A_{11} is a $(k-1) \times (k-1)$ upper-triangular matrix with $a_{22}, \ldots, a_{nn}$ on its diagonal.

By the inductive hypothesis, $\det A_{11} = a_{22} \cdots a_{nn}$.
So, $\det A = a_{11} \cdot a_{22} \cdots a_{nn}$.

26.12 According to Fact 26.11, the determinant of an upper- or lower-triangular matrix is 0 if even one diagonal element is 0. In this case, by Theorem 9.3, the matrix is singular.

26.13 *a*) *i*) $\det A = \det \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 3 \end{pmatrix} = -6.$

ii) $\det A = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{pmatrix} = 2 \cdot 1 \cdot 2 \cdot 2 = 8$

iii) $\det A = \det \begin{pmatrix} 2 & 6 & 0 & 5 \\ 0 & 3 & 8 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 19 \end{pmatrix} = 2 \cdot 3 \cdot (-4) \cdot 19 = -456.$

b) All three are nonsingular.

26.14 $\det \begin{pmatrix} 1 & k \\ k & 1 \end{pmatrix} = 1 - k^2 = 0 \Longleftrightarrow k = 1, -1;$

$$\det \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix} = k^3 - 3k + 2 = (k-1)(k^2 + k - 2)$$

$$= (k-1)(k-1)(k+2) = 0 \qquad \text{if and only if } k = 1 \text{ or } -2.$$

26.15 Suppose A is invertible. Then, $A \cdot A^{-1} = I$.

By Theorem 26.4, $\det A \cdot \det A^{-1} = \det I = 1$. So, $\det A^{-1} = 1/\det A$.

26.16 *a*) Let $\mathbf{a}_i$ denote the *i*th row vector of A. Then according to Fact 26.4,

$$\det rA = \det \begin{pmatrix} r\mathbf{a}_1 \\ r\mathbf{a}_2 \\ \vdots \\ r\mathbf{a}_n \end{pmatrix} = r \det \begin{pmatrix} \mathbf{a}_1 \\ r\mathbf{a}_2 \\ \vdots \\ r\mathbf{a}_n \end{pmatrix} = \cdots = r^n \det \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_n \end{pmatrix}.$$

b) Let $r = -1$ in part *a*.

c) Use induction of r. If $r = 1$, trivial. If $r = 2$, Theorem 26.4 applies. Assume true for $r - 1$. Then,

$$\begin{aligned} \det(A_1 \cdots A_r) &= \det A_1 \cdot (A_2 \cdots A_r) \\ &= \det A_1 \cdot \det(A_2 \cdots A_r) && \text{by Theorem 26.4} \\ &= \det A_1 \cdot \det A_2 \cdots \det A_r && \text{by induction hypothesis.} \end{aligned}$$

d) Apply *c* with $A_1 = \cdots = A_r = A$ and $k = r$.

e) If $k = -m$ a negative integer, $A^k = (A^{-1})^m$ and

$$\begin{aligned}\det(A^k) &= \det(A^{-1})^m = (\det A^{-1})^m \\ &= [(\det A)^{-1}]^m = (\det A)^{-m} \\ &= (\det A)^k.\end{aligned}$$

26.17 $E_{ij}(r) \cdot B$ is B with r times row i added to row j. By Fact 26.7, $\det(E_{ij}(r) \cdot B) = \det B$.

On the other hand, since $E_{ij}(r)$ is a triangular matrix with only 1s on its diagonal, $\det E_{ij}(r) = 1$. So $\det E_{ij}(r) \cdot \det B = \det B = \det(E_{ij}(r) \cdot B)$.

26.18 *a*) If $A^T = A^{-1}$, $\det A = \det A^T = \det A^{-1} = 1/\det A$.

Therefore $(\det A)^2 = 1$ and $\det A = \pm 1$.

b) If $A^T = -A$, $\det A = \det A^T = \det(-A) = (-1)^n \det A$.

If n is odd, $(-1)^n = -1$ and $\det A = -\det A$.

Since $\det A = 0$, A is singular.

c) $\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ is orthogonal; $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew-symmetric (and orthogonal). If U is any upper-triangular matrix, then $U - U^T$ is skew-symmetric.

26.19 Let R denote the row echelon form of A. If no row interchanges were used in the Gaussian elimination process, then by Fact 26.7, $\det R = \det A$. But R is an upper-triangular matrix with the pivots of A on its diagonal. By Fact 26.11, $\det R =$ the product of the pivots of A. If Gaussian elimination required some row interchanges, then $\det A = \pm \det R$ as in the proof of Theorem 26.3. In this case, $\det A = \pm$ the product of its pivots.

26.20 $\det(AB) = \det A \cdot \det B$.

So $\det AB \neq 0$ if and only if $\det A \neq 0$ and $\det B \neq 0$.

But a matrix is nonsingular if and only if its determinant is nonzero.

26.21 *a*) Notation: A^{ij} = the submatrix of A obtained by deleting row i and column j of A.

A_{11}^{ij} is the submatrix of A_{11} obtained by deleting row i and column j from A_{11}.

We will use induction on the size of A_{11}.

If A_{11} is a 1×1 matrix a_{11},

$$\det \begin{pmatrix} a_{11} & \mathbf{0} \\ \mathbf{0} & A_{22} \end{pmatrix} = a_{11} \cdot \det A_{22}, \qquad \text{by definition of determinant,}$$
$$= \det A_{11} \cdot \det A_{22}.$$

Assume true for $(n-1) \times (n-1) A_{11}$'s.

Suppose A_{11} is $n \times n$. Expand $\det A$ across its first row.

$$\begin{aligned}
\det A &= a_{11} \cdot \det A^{11} - a_{12} \cdot \det A^{12} + \cdots \\
&= a_{11} \cdot \det \begin{pmatrix} A_{11}^{11} & 0 \\ 0 & A_{22} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} A_{11}^{12} & 0 \\ 0 & A_{22} \end{pmatrix} + \cdots \\
&= a_{11} \cdot \det A_{11}^{11} \cdot \det A_{22} - a_{12} \cdot \det A_{11}^{12} \cdot \det A_{22} + \cdots \\
&\qquad \text{(inductive hypothesis)} \\
&= (a_{11} \cdot \det A_{11}^{11} - a_{12} \cdot \det A_{11}^{12} + \cdots) \det A_{22} \\
&= \det A_{11} \cdot \det A_{22}.
\end{aligned}$$

b) Same proof as in *a*.

c) Carry out the multiplication on the right side:

$$\begin{aligned}
&\begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{pmatrix} \\
&\qquad = \begin{pmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} + A_{12}A_{22}^{-1}A_{21} & A_{12} \\ A_{22}A_{22}^{-1}A_{21} & A_{22} \end{pmatrix} \\
&\qquad = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A.
\end{aligned}$$

d) Combine *b* and *c*.

e) $$A_{22} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \qquad A_{22}^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix},$$

$$A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -89 & -2 \\ 14 & 0 \end{pmatrix} \quad \text{with} \quad \det = 28.$$

Then $\det A = 28 \cdot \det A_{22} = 28.$

26.22

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{pmatrix}$$
$$\Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-a)(c-b) \end{pmatrix}.$$

$\det = 1 \cdot (b-a)(c-a)(c-a)$.

26.23 $\begin{pmatrix} 1/59 & -63/59 & 54/59 \\ -1/59 & 4/59 & 5/59 \\ 6/59 & 35/59 & -30/59 \end{pmatrix}$.

26.24

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \Longrightarrow \operatorname{adj} A = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix}$$
$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{pmatrix},$$

as in Theorem 8.8.

26.25 *a*) $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$.

b, c) Not invertible.

d) $\begin{pmatrix} -3/2 & 1/4 & 1/2 \\ 5/4 & -1/2 & -1/4 \\ 0 & 1/4 & 0 \end{pmatrix}$.

e) $\begin{pmatrix} -1/80 & 9/40 & 3/80 & -7/20 \\ -1/10 & -1/5 & 3/10 & 1/5 \\ 27/80 & -3/40 & -1/80 & -11/20 \\ 1/20 & 1/10 & -3/20 & 2/5 \end{pmatrix}$.

26.26 *a*) Nine 2×2 determinants at 3 steps each $= 27$ steps.

b) $6 + 6 + 4 = 16$ steps.

c) Gaussian elimination:

$$\begin{aligned} &2n \cdot (n-1) + 2 \cdot (n-1) \cdot (n-2) + \cdots + 2 \cdot 2 \cdot 1 \\ &\quad = 2(1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n) \\ &\quad \leq 2 \cdot (2^2 + 3^2 + \cdots + n^2) \leq \frac{n(n+1)(2n+1)}{3}. \end{aligned}$$

Adjoint method:

Each $(n-1)\times(n-1)$ determinant requires $(n-1)!\cdot(n-2)$ steps and there are n^2 such determinants: $n^2(n-2)\cdot(n-1)!$

26.27 *a*) $x_1 = -13/(-13) = 1,\ x_2 = 52/(-13) = -4.$

b) $x_1 = -12/12 = -1,\ x_2 = 12/12 = 1,\ x_3 = 24/12 = 2.$

c) $x_1 = 0/(-5) = 0, x_2 = 5/(-5) = 1, x_3 = 5/(-5) = -1.$

26.28 *a*) If entries of A are all integers, each $\det A_{ij}$ is an integer. Since $\det A = \pm 1$, each $(\det A_{ij})/(\det A)$ is an integer. But this is $\pm(ji)$th entry of A^{-1}.

b) If all entries of A and A^{-1} are integers, $\det A$ is an integer a and $\det A^{-1}$ is an integer b. By Theorem 26.5, $b = 1/a$. If a and $1/a$ are integers, $a = \pm 1$.

26.29 *a*) Since the leading principal minors are 1, −6, and 28, the pivots are 1, $-6/1 = -6$, and $28/(-6) = -14/3$.

b) $|A_1| = 2,\ |A_2| = 2,\ |A_3| = -4,\ |A_4| = 4.$

So, the pivots of A are $2, 2/2 = 1, -4/2 = -2$ and $4/-4 = -1$.

This agrees with the result of Gauss elimination:

$$\begin{pmatrix} 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

26.30 *b*) Log-demand equations:

$$q_1^d = k_1 + a_{12}\ln(1+t) + a_{11}p_1 + a_{12}p_2 + b_1 y,$$
$$q_2^d = k_2 + a_{22}\ln(1+t) + a_{21}p_1 + a_{22}p_2 + b_2 y.$$

Equilibrium equations:

$$(a_{11} - n_1)p_1 + a_{12}p_2 = m_1 - k_1 - a_{12}\ln(1+t) - b_1 y,$$
$$a_{21}p_1 + (a_{22} - n_2)p_2 = m_2 - k_2 - a_{22}\ln(1+t) - b_2 y.$$

In the expressions for p_1 and p_2 on page 742, replace $a_{11}\ln(1+t)$ and $a_{21}\ln(1+t)$ by $a_{12}\ln(1+t)$ and $a_{22}\ln(1+t)$, respectively.

26.31 Take $\partial/\partial a_{22}$ of both sides of both equations in (20):

$$(a_{11} - n_1)\frac{\partial p_1}{\partial a_{22}} + a_{12}\frac{\partial p_2}{\partial a_{22}} = 0,$$
$$a_{21}\frac{\partial p_1}{a_{22}} + (a_{22} - n_2)\frac{\partial p_2}{\partial a_{22}} + p_2 = 0.$$

Solve via Cramer's rule:

$$\frac{\partial p_1}{\partial a_{22}} = \frac{p_2 a_{12}}{D} \quad \text{and} \quad \frac{\partial p_2}{\partial a_{22}} = \frac{p_2(a_{11} - n_1)}{D}.$$

Do the same using $\partial/\partial n_2$:

$$(a_{11} - n_1)\frac{\partial p_1}{\partial n_2} + a_{12}\frac{\partial p_2}{\partial n_2} = 0,$$
$$a_{21}\frac{\partial p_1}{\partial n_1} + (a_{22} - n_2)\frac{\partial p_2}{\partial n_2} - p_2 = 0.$$

$$\frac{\partial p_1}{\partial n_2} = \frac{-a_{12} p_2}{D}.$$

26.32 Let $q_1^s = m_1 + n_1 p_1 + n_1 \ln(1-t)$ in (19). Write $m_1' = m_1 + n_1 \ln(1-t) < m_1$ since $\ln(1 - t) < 0$.

In (21), $\dfrac{\partial p_1}{\partial m_1} = -\dfrac{(-a_{22} + n_2)}{D} < 0.$

As m_1 decreases to m_1', p_1 rises.

In (22), $\dfrac{\partial p_2}{\partial m_1} = -\dfrac{a_{21}}{D}.$

If goods are complements so that $a_{21} < 0$, $\partial p_2/\partial m_1 > 0$. As m_1 drops to m_1', p_2 falls.

If goods are substitutes so that $a_{21} > 0$, $\partial p_2/\partial m_1 < 0$. As m_1 drops to m_1', p_2 rises.

26.33

$$(q - 1)x + py = 0,$$
$$x + y = 1.$$

Applying Cramer's rule,

$$x = \frac{\det\begin{pmatrix} 0 & p \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} q-1 & p \\ 1 & 1 \end{pmatrix}} = -\frac{p}{q-1-p} = \frac{p}{1+p-q},$$

$$y = \frac{\det\begin{pmatrix} q-1 & 0 \\ 1 & 1 \end{pmatrix}}{\det\begin{pmatrix} q-1 & p \\ 1 & 1 \end{pmatrix}} = \frac{q-1}{1+p-q}.$$

26.34 Suppose that for $(n-1) \times (n-1)$ matrices A, each term in the expansion of $\det A$ contains one and only one entry from each row of A and one and only one entry from each column of A. Let B be an $n \times n$ matrix.

$$\det B = b_{11} \cdot \det B_{11} - b_{12} \cdot \det B_{12} + \cdots \pm b_{1n} \cdot \det B_{1n}.$$

Look at the jth term $\pm b_{ij} \cdot \det B_{ij}$ in this expansion, where B_{ij} is B excluding row 1 and column j. By the induction hypothesis, each term in $\det B_{1j}$ contains exactly one entry from each of rows $2, \ldots, n$ of B and exactly one entry from each of columns $1, \ldots, j-1, j+1, \ldots, n$ of B. Then $b_{1j} \cdot \det B_{ij}$ contains exactly one entry from each row of B and exactly one entry from each column of B. This holds for each $j = 1, \ldots, n$.

26.35 *a*) 2×2: even: $(1,2) \to (1,2)$; odd: $(1,2) \to (2,1)$. 3×3: even: $(1,2,3) \to (1,2,3)$ or $(2,3,1)$ or $(3,1,2)$;

odd: $(1,2,3) \to (2,1,3)$ or $(1,3,2)$ or $(3,2,1)$.

b) Therefore, using (36), det of 2×2 is $a_{11}a_{22} - a_{12}a_{21}$.

$$\begin{aligned}\text{det for } 3 \times 3 = {} & a_{11}a_{22}a_{32} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{12}a_{21}a_{33} - a_{11}a_{32}a_{23} - a_{13}a_{22}a_{31}.\end{aligned}$$

These match (1) and (5), respectively.

Chapter 27

27.1 *a*) Yes.

b) No: $(1,0) + (1,1) = (2,1)$.

c) Yes.

d) No: $(1,1) + (1,-1) = (2,0)$.

e) No: $3 \cdot (0, 1) = (0, 3)$.

f) Yes, $\{(0, 0)\}$.

27.2 If (x_1, x_2) and (y_1, y_2) are nonnegative vectors, then so is $(x_1 + y_1, x_2 + y_2)$. Thus, the nonnegative orthant is closed under addition. But $(-1)(1, 1) = (-1, -1)$, so the nonnegative orthant is not closed under scalar multiplication.

27.3 $\alpha(x_1, 0) = (\alpha x_1, 0)$ and $\beta(0, x_2) = (0, \beta x_2)$. But $(1, 0) + (0, 1) = (1, 1) \notin W$.

27.4 $(a_1, b_1, b_1, c_1) + (a_2, b_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, b_1 + b_2, c_1 + c_2)$, and $r(a, b, b, c) = (ra, rb, rb, rc)$.

27.5 2*a*) $(2, 1), (1, 2)$.

2*b*) $(2, 1)$.

2*c*) $(1, 1, 0), (0, 1, 1)$.

2*d*) $(1, 1, 0), (0, 1, 1), (1, 0, 1)$.

3*a*) $(1, 0, 1, 0), (1, 0, 0, 1), (0, 0, 1, 1)$.

3*b*) $(1, 0, 1, 0), (1, 0, -1, 0)$.

27.6 1) $(1, 1, 1)$.

2) Any maximal linearly independent subset of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$.

3) $(1, -1)$.

4) Not a subspace.

5) $(1, 0, 0, 0), (0, 1, 1, 0), (0, 0, 0, 1)$.

6) No basis.

7) $(1, 1 \ldots, 1)$.

27.7 *a*) $(2, -1)$.

b) $(2, -1, 3), (0, 0, 2)$.

c) $(2, 1), (0, 4)$.

d) $(4, 1, -5, 1), (0, 3, 0, 6), (0, 0, 2, 0)$.

27.8 The corresponding matrix is $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 3 & 5 & 1 & 7 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 2 \end{pmatrix}$. Its row echelon form is $\begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

The basis is: $(1, 2, 0, 3)$ and $(0, -1, 1, -2)$.

27.9 The m rows of A are vectors in $\mathbf{R^n}$.

By Theorem 11.3, if $m > n$, any set of m vectors in $\mathbf{R^n}$ is linearly independent.

27.10 *a*) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

b) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

c) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

d) $\begin{pmatrix} 4 \\ 8 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} -5 \\ -10 \\ 7 \end{pmatrix}$.

27.11 $\begin{pmatrix} 2 \\ 3 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 1 \\ 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}$.

27.12 *a*) $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

b) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

c) The nullspace contains only the vector $\mathbf{0}$.

d) The reduced row echelon form is $\begin{pmatrix} 1 & 0 & 0 & -1/4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. So, $x_1 = .25x_4$, $x_2 = -2x_4$, $x_3 = 0$.

The basis is $\begin{pmatrix} 1/4 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

27.13 *a*) $\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\end{pmatrix}=0.$

b) $\begin{pmatrix}2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=0$ and $\begin{pmatrix}0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix}=0.$

c) $\begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\end{pmatrix}=0$ and $\begin{pmatrix}0\\-4\end{pmatrix}\cdot\begin{pmatrix}0\\0\end{pmatrix}=0.$

d) $\begin{pmatrix}4\\1\\-5\\1\end{pmatrix}\cdot\begin{pmatrix}1/4\\-2\\0\\1\end{pmatrix}=0,\ \begin{pmatrix}0\\3\\0\\6\end{pmatrix}\cdot\begin{pmatrix}1/4\\-2\\0\\1\end{pmatrix}=0,$ and $\begin{pmatrix}0\\0\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1/4\\-2\\0\\1\end{pmatrix}=0.$

27.14 *a*) $A\mathbf{x}=\mathbf{b}$ has a solution iff $\mathbf{b}=c\begin{pmatrix}2\\4\end{pmatrix}$. For such a $\mathbf{b}$, the solution is: $a\begin{pmatrix}1\\2\end{pmatrix}+c\begin{pmatrix}1\\0\end{pmatrix}$ for all a.

b) $A\mathbf{x}=\mathbf{b}$ has a solution iff $\mathbf{b}=c_1\begin{pmatrix}2\\4\end{pmatrix}+c_2\begin{pmatrix}3\\5\end{pmatrix}$.

For such a $\mathbf{b}$, the general solution is $a\begin{pmatrix}1\\2\\0\end{pmatrix}+c_1\begin{pmatrix}1\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\0\\1\end{pmatrix}$ for all a.

c) $A\mathbf{x}=\mathbf{b}$ always has a solution: $\mathbf{x}=A^{-1}\mathbf{b}$.

d) $A\mathbf{x}=\mathbf{b}$ always has a solution.

If $\mathbf{b}=c_1\begin{pmatrix}4\\8\\-4\end{pmatrix}+c_2\begin{pmatrix}1\\5\\2\end{pmatrix}+c_3\begin{pmatrix}-5\\-10\\7\end{pmatrix}$, the general solution is $a\begin{pmatrix}1/4\\-2\\0\\1\end{pmatrix}+c_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+c_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+c_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ for all a.

27.15 *a*) $\text{Col}(C)=\{\mathbf{b}:C\mathbf{x}=\mathbf{b}\text{ for some }\mathbf{x}\}$.

$$\begin{aligned}\mathbf{b}\in\text{Col}(AB)&\Longrightarrow (AB)\mathbf{x}=\mathbf{b}\quad\text{for some}\quad\mathbf{x}\\&\Longrightarrow A(B\mathbf{x})=\mathbf{b}\quad\text{for some}\quad\mathbf{x}\\&\Longrightarrow \mathbf{b}\in\text{Col}(A).\end{aligned}$$

b) Suppose $\mathbf{b}\in\text{Col}(A)$. So, $\mathbf{b}=A\mathbf{z}$ for some $\mathbf{z}$.
Let $\mathbf{x}=B^{-1}\mathbf{z}$, so that $\mathbf{z}=B\mathbf{x}$. Then $\mathbf{b}=A\mathbf{z}=A(B\mathbf{x})=(AB)\mathbf{x}$ and $\mathbf{b}\in\text{Col}(AB)$.

27.16 *a*)

$$\begin{aligned}\mathbf{x} \in \text{nullspace}(A) &\implies A\mathbf{x} = \mathbf{0}\\ &\implies B(A\mathbf{x}) = B \cdot \mathbf{0} = \mathbf{0}\\ &\implies (BA)\mathbf{x} = \mathbf{0}\\ &\implies \mathbf{x} \in \text{nullspace}(BA)\end{aligned}$$

b) If $\mathbf{x} \in \text{nullspace}(BA)$, $BA\mathbf{x} = \mathbf{0}$. Since B is invertible, $A\mathbf{x} = B^{-1}\mathbf{0} = \mathbf{0}$ and $\mathbf{x} \in \text{nullspace}(A)$.

27.17 *a*) Reduced Row Echelon Form (RREF) $= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Basis of Row$(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Basis of Col$(A) = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ (or any basis of $\mathbf{R}^2$).

Basis of nullspace$(A) = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$.

b) RREF $= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Basis of Row$(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Basis of Col$(A) = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix} \right\}$ (or any basis of $\mathbf{R}^2$).

Basis of nullspace$(A) = \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$.

c) RREF $= \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1/3 \end{pmatrix}$. Basis of Row$(A) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1/3 \end{pmatrix} \right\}$.

Basis of Col$(A) = \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 9 \end{pmatrix} \right\}$ (or any basis of $\mathbf{R}^2$).

Basis of nullspace$(A) = \left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} \right\}$.

d) RREF $= \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Basis of Row$(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Basis of Col(A) $= \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ (or any basis of $\mathbf{R^2}$).

Basis of nullspace(A) $= \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right\}$.

27.18 Let A be an $n \times m$ matrix with column vectors $\mathbf{a}_1, \ldots, \mathbf{a}_m$.

According to Theorem 27.6, $\dim \operatorname{col}(A) = \dim \operatorname{Row}(A)$, which in this case equals n. Thus, the set of $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ is a solution is an n-dimensional subspace of $\mathbf{R^n}$, in other words, all of $\mathbf{R^n}$. This proves (1).

The matrix product $A\mathbf{x}$ is the linear combination $x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m$. Thus, the set of matrix products $A\mathbf{x}$ for all $\mathbf{x} \in \mathbf{R^m}$ is the column space of A. Consequently, $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the column space of A. This proves (2). (See Theorem 27.7*a*.)

If $\operatorname{rank}(A) = m$, then according to Theorem 27.10, $\dim \operatorname{nullspace}(A) = m - m = 0$. If $\mathbf{x}$ and $\mathbf{y}$ are both solutions to $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{x} - \mathbf{y}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$, so $\mathbf{x} - \mathbf{y} \in$ nullspace(A). But nullspace(A) contains only the vector $\mathbf{0}$, so $\mathbf{x} = \mathbf{y}$. This proves the first statement in (3); the second follows from Theorem 27.9.

According to Theorem 27.9, the solution set to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_0 + \operatorname{nullspace}(A)$ where $\mathbf{x}_0$ is any particular solution of the equation. Thus, the dimension of the solution set is $\dim \operatorname{nullspace}(A) = m - n$ according to Theorem 27.10. This proves (4).

27.19 *a*) Yes; $\{(x_1, x_2, x_3, x_4) : x_2 - x_3 = 0\} = \{(0 \quad 1 \quad -1 \quad 0)\}\,4$.

b) Yes; $\{(x_1, x_2, x_3, x_4) : x_2 = x_3 = 0\} = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$.

c) No; $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

d) Yes.

e) No. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not in the set.

27.20 *a*) Check properties 1 to 10. Straightforward.

b) Proof 1: Suppose $c_0 \cdot 1 + c_1 x + c_2 x^2 + \cdots + c_n x^n = 0$, the zero function.
Let $f(x) \equiv c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$.
$f(0) = c_0 \Longrightarrow c_0 = 0$.
$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1}$.
$f'(0) = 0 \Longrightarrow c_0 = 0$.

$f''(x) = 2c_2 + 2 \cdot 3c_3x + \cdots + n(n-1)c_nx^{n-2}$.
$f''(0) = 2c_2 = 0 \Longrightarrow c_2 = 0$.
Similarly, $c_3 = \cdots = c_n = 0$.

Proof 2: Use induction on n, the degree of the highest monomial. Note first that the statement must be true when $n = 0$. Suppose now that $\sum_{k=0}^{n-1} a_kx^k \equiv 0$ implies $a_0 = \cdots = a_{n-1} = 0$. Now suppose that for some linear combination, $\sum_{k=0}^{n} a_kx^k \equiv 0$. Observe first that $a_0 = 0$, else the statement would be false for $x = 0$. Therefore $\sum_{k=1}^{n} a_kx^k \equiv 0$. Factoring, $x\sum_{k=0}^{n-1} a_{k+1}x^k \equiv 0$. It follows that for all $x \neq 0$, $\sum_{k=0}^{n-1} a_{k+1}x^k \equiv 0$. By continuity this must hold true for $x = 0$ as well. Finally, the induction hypothesis implies that $a_1 = \cdots = a_n = 0$.

27.21 $0 + 0 = 0$
$(0 + 0)\mathbf{x} = 0\mathbf{x}$
$0\mathbf{x} + 0\mathbf{x} = 0\mathbf{x}$
$0\mathbf{x} + 0\mathbf{x} = \mathbf{0} + 0\mathbf{x}$
$0\mathbf{x} + 0\mathbf{x} - 0\mathbf{x} = \mathbf{0} + 0\mathbf{x} - 0\mathbf{x}$
$0\mathbf{x} = \mathbf{0}$.

27.22 (1) $u, v \in \mathbf{R}_+ \Longrightarrow u \cdot v \in R_+$.

(2) $u \cdot v = v \cdot u$.

(3) $u \cdot (v \cdot w) = (u \cdot v) \cdot w$.

(4) $1 \cdot v = v$ for all v.

(5) $(1/v) \cdot v = v(1/v) = 1$.

(6) $v^r \in \mathbf{R}_+$ for $v \in \mathbf{R}_+$ and $r \in \mathbf{R}$.

(7) $(u \cdot v)^r = u^r \cdot v^r$.

(8) $u^{r+s} = u^r \cdot u^s$.

(9) $(u^r)^s = u^{r \cdot s}$.

(10) $u^1 = u$.

27.23 Let $\mathbf{x} = \mathbf{e}_1 = (1, 0, \ldots, 0)$.
$A\mathbf{e}_1$ = first column of $A = (0, 0, \ldots, 0)$.
So each entry in the first column of A is zero.
$A\mathbf{e}_i$ = ith column of $A = (0, 0, \ldots, 0) \Longrightarrow$ each entry in the ith column is zero.

27.24 Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be a linearly independent set in vector space V.

If $\mathbf{w} \in V$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$,

then $\mathbf{w}_1, \mathbf{v}_1, \ldots, \mathbf{v}_k$ is a linearly dependent set.

Proof: Since $\mathbf{w} = c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k$, $1\cdot\mathbf{w} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_k\mathbf{v}_k = \mathbf{0}$.

Since the coefficient of $\mathbf{w}$ is not zero, $\mathbf{w}, \mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent.

Chapter 28

28.1 Each equation defines a line in the plane, so three equations describe three lines. The figure shows how they can be configured: The three lines can be coincident as in *a*, in which case the entire line is the (one-dimensional) solution set. If two lines are coincident, the third line can either be parallel as in *b*, in which case there are no solutions, or intersect at a single point as in *c*, in which case there is a unique solution. When no lines are coincident, all three can be parallel as in *d*, two can be parallel as in *e*, or no two can be parallel. In cases *d* and *e*, there is no solution. If no two lines are parallel, each pair of lines may intersect at a distinct point as in *f*, in which case there is no solution; or all lines may intersect at a common point. In this last case the equation system has a unique solution, at the point of intersection. See figure.

28.3 For 10 alternatives, M is 45 by 3,628,000. For n alternatives, M is $n(n-1)/2$ by $n!$.

28.4 The eight aggregate pairwise rankings are:

a)	$1 > 2$,	$2 > 3$,	$1 > 3$.
b)	$1 > 2$,	$2 > 3$,	$3 > 1$.
c)	$1 > 2$,	$3 > 2$,	$1 > 3$.
d)	$1 > 2$,	$3 > 2$,	$3 > 1$.
e)	$2 > 1$,	$2 > 3$,	$1 > 3$.
f)	$2 > 1$,	$2 > 3$,	$3 > 1$.
g)	$2 > 1$,	$3 > 2$,	$1 > 3$.
h)	$2 > 1$,	$3 > 2$,	$3 > 1$.

All voters $1 > 2 > 3$ will achieve *a*.

All voters $1 > 3 > 2$ will achieve *c*.

All voters $3 > 1 > 2$ will achieve *d*.

All voters $2 > 1 > 3$ will achieve *e*.

All voters $2 > 3 > 1$ will achieve *f*.

All voters $3 > 2 > 1$ will achieve *h*.

Six voters with $1 > 2 > 3$, five voters with $2 > 3 > 1$, and four voters with $3 > 1 > 2$ will achieve *b*.

Six voters with $3 > 2 > 1$, five voters with $1 > 3 > 2$, and four voters with $2 > 1 > 3$ will achieve *g*.

More generally, consider the system of equations (8). To achieve an aggregate $1 > 2$ ($2 > 1$), choose $y_{12} > 0$ ($y_{12} < 0$), etc. Then, solve system (8) for the vector of profiles $(N_1, \ldots, N_6)$. One way to do this is to find (x_1, x_2, x_3) that satisfies:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_{12} \\ y_{13} \\ y_{23} \end{pmatrix};$$

that is,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (0.5) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_{12} \\ y_{13} \\ y_{23} \end{pmatrix}.$$

Then, for an appropriate N, $\begin{pmatrix} N + x_1 \\ N + x_2 \\ N + x_3 \\ N \\ N \\ N \end{pmatrix}$ will give the appropriate number of voters with each profile to achieve the aggregate paired rankings.

28.6 *a*) Let $w_i = 1$ if $i = 1$ and 0 otherwise.

b) Let N_{ik} denote the number of voters who put alternative i in position k. Then

$$\begin{aligned} i \text{ is preferred to } j \text{ iff} \quad & \sum_k w_k N_{ik} > \sum_k w_k N_{jk} \\ \text{iff} \quad & a \sum_k w_k N_{ik} + b \sum_k N_{ik} > a \sum_k w_k N_{jk} + b \sum_k N_{jk} \end{aligned}$$

since $\sum_k N_{ik} = \sum_k N_{jk} = N$, the total number of voters.

c) For a given vector of weights $\mathbf{w}$, define a new vector of weights $w'_i = w_i + b$ where $b = -w_n$. Then, let $\mathbf{v} = a\mathbf{w}'$ where $a = 1/\sum_i w'_i$. According to part *b*, $\mathbf{w}$, $\mathbf{w}'$ and $\mathbf{v}$ give identical outcomes. But $w'_n = v_n = 0$ and $\sum_j v_j = 1$.

Chapter 29

29.1 *b*), *d*), *e*), *f*) *g*) and *h*) are bounded sequences, with least upper bounds 1, 1, 1, 2, π, and 9, respectively.

29.2 Let $\{x_n\}_{n=1}^{\infty}$ denote a bounded, decreasing sequence with greatest lower bound b. We want to show that $x_n \to b$. For all $\varepsilon > 0$, there is an N such that $b \le x_N < b + \varepsilon$. Since the sequence is decreasing with greatest lower bound b, it follows that for all $n > N$, $b \le x_n \le x_N < b + \varepsilon$. Thus, for all $\varepsilon > 0$ there is an N such that for all $n \ge N$, $|x_n - b| < \varepsilon$, that is, b is the limit of $\{x_n\}_{n=1}^{\infty}$.

29.3 Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x, and that $\{y_m\}_{m=1}^{\infty}$ is a subsequence. Let $n(m)$ denote the index in the original x-sequence of the mth element of the y-sequence. The function $n(m)$ is strictly increasing. Clearly, $n(m) \ge m$. For all $\varepsilon > 0$ there is an N such that for all $n \ge N$, $|x_n - x| < \varepsilon$. Choose $N_1 \ge N$ with $N_1 = n(M_1)$ for some M_1. Then, for all $m \ge M_1 = n^{-1}(N_1)$, $|y_m - x| < \varepsilon$.

29.4 Suppose that b and c are least upper bounds for a set S. Since b is a least upper bound and c is an upper bound, $b \le c$. Since c is a least upper bound and b is an upper bound, $c \le b$. Consequently, $c = b$.

29.5 Suppose x is an accumulation point of a sequence $\{x_n\}_{n=1}^{\infty}$. Then, for all $\varepsilon > 0$ there are infinitely many elements x_n such that $|x_n - x| < \varepsilon$. Let $y_1 = x_1$. Let $n_1 = 1$, and for $i = 1, 2, 3, \ldots$, let n_i denote the first element x_k in the sequence $\{x_n\}_{n=1}^{\infty}$ after $x_{n_{i-1}}$ such that $|x_k - x| < 1/i$. Then the sequence defined by $y_j = x_{n_j}$ converges to x, since $|y_j - x| < 1/j$.

29.6 If a sequence of vectors is Cauchy, then each sequence of coordinate vectors is a Cauchy sequence of real numbers. Thus, according to Theorem 29.3, each coordinate sequence converges; and so the sequence of vectors converges.

29.7 A set S is connected if for each pair of open sets U_1 and U_2, $S \cap U_1 \neq \emptyset$ and $S \cap U_2 \neq \emptyset$ and $S \subset U_1 \cup U_2$ implies that $U_1 \cap U_2 \neq \emptyset$.

29.8 Suppose $\operatorname{cl} S$ is not connected. Then, there exist disjoint open sets U_1 and U_2, each of which intersects $\operatorname{cl} S$, that satisfy $\operatorname{cl} S \subset U_1 \cup U_2$. These two sets cover S since $S \subset \operatorname{cl} S$. Furthermore, $x \in \operatorname{cl} S$ if and only if every open set containing x intersects S, so U_1 and U_2 each intersect S. Hence, S is not connected.

29.9 Let $S = B_{(+1,0)} \cup B_{(-1,0)}$, where $B_{\mathbf{x}}$ is the closed ball of radius 1 about $\mathbf{x}$ in $\mathbf{R}^2$. S is closed and connected. However, when you remove its figure-8-shaped boundary, you find that its interior is the *disjoint* union of two open balls, a disconnected set. See figure.

29.10 The closure of the graph G is $G \cup \{(0, x) : -1 \leq x \leq 1\}$. The fact that this set is connected follows from Problem 29.8 and the fact that the curve G is connected. See figure.

29.11 *i* is closed and connected. *ii* is closed, compact and connected. *iii* is closed and connected. *iv* is the union of the x- and y-axes less the origin. It is not connected, and neither open nor closed. See figure.

29.12 *i* is a closed, connected subspace. *ii* is a connected, open annulus. *iii* is the closed unit simplex in $\mathbf{R}^3$, and is compact and connected.

29.13 $N_{(a_1,\ldots,a_n)}(\mathbf{x})$ is the square root of the sum of nonnegative numbers, and hence nonnegative. If any one x_i is strictly positive, so is the sum and hence so is the square root. Thus a and b hold. Then

$$N_{(a_1,\ldots,a_n)}(r\mathbf{x}) = \sqrt{\sum_i a_i r^2 x_i^2} = |r|\sqrt{\sum_i a_i x_i^2},$$

proving c. Finally,

$$N_{(a_1,\ldots,a_n)}(\mathbf{x} + \mathbf{y}) = \sqrt{\sum_i a_i(x_i + y_i)^2} = \sqrt{\sum_i (\sqrt{a_i} x_i + \sqrt{a_i} y_i)^2}.$$

From Theorem 10.5 it follows that the last term is less than or equal to

$$\sqrt{\sum_i a_i x_i^2} + \sqrt{\sum_i a_i y_i^2} = N_{(a_1,\ldots,a_n)}(\mathbf{x}) + N_{(a_1,\ldots,a_n)}(\mathbf{y}),$$

which proves d. (In Theorem 10.5, take $\mathbf{u} = (a_1 x_1, \ldots, a_n x_n)$ and $\mathbf{v} = (a_1 y_1, \ldots, a_n y_n)$.)

29.14 a, b, c, and d all follow from the corresponding property of the absolute value. For example,

$$\begin{aligned} N_1(\mathbf{x} + \mathbf{y}) &= |x_1 + y_1| + \cdots + |x_n + y_n| \\ &\leq |x_1| + |y_1| + \cdots + |x_n| + |y_n| \\ &= (|x_1| + \cdots + |x_n|) + (|y_1| + \cdots + |y_n|) \\ &= N_1(\mathbf{x}) + N_1(\mathbf{y}). \end{aligned}$$

29.15 $N_0(\mathbf{x})$ is the max of nonnegative elements, hence nonnegative. If $N_0(\mathbf{x}) = 0$, then each $|x|_j = 0$, and so $\mathbf{x} = \mathbf{0}$. $N_0(r\mathbf{x}) = \max\{|rx_1|, \ldots, |rx_n|\} =$

$|r|N_0(\mathbf{x})$, proving c. Finally,

$$\begin{aligned} N_0(\mathbf{x}+\mathbf{y}) &= \max\{|x_1+y_1|, \ldots, |x_n+y_n|\} \\ &\le \max\{|x_1|+|y_1|, \ldots, |x_n|+|y_n|\} \\ &\le \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\}, \end{aligned}$$

proving d.

29.16 Since $|x_i| \le N_0(\mathbf{x})$ for all i, $N_1(\mathbf{x}) = \sum_i |x_i| \le nN_0(x)$. It is obvious from the definitions that $N_0(\mathbf{x}) \le N_1(\mathbf{x})$.

29.17 Suppose that $N \sim N'$. Then, there are $a, b > 0$ such that $aN(x) \le N'(x) \le bN(x)$ for all x. Then $(1/b)N'(x) \le N(x) \le (1/a)N'(x)$, so $N' \sim N$.

Take $a = b = 1$ to see that $N \sim N$ for any norm N.

Suppose $N \sim N'$ and $N' \sim N''$. Then there are $a, b, a', b' > 0$ such that

$$aN(x) \le N'(x) \le bN(x) \quad \text{and} \quad a'N'(x) \le N''(x) \le b'N'(x) \quad \text{for all } x.$$

Then, $aa'N(x) \le N''(x) \le bb'N(x)$, so $N \sim N''$.

29.18 Let $\{b_n\}_{n=1}^{\infty}$ be a sequence, and suppose $N \sim N'$. Then for some positive B and all n, $0 < N'(b_n - b) < BN(b_n - b)$. If the sequence converges to b in the N-norm, then the right-hand side converges to 0. Hence, so does the left; so the sequence converges to b in the N'-norm.

29.19 For $\mathbf{x} \in \mathbf{R^n}$,

$$\begin{aligned} N_0(\mathbf{x}) &= \max_k\{|x_k|\} \\ &\le \sum_k |x_k| \\ &= \sum_k \sqrt{x_k^2} \\ &\le \sqrt{\sum_k x_k^2} = N_2(\mathbf{x}) \end{aligned}$$

where the last inequality follows from the concavity of the square-root function. On the other side,

$$\begin{aligned} N_2(\mathbf{x}) \equiv \sqrt{\sum_k x_k^2} &\leq \sqrt{n \max_k x_k^2} \\ &= \sqrt{n} \max_k |x_k| \\ &= \sqrt{n} N_0(\mathbf{x}). \end{aligned}$$

29.20 The N_0-unit ball in the Euclidean plane is a box: $-1 \leq x_i \leq 1$ for $i = 1, 2$.

The N_1-unit ball is the set of all points $\mathbf{x}$ such that $|x_1| + |x_2| \leq 1$. This set is diamond-shaped, given by the intersection of the following four half-spaces: $x_1 + x_2 \leq 1$, $x_1 - x_2 \leq 1$, $-x_1 + x_2 \leq 1$, and $-x_1 - x_2 \leq 1$.

The N_2-unit ball satisfies the inequality $\|\mathbf{x}\|^2 \leq 1$, which in this case is $x_1^2 + x_2^2 \leq 1$. This inequality describes a disk of radius 1 around the origin. The various weighted Euclidean norms have unit balls given by the inequality $a_1 x_1^2 + a_2 x_2^2 \leq 1$ with $a_1, a_2 > 0$. These are ellipses. See figure.

29.21 This computation is contained in Exercise 19, where it is shown that

$$\sqrt{\sum_k x_k^2} \leq \sqrt{n} \max_k |x_k| = \sqrt{n} N_0(\mathbf{x}).$$

Thus if $N_0(\mathbf{x} - \mathbf{x}_0) < r/\sqrt{n}$, $N_2(\mathbf{x} - \mathbf{x}_0) < r$.

29.22 Suppose the set S is bounded in the N-norm; i.e., $N(\mathbf{y}) \leq a$ for all $\mathbf{y} \in S$. Since all norms in $\mathbf{R^n}$ are equivalent, for any other norm N', $N'(\mathbf{x}) < BN(\mathbf{x})$ for all $\mathbf{x}$. Then for any $\mathbf{y} \in S$, $N'(\mathbf{y}) < Ba$, and S is bounded in the N'-norm.

Similarly, suppose $AN(\mathbf{x}) \leq N'(\mathbf{x}) \leq BN(\mathbf{x})$ for all $\mathbf{x}$. Then, the open N'-ball of radius ε is contained in the open N-ball of radius ε/A and contains the open N-ball of radius ε/B. Thus, every N-open set is an N'-open set, and vice versa. Since closed sets are complements of open sets, the closed sets for the two norms must be identical too. Since both norms have the same open sets, a set S is N-connected if and only if it is N'-connected. It follows from the definition and the first part of this question that a set S is N'-closed and bounded if and only if it is N-closed and bounded.

29.23 *a*) N_0-norm: For any norm N, write $B(\mathbf{x}, \varepsilon, N)$ for the open set

$$B(\mathbf{x}, \varepsilon, N) \equiv \{\mathbf{y} \in \mathbf{R^n} : N(\mathbf{y} - \mathbf{x}) < \varepsilon\}.$$

Now, $\mathbf{x} \in \mathbf{R}^{\mathbf{n}}_{++}$ implies that each component $x_i > 0$ and $\min x_i = \varepsilon > 0$. We'll show that the open set $B(\mathbf{x}, \varepsilon/2, N_0)$ lies in $\mathbf{R}^{\mathbf{n}}_{++}$. Let $\mathbf{y} \in B(\mathbf{x}, \varepsilon/2, N_0)$. Then, $\max_i |y_i - x_i| < \varepsilon/2$, i.e.,

$$-\frac{\varepsilon}{2} < y_i - x_i < \frac{\varepsilon}{2} \quad \text{for all } i,$$

or

$$y_i > x_i - \frac{\varepsilon}{2} > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} > 0 \quad \text{for all } i.$$

So, $B(\mathbf{x}, \varepsilon/2, N_0) \subset \mathbf{R}^{\mathbf{n}}_{++}$.

b) N_1-norm: As above, $\mathbf{x} \in \mathbf{R}^{\mathbf{n}}_{++}$ implies that $\min x_i = \varepsilon > 0$. We'll show that the open set $B(\mathbf{x}, \varepsilon/2, N_1)$ lies in $\mathbf{R}^{\mathbf{n}}_{++}$. Since $\max |y_i - x_i| \leq \sum_j |y_j - x_j|$, $B(\mathbf{x}, \varepsilon/2, N_1) \subset B(\mathbf{x}, \varepsilon/2, N_0) \subset \mathbf{R}^{\mathbf{n}}_{++}$.

29.24 Suppose that a finite subcollection covers $(0, 1)$. Let N denote the largest n such that the interval $\big(1/(n+1), n/(n+1)\big)$ is in the collection. Since the sets in the collection are nested in each other, the union of all elements in the collection is $\big(1/(n+1), n/(n+1)\big)$. If $0 < x < 1/(n+1)$, then x is not in the union; so $(0, 1)$ does not have the finite covering property.

29.25 The sets with $n = 1, \ldots, 10$ together with the two additional sets cover $[0, 1]$. The finite covering property requires that *every* open cover of $[0, 1]$ has a finite subcover.

29.26 Closed: Let $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ be a finite set in $\mathbf{R}^{\mathbf{n}}$. let $\varepsilon_0 = \min\{\|\mathbf{a}_i - \mathbf{a}_j\| : \mathbf{a}_i \neq \mathbf{a}_j \in S\}$. Let $\{\mathbf{z}_j\}_{j=1}^{\infty}$ be a convergent sequence in S with limit $\mathbf{z}_0$. Since the sequence is Cauchy, there exists N such that $j, k > N$ implies $\|\mathbf{z}_j - \mathbf{z}_k\| < \varepsilon/2$. By the definition of ε, this requires that $\mathbf{z}_N = \mathbf{z}_{N+1} = \ldots$. So, $\mathbf{z}_0$ is this common value and lies in S. Therefore, S is closed.

Bounded: Let $B = \max\{\|\mathbf{a}_i\| : \mathbf{a}_i \in S\}$; B makes S bounded.

Sequential compactness: Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence in S. Since S is finite, there is some point $\mathbf{y} \in S$ such that $\mathbf{x}_n = \mathbf{y}$ for infinitely many n. The subsequence consisting of all $\mathbf{x}_n$ such that $\mathbf{x}_n = \mathbf{y}$ is obviously convergent.

Finite subcover: Let S denote an open cover of S. For each $\mathbf{x} \in S$ choose a single set $U_{\mathbf{x}} \in S$ such that $\mathbf{x} \in U_{\mathbf{x}}$. The collection $\{U_{\mathbf{x}}, \mathbf{x} \in S\}$ is a finite subcover.

29.27 According to Theorem 29.14, for each $\mathbf{x} \in K_1$ there is an open set $U_{\mathbf{x}}$ containing $\mathbf{x}$ and an open set $V_{\mathbf{x}}$ containing all of K_2 such that $U_{\mathbf{x}} \cap V_{\mathbf{x}} = \emptyset$. The collection $\{U_{\mathbf{x}}\}_{\mathbf{x} \in K_1}$ is an open cover of K_1. Choose a finite subcover

$\{U_{\mathbf{x}_1}, \ldots, U_{\mathbf{x}_n}\}$. The union U of these sets contains K_1. Let $V = \cap_{k=1}^{n} V_{\mathbf{x}_k}$. Since each $V_{\mathbf{x}_k}$ is open and the intersection is a finite one, V is open. Since each $V_{\mathbf{x}_k}$ contains K_2, so does V. Finally, $U \cap V = \emptyset$.

29.28 The intersection and finite union of closed and bounded sets are closed and bounded.

Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ denote a sequence of points in either the intersection or finite union of compact sets. In either case there is a compact set A containing an infinite number of points in the sequence. Let $\{\mathbf{y}_m\}_{m=1}^{\infty}$ denote the subsequence containing those points in the original sequence which are in the compact set A. Since A is compact, the sequence $\{\mathbf{y}_m\}_{m=1}^{\infty}$ has a convergent subsequence with limit $\mathbf{y}$. This limit point is obviously in the finite union of the compact sets, so the finite union is compact. In the intersection case, this convergent subsequence is contained in every compact set, so its limit $\mathbf{y}$ is contained in every compact set, and so $\mathbf{y}$ is in the intersection.

Let $\{K_1, \ldots, K_n\}$ denote a finite collection of compact sets, whose union is K. Let S be an open cover of K. Let S_m denote the collection of all elements of S which intersect K_m. Since K_m is compact, each S_m has a finite subcover of K_m. The (finite) union of these finite subcovers is a finite subcover of K.

Let $\{K_a\}_{a \in A}$ denote a collection of compact sets with intersection K. K is closed. Let S be an open cover of K, and let K_a be a compact set in the collection of compact sets. The collection $S \cup \{K^c\}$ is an open cover of K_a. Since K_a is compact, it has a finite subcover S'. Since this subcover covers K_a, it has to cover K. The finite collection $S' \setminus \{K^c\}$ consists only of sets in S, and covers K.

29.29 Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ denote a sequence of points in a closed subset C of a compact set K. The sequence has a convergent subsequence in K with limit $\mathbf{x}$. Since C is closed, $\mathbf{x} \in C$; so C is sequentially compact.

A closed subset of a bounded set is bounded, so C is compact.

Let S be a cover of C. Then $S \cup C^c$ is an open cover of K. Hence, it has a finite subcover S'. Finally, observe that $S' \setminus C^c$ covers C, so C has the Heine-Borel property.

Chapter 30

30.1 Apply the argument of the text to the continuous function $|F(x)|$ to show that $|F(x)|$ is bounded. Thus there is a $b > -\infty$ such that $F(x) > b$ for all $x \in C$. Let B denote the greatest lower bound of the values that F takes in C. Since B is the *greatest* lower bound, for all n there is an $x_n \in C$ such that $F(x_n) < B + (1/n)$. Since C is compact, we can extract a convergent subsequence $\{w_n\}_{n=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ with limit w. Since F is continuous, $\lim_n F(w_n) = F(w) = B$, so w is the global min of F in C. Alternatively, apply the proof for the existence of a global max to the function $-F$.

30.2 Denote the two sets $A_=$ and $A_\leq$. To prove $A_=$ is closed, we begin with a sequence of points $\{x_n\}_{n=1}^\infty$ in $A_=$ with limit x. We need to show that $x \in A_=$. Suppose (without loss of generality) that $F(x) \leq c$. Since F is continuous, $F(x_n) \to F(x)$. Now $F(x_n) \geq c$ for all n, so it follows from Theorem 12.4 that $F(x) \geq c$, and therefore that $F(x) = c$; so $x \in A_=$. Exactly the same argument applies to $A_\leq$.

30.3 The function $F(x) = 1 - \dfrac{1}{(1+x)}$ is a bounded function on its closed domain on $[0, \infty)$, but it does not achieve its supremum of 1.

30.4 *a*) Let N_0, N_1 and $N_2 = \|\ \|$ denote the standard norms on $\mathbf{R^n}$, as described in Section 29.4. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ denote the canonical basis of $\mathbf{R^n}$. Let $q_i = N(\mathbf{e}_i)/\|\mathbf{e}_i\|$ and $q = \max_i q_i$. For $\mathbf{x} = \sum_i x_i \mathbf{e}_i$,

$$\begin{aligned}
N(\mathbf{x}) &\leq \sum_i |x_i| N(\mathbf{e}_i) && \text{by properties } c \text{ and } d \text{ of a norm} \\
&= \sum_i |x_i|\, q_i \,\|\mathbf{e}_i\| && \text{by the definition of } q_i \\
&\leq q \sum_i |x_i|\, \|\mathbf{e}_i\| && \text{by definition of } q \\
&= qN_1(\mathbf{x}) && \text{by definition of } N_1 \\
&\leq qnN_0(\mathbf{x}) && \text{since } N_1(\mathbf{x}) \leq nN_0(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbf{R^n} \\
&\leq qn\|\mathbf{x}\| && \text{by equation (29.8).}
\end{aligned}$$

From this and property *a* it follows that if $\{\mathbf{x}_n\}_{n=1}^\infty$ converges to $\mathbf{0}$ (in N_2-norm), then $0 \leq N(\mathbf{x}_n) \leq qn\|\mathbf{x}_n\| \to 0$, so $N(\mathbf{x}_n) \to N(\mathbf{0})$.

Now suppose that $\{\mathbf{x}_n\}_{n=1}^\infty$ converges to $\mathbf{x}$ in the Euclidean norm. Then, $\mathbf{x}_n - \mathbf{x}$ converges to 0, so $N(\mathbf{x}_n - \mathbf{x})$ converges to 0. Then $N(\mathbf{x}_n) = N(\mathbf{x} + \mathbf{x}_n - \mathbf{x}) \leq N(\mathbf{x}) + N(\mathbf{x}_n - \mathbf{x})$ from property *d* of a norm; so $\limsup N(\mathbf{x}_n) \leq N(\mathbf{x})$. Similarly, $N(\mathbf{x}) = N(\mathbf{x}_n + \mathbf{x} - \mathbf{x}_n) \leq N(\mathbf{x}_n) + N(\mathbf{x}_n - \mathbf{x})$; so $\liminf N(\mathbf{x}_n) \geq N(\mathbf{x})$. Thus, $N(\mathbf{x}_n)$ converges to $N(\mathbf{x})$ as $n \to \infty$. Therefore, the norm N is a continuous function.

b) The unit sphere S is a compact subset of $\mathbf{R^n}$. It follows from part *a* and Weierstrass's theorem that N achieves its minimum value m_1 on S at some point $\mathbf{x}^* \in S$ and its maximum value m_2 at some $\mathbf{y}^* \in S$. Since $\mathbf{0}$ is not in the unit sphere (sphere, not ball), neither $\mathbf{x}^*$ nor $\mathbf{y}^*$ is $\mathbf{0}$. By property *b* of a norm, $N(\mathbf{y}^*) \geq N(\mathbf{x}^*) > 0$.

c) For any $\mathbf{x} \in \mathbf{R^n}$, write $\mathbf{x} = r\mathbf{v}$ where $r = \|\mathbf{x}\|$ and $\mathbf{v} = \mathbf{x}/\|\mathbf{x}\|$. By property *c* of a norm, $\|\mathbf{v}\| = 1$. By part *b* of this exercise, $m_1 \le N(\mathbf{v}) \le m_2$, and therefore $rm_1 \le rN(\mathbf{v}) \le rm_2$. Since $N(\mathbf{x}) = N(r\mathbf{v}) = rN(\mathbf{v})$ and $r = \|\mathbf{x}\|$, $m_1\|\mathbf{x}\| \le N(\mathbf{x}) \le m_2\|\mathbf{x}\|$.

d) Part *c* proves that all norms in $\mathbf{R^n}$ are equivalent to the Euclidean norm N_2. If N and N' are two arbitrary norms on $\mathbf{R^n}$, then there are positive constants a_1, a_2, b_1, b_2 such that

$$a_1 N(\mathbf{x}) \le N_2(\mathbf{x}) \le b_1 N(\mathbf{x}) \quad \text{and} \quad a_2 N'(\mathbf{x}) \le N_2(\mathbf{x}) \le b_2 N'(\mathbf{x})$$

for all vectors $\mathbf{x} \in \mathbf{R^n}$ by part *c*. It follows that

$$\frac{a_1}{b_2} N(\mathbf{x}) \le N'(\mathbf{x}) \quad \text{and} \quad \frac{a_2}{b_1} N'(\mathbf{x}) \le N(\mathbf{x}).$$

So, all norms on $\mathbf{R^n}$ are equivalent. In particular, a sequence that converges in one norm converges in any other norm.

30.5 The third order approximation is: $e^h \approx 1 + h + (1/2)h^2 + (1/6)h^3$,
and the fourth order one is: $e^h \approx 1 + h + (1/2)h^2 + (1/6)h^3 + (1/24)h^4$.
The third order approximation at $h = 0.2$ is 1.22133 with error 0.0000694.
The fourth order approximation at $h = 0.2$ is 1.22140 with error 0.00000276.
The third order approximation at $h = 1$ is 2.66667 with error 0.0516152.
The fourth order approximation at $h = 1$ is 2.718282 with error 0.009978.

30.6 $(x + h)^{3/2} \approx x^{3/2} + (3/2)x^{1/2}h + (1/2)(3/4)x^{-1/2}h^2$. Taking $x = 4$ and $h = 0.2$ gives $(4.2)^{3/2} \approx 8 + 3 \cdot 0.2 + (3/16) \cdot 0.04 = 8.6075$. The actual value to four decimal places is 8.6074.

30.7 For $F(x) = \sqrt{1 + x}$,

$$P_1(h) = 1 + \frac{h}{2}, \quad P_2(h) = 1 + \frac{h}{2} - \frac{h^2}{8}, \quad P_3(h) = 1 + \frac{h}{2} - \frac{h^2}{8} + \frac{h^3}{16}.$$

For $h^* = 0.2$, $F(h^*) = 1.09545$ to 5 decimal places.
$P_1(h^*) = 1.1$ and the error is 0.00455488.
$P_2(h^*) = 1.095$ and the error is -0.000445115.
$P_3(h^*) = 1.0955$ and the error is 0.000054885.
For $h^* = 1.0$, $F(h^*) = 1.41421$ to 5 decimal places.
$P_1(h^*) = 1.5$ and the error is 0.0857864.
$P_2(h^*) = 1.375$ and the error is -0.0392136. $P_3(h^*) = 1.4375$ and the error is 0.0232864.

For $F(x) = \ln x + 1$,

$$P_1(h) = h, \quad P_2(h) = h - \frac{h^2}{2}, \quad P_3(h) = h - \frac{h^2}{2} + \frac{h^3}{3}.$$

For $h^* = 0.2$, $F(1 + h^*) = 0.182322$ to six decimal places.
$P_1(h^*) = 0.2$ and the error is 0.00177.
$P_2(h^*) = 0.18$ and the error is -0.00232156.
$P_3(h^*) = 0.182667$ and the error is 0.00023511.
For $h^* = 1.0$, $F(1 + h^*) = 0.693147$ to six decimal places.
$P_1(h^*) = 1.0$ and the error is 0.306853.
$P_2(h^*) = 0.5$ and the error is -0.193147.
$P_3(h^*) = 0.833333$ and the error is 0.140186.

30.8 Suppose ℓ_1 and ℓ_2 are two such numbers, with corresponding remainder terms $R_1(h)$ and $R_2(h)$. Then $\ell_1 - \ell_2 = R_2(h)/h - R_1(h)/h$. Taking limits as $h \to 0$, $\ell_1 - \ell_2 = 0$.

30.9 Proof of Theorem 30.5:

$$g_1(a) = f(a) - f(a) - f'(a)(a - a) - M_1(a - a)^2 = 0.$$

Also,

$$g'(t) = f'(t) - f'(a) - 2M_1(t - a).$$

At $t = a$, $g'(a) = f'(a) - f'(a) - 2M_1(a - a) = 0$.

Proof of Theorem 30.6:

$$\begin{aligned} g_2(a + h) &= f(a + h) - f(a) - f'(a)h - (1/2)f''(a)h^2 \\ &\quad - f(a + h) + f(a) + f'(a)h + (1/2)f''(a)h^2 = 0 \\ g_2(a) &= f(a) - f(a) - f'(a)(a - a) - (1/2)f''(a)(a - a)^2 - M_2(a - a)^3 \\ &= 0 \\ g_2'(a) &= f'(a) - f'(a) - f''(a)(a - a) - 3M_2(a - a)^2 = 0 \\ g_2''(a) &= f''(a) - f''(a) - 6M_2(a - a) = 0. \end{aligned}$$

30.12 *a*)

$$\begin{aligned} P_1(h_1, h_2) &= h_1 \\ P_2(h_1, h_2) &= h_1 - \frac{h_1 h_2}{2} \\ P_3(h_1, h_2) &= h_1 - \frac{h_1 h_2}{2} + \frac{h_1 h_2^2}{3}. \end{aligned}$$

b)

$$P_1(h_1, h_2) = 1 + h_1$$

$$P_2(h_1, h_2) = 1 + h_1 + \frac{h_1^2}{2} + \frac{h_2^2}{2}$$

$$P_2(h_1, h_2) = 1 + h_1 + \frac{h_1^2}{2} + \frac{h_2^2}{2} + \frac{h^3}{6} + \frac{hk^2}{6}.$$

d)

$$P_1(h_1, h_2) = k[a(h_1 - 1) + b(h_2 - 1)]$$

$$P_2(h_1, h_2) = k[a(h_1 - 1) + b(h_2 - 1)] + \frac{a(a-1)(h_1-1)^2 + ab(h_1-1)(h_2-1) + b(b-1)(h_2-1)^2}{2}$$

$$P_3(h_1, h_2) = k[a(h_1 - 1) + b(h_2 - 1)] + \frac{a(a-1)(h_1-1)^2 + ab(h_1-1)(h_2-1) + b(b-1)(h_2-1)^2}{2} + \frac{1}{6}[a(a-1)(a-2)(h_1-1)^3 + a(a-1)b(h_1-1)^2(h_2-1)] + \frac{1}{6}[ab(b-1)(h_1-1)(h_2-1)^2 + b(b-1)(b-2)(h_2-1)^3].$$

30.17 From the Chain Rule, at $t = 0$

$$\frac{d}{dt} f(\mathbf{x} + t\mathbf{w}) = Df(\mathbf{x}) \cdot \mathbf{w} \quad \text{and} \quad \frac{d^2}{dt^2} f(\mathbf{x} + t\mathbf{w}) = \mathbf{w}^t \cdot D^2 f(\mathbf{x}) \cdot \mathbf{w}.$$

The claim now follows from Theorem 3.4.

30.20

$$f(x_0 + h) - f(x_0) = \frac{1}{k!} f^{[k]}(x_0) h^k + R_k(h)$$

where $R_k(h)/h^k \longrightarrow 0$ as $h \longrightarrow 0$. For small $|h|$, if $f^{[k]}(x_0) \neq 0$, then

$$\frac{\text{sgn}\, f(x_0 + h) - f(x_0)}{h^k} = \text{sgn}\, f^{[k]}(x_0)/k!.$$

Suppose k is even. Then $h^k > 0$ for all nonzero h. If $f^{[k]}(x_0) > 0$, then for all sufficiently small h, $f(x_0 + h) > f(x_0)$ so x_0 is a local min. If $f^{[k]}(x_0) < 0$, then for all sufficiently small h, $f(x_0 + h) < f(x_0)$ so x_0 is a local max.

Suppose k is odd and $f^{[k]}(x_0) \neq 0$. If $h < 0$ and small, then

$$\text{sgn}\, f(x_0 + h) - f(x_0) = -\,\text{sgn}\, \frac{f(x_0 + h) - f(x_0))}{h^k} = -\,\text{sgn}\, f^{[k]}(x_0)/k!.$$

If $h > 0$ and small, then

$$\operatorname{sgn} f(x_0 + h) - f(x_0) = \operatorname{sgn} \frac{f(x_0 + h) - f(x_0))}{h^k} = \operatorname{sgn} f^{[k]}(x_0)/k!.$$

The sign of $f(x_0 + h) - f(x_0)$ changes as h passes through 0, so x_0 is neither a local maximum nor a local minimum.

Appendix 1

A1.1 $C \cup A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 18, 21, \ldots\}$

$\cup \{0, -2, -4, -6, \ldots\}$.

$C \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 15, 17, 19, 21, \ldots\}$

$\cup \{-1, -3, -5, -7, \ldots\}$.

$C - B = \{2, 4, 6, 8, 10\}$.

$A \cap D = \{0, 2, 4, 6, 8, 10, \ldots\}$.

$B \cup D = \mathbf{R}_+ \cup \{-1, -3, -5, \ldots\}$.

$A \cup B =$ all integers.

$A \cap B = \emptyset$.

A1.2 *a*) Glb $= \{1\}$, no lub.

b) Glb $= \{0\}$, lub $= \{1\}$.

c) Glb $= \{-1\}$, lub $= \{+1\}$.

d, e, f) Glb $= \{0\}$, lub $= \{1\}$.

A1.3 *a*) $x \in (A \cap B)^c \iff x \notin A \cap B \iff x \notin A$ or $x \notin B \iff x \in A^c$ or $x \in B^c$ $x \in A^c \cup B^c$.

b) $x \in (A \cup B)^c \iff x \notin A \cup B \iff x \notin A$ and $x \notin B \iff x \in A^c$ and $x \in B^c \iff x \in A^c \cap B^c$.

c) $x \in A \cap (B \cup C)$

$\iff x \in A \quad$ and $\quad x \in B \cup C$

$\iff (x \in A \quad$ and $\quad x \in B)$ or $(x \in A \quad$ and $\quad x \in C)$

$\iff x \in A \cap B$ or $x \in A \cap C$

$\iff x \in (A \cap B) \cup (A \cap C)$.

A1.4 **Lemma**: If 3 divides n^2, 3 divides n.

Proof: Any natural number n can be written as $n = 3k + a$ where $k \in N$ and $a \in \{0, 1, 2\}$. In particular, 3 divides n iff $a = 0$. For $n = 3k + a$, $n^2 = 9k^2 + 6ka + a^2$. If 3 divides n^2, $n^2 = 3r = 9k^2 + 6ka + a^2$ for some $r \in N$. So, $r = 3k^2 + 2ka + (a^2/3)$.
But, r is an integer iff $a^2/3$ is an integer iff $a = 0$ iff 3 divides n. □

Theorem: $\sqrt{3}$ is irrational.

Proof: Suppose $\sqrt{3}$ is rational, i.e., a quotient of integers: $\sqrt{3} = p/q$. We can assume that 3 does not divide both p and q. Then, $\sqrt{3} \cdot q = p$ or $3 \cdot q^2 = p^2$.
This implies 3 divides p^2. By the lemma, 3 divides p. So p can be written $p = 3m$ for some m in N, the set of natural numbers.

Then $3q^2 = p^2 = 9m^2$ or $q^2 = 3m^2$. This implies 3 divides q^2. By the lemma, 3 divides q. This contradicts our choice of p and q—not both divisible by 3.
We conclude that $\sqrt{3}$ cannot be written as p/q, i.e., is irrational. □

A1.5 True for $n = 1$: $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$.

Assume true for $n = k$:

$$1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \qquad (IH)$$

Add $(k+1)^2$ to both sides of (IH):

$$\begin{aligned} 1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)(k+2)\big(2(k+1)+1\big)}{6}, \end{aligned}$$

which is (IH) with $k + 1$ replacing k. This establishes the induction.

A1.6 $\dfrac{1}{1 \cdot 2} = \dfrac{1}{2}, \qquad \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} = \dfrac{2}{3}, \qquad \dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} = \dfrac{3}{4}, \qquad \ldots$

Guess:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}. \qquad (G)$$

We have seen that (G) is true for $n = 1, 2, 3$.

Assume (G) holds for $n = k$ and add $1/[(k+1)(k+2)]$ to both sides:

$$\begin{aligned}\frac{1}{1\cdot 2}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}.\end{aligned}$$

That is, (G) holds for $n = k + 1$. This establishes the induction.

A1.7 True for $n = 1$ since $1 < 2^1 = 2$.
Assume true for $n = k$: $k < 2^k$.
Multiply both sides by $(k+1)/k = 1 + 1/k$ which is less than 2 for $k > 1$):

$$\begin{aligned}k+1 &< 2^k\cdot\frac{(k+1)}{k}\\ &= 2^k\left(1+\frac{1}{k}\right)\\ &< 2^k\cdot 2\\ &= 2^{k+1}.\end{aligned}$$

Appendix 2

A2.1 Using the notation of Figure A2.2,

$$\tan\theta=\frac{\|BC\|}{\|AC\|}=\frac{\|BC\|}{\|AB\|}\bigg/\frac{\|AC\|}{\|AB\|}=\frac{\sin\theta}{\cos\theta}$$

$$\cot\theta=\frac{\|AC\|}{\|BC\|}=\frac{1}{\tan\theta}=\frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{\|AB\|}{\|AC\|} = 1\Big/\frac{\|AC\|}{\|AB\|} = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{\|AB\|}{\|BC\|} = 1\Big/\frac{\|BC\|}{\|AB\|} = \frac{1}{\sin\theta}.$$

A2.2 For graphs of cotangent, cosecant X, secant X. See figures.

A2.3 Use Figure A2.2 for angles between 0 and 90 degrees. If $\theta = \angle BAC$, then $90 - \theta = \angle ABC$.

$$\cos\theta = \|AC\|\Big/\|AB\|$$

$$\sin(90-\theta) = \sin(\angle ABC) = \frac{\text{opposite leg}}{\text{hypotenuse}}$$

$$= \frac{\|AC\|}{\|AB\|} = \cos\theta.$$

Similarly,

$$\cos(90-\theta) = \cos(\angle ABC) = \frac{\text{adjacent leg}}{\text{hypotenuse}}$$

$$= \frac{\|BC\|}{\|AB\|} = \sin\theta.$$

A2.4 *a*) $\sin 120^\circ = \sin 60^\circ = \sqrt{3}/2$, $\cos 120^\circ = -\cos 60^\circ = -1/2$, $\tan 120^\circ = -\sqrt{3}$.

b) $\sin 135^\circ = \sin 45^\circ = 1/\sqrt{2}$, $\cos 135^\circ = -\cos 45^\circ = -1/\sqrt{2}$, $\tan 135^\circ = 1$.

f) $\sin 240^\circ = -\sin 60^\circ = -\sqrt{3}/2$, $\cos 240^\circ = -\cos 60^\circ = -1/2$, $\tan 240^\circ = \sqrt{3}$.

A2.5 *a*) $\cot 30^\circ = \cos 30^\circ/\sin 30^\circ = \sqrt{3}$, $\sec 30^\circ = 2/\sqrt{3}$, $\csc 30^\circ = 2$.

b) $\cot 60^\circ = 1/\sqrt{3}$, $\sec 60^\circ = 2$, $\csc 60^\circ = 2/\sqrt{3}$.

A2.6 Let $X = \sin a$, $Y = \cos a$, $x = \cos a/2$, $y = \sin a/2$. By Theorem A2.2, $X = 2xy$ and $Y = x^2 - y^2$. Solve for x and y in terms of X and Y (compare with Exercise 15.33) to find $x = X/2y$, $Y = (X^2/4y^2) - y^2 = (X^2 - 4y^4)/4y^2$.

$$4y^4 + 4Yy^2 - X^2 = 0 \implies y^2 = \frac{-4Y \pm \sqrt{16Y^2 + 16X^2}}{8} = -\frac{Y}{2} \pm \frac{1}{2} = \frac{\pm 1 - Y}{2}$$

since $X^2 + Y^2 = 1$.

$$y^2 \geq 0 \implies y^2 = \frac{1}{2}(1 - Y) \implies y = \pm\sqrt{0.5(1 - Y)}.$$

Substitution gives: $\sin(a/2) = \pm\sqrt{0.5(1 - \cos a)}$. Similarly, $\cos(a/2) = \pm\sqrt{0.5(1 + \cos a)}$.

A2.7 *a*) $\sin 15° = \sin(60° - 45°) = 2^{-3/2}(\sqrt{3} - 1)$.

d) $\cos 22.5° = \cos\left(\frac{1}{2} \cdot 45\right) = 2^{-3/4}\left(\sqrt{2} + 1\right)^{1/2}$.

A2.8 See figure.

A2.9

0.01	$\frac{\sin x}{x}$ (x radians)	$\frac{\sin x}{x}$ (x degrees)
1	0.84	0.01745
0.1	0.998334	0.01745
0.01	0.999983	0.01745

A2.10

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\sin' x \cdot \cos x - \sin x \cos' x}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

$$(\cot x)' = \left(\frac{1}{\tan x}\right)' = -\frac{1}{\tan^2 x} \cdot \sec^2 x = -\frac{\cos^2 x}{\sin^2 x} \cdot \frac{1}{\cos^2 x} = -\frac{1}{\sin^2 x} = -\csc^2 x.$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{1}{\cos^2 x}(-\sin x)$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$
$$= \tan x \cdot \sec x.$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{1}{\sin^2 x}(\cos x)$$
$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$
$$= -\cot x \cdot \csc x.$$

A2.11

$$\cos\frac{\pi}{4} \approx 1 - \frac{1}{2!}\left(\frac{\pi}{4}\right)^2 + \frac{1}{4!}\left(\frac{\pi}{4}\right)^4 \approx .707429,$$

which compares well with 0.707107.

$$\sin\frac{\pi}{3} \approx \frac{\pi}{3} - \frac{1}{3!}\left(\frac{\pi}{3}\right)^3 + \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx .866295,$$

which compares well with 0.866025.

A2.12

$$\begin{array}{ll} f(x) = \sin x & f(0) = 0 \\ f'(x) = \cos x & f'(0) = 1 \\ f''(x) = -\sin x & f''(0) = 0 \\ f^{[3]}(x) = -\cos x & f^{[3]}(0) = -1 \\ f^{[4]}(x) = \sin x & f^{[4]}(0) = 0 \\ f^{[5]}(x) = \cos x & f^{[5]}(0) = 1 \end{array}$$

Taylor series:

$$0 + \frac{1}{1!}\cdot 1 \cdot x + \frac{1}{2!}\cdot 0 \cdot x^2 + \frac{1}{3!}(-1)\cdot x^3 + \frac{1}{4!}\cdot 0 \cdot x^4 + \frac{1}{5!}\cdot 1 \cdot x^5 + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Similarly, for $g(x) = \cos x$.

A2.13 Let N be any fixed integer $> a$.Then, $a/N < 1$ and $(a/N)^m \to 0$ as $m \to \infty$. Choose $n > N$.

$$\frac{a^n}{n!} = \frac{a^N}{N!}\left(\frac{a}{N+1} \cdot \frac{a}{N+2} \cdots \frac{a}{N+(n-N)}\right)$$

$$\leq \frac{a^N}{N!}\left(\frac{a}{N}\cdot\frac{a}{N}\cdots\frac{a}{N}\right)$$
$$= \frac{a^N}{N!}\cdot\left(\frac{a}{N}\right)^{n-N} = \left(\frac{a^N}{N!}\cdot\left(\frac{N}{a}\right)^N\right)\cdot\left(\frac{a}{N}\right)^n$$
$$= \frac{N^N}{N!}\cdot\left(\frac{a}{N}\right)^N.$$

Let $n \longrightarrow \infty$; keep in mind that N is fixed. $(a/N)^n \longrightarrow 0$. Therefore, $a^n/n! \longrightarrow 0$.

Appendix 3

A3.1 By (4) solutions are: $\dfrac{4}{2} \pm \dfrac{\sqrt{-36}}{2} = 2 \pm \dfrac{i\cdot 6}{2} = 2 \pm 3i.$

$$\begin{aligned}(2+3i)^2 - 4(2+3i) + 13 &= 4 + 12i + 9i^2 - 8 - 12i + 13\\ &= (4-9-8+13) + i(12-12)\\ &= 0.\end{aligned}$$

A3.2 $z_1 = 2 - 3i, \quad z_2 = 3 + 4i, \quad z_3 = 1 + i.$

$$z_1 + z_2 = 5 + i \qquad z_1 - z_3 = 1 - 4i$$

$$z_1 \cdot z_2 = 18 - i \qquad \frac{z_1}{z_3} = \frac{(2-3i)(1-i)}{(1+i)(1-i)} = \frac{(2-3)-i(3+2)}{1-i^2} = \frac{1}{2}(-1-5i)$$

$$z_1 \cdot \overline{z_1} = (2-3i)(2+3i) = 4 - 9i^2 = 13 \qquad z_1 \cdot \overline{z_3} = (2-3i)(1-i) = (-1) - 5i.$$

A3.3 $\dfrac{1}{a+bi}\cdot\dfrac{a-bi}{a-bi} = \dfrac{a-bi}{a^2+b^2} = \left(\dfrac{a}{a^2+b^2}\right) - i\left(\dfrac{b}{a^2+b^2}\right).$

A3.4 *b*) Let $z_1 = x_1 + iy_1$, and $z_2 = x_2 + iy_2$.

$$\begin{aligned}(\overline{z_1} + \overline{z_2}) &= (x_1 - iy_1) + (x_2 - iy_2)\\ &= (x_1 + x_2) - i(y_1 + y_2)\\ &= \overline{z_1 + z_2}.\end{aligned}$$

c)

$$\begin{aligned} z_1 \cdot z_2 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ \overline{z_1} \cdot \overline{z_2} &= (x_1 - iy_1)(x_2 - iy_2) \\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) \\ &= \overline{z_1 \cdot z_2}. \end{aligned}$$

A3.5 *a*) $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$

$$\begin{aligned} \overline{z_1 - z_2} &= \overline{(x_1 - x_2) + i(y_1 - y_2)} \\ &= (x_1 - x_2) - i(y_1 - y_2) \\ &= (x_1 - i\,y_1) - (x_2 - i\,y_2) \\ &= \overline{z_1} - \overline{z_2} \end{aligned}$$

b) $$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\frac{\overline{z_1}}{\overline{z_2}} = \frac{x_1 - iy_1}{x_2 - iy_2} \cdot \frac{x_2 + iy_2}{x_2 + iy_2} = \frac{(x_1x_2 + y_1y_2) + i(x_1y_2 - x_2y_1)}{x_2^2 + y_2^2} = \overline{\left(\frac{z_1}{z_2}\right)}.$$

A3.6 Solutions of $ax^2 + bx + c = 0$ are:

$$\begin{aligned} x &= \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a} \\ &= \alpha \pm i\beta, \end{aligned}$$

if $b^2 - 4ac < 0$.

$$\alpha = -\frac{b}{2a}; \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}.$$

A3.7 *a*) $x^3 - 1 = 0$: Since $x = 1$ is a solution, $x - 1$ divides $x^3 - 1$: $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

$$x^2 + x + 1 = 0 \Longleftrightarrow x = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Solutions are $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

b) $x^3 + 1 = 0$; Since $x = -1$ is a solution, $x + 1$ divides $x^3 + 1$: $x^3 + 1 = (x + 1)(x^2 - x + 1)$.

$$x^2 - x + 1 = 0 \Longleftrightarrow x = \frac{1 \pm \sqrt{1 - 4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Solutions are $-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

A3.8

$$e^{1+i} = e^1 \cdot e^i = e(\cos 1 + i \sin 1).$$
$$e^{\pi i/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$$
$$e^{2-\pi i} = e^2 \cdot e^{-\pi i} = e^2(\cos(-\pi) + i\sin(-\pi)) = -e^2.$$

A3.9 $e^{z_1} \cdot e^{z_2}$

$$= \left(1 + z_1 + \frac{z_1^2}{2!} + \frac{z_1^3}{3!} + \cdots\right)\left(1 + z_2 + \frac{z_2^2}{2!} + \frac{z_3^3}{3!} + \cdots\right)$$
$$= 1 + (z_1 + z_2) + \left(\frac{z_1^2}{2!} + z_1 z_2 + \frac{z_2^2}{2!}\right)$$
$$+ \frac{1}{3!}\left(z_1^3 + \frac{3!}{1!\,2!}z_1 z_2^2 + \frac{3!}{2!\,1!}z_1^2 z_2 + z_2^3\right)$$
$$+ \frac{1}{4!}\left(z_1^4 + \frac{4!}{3!\,1!}z_1^3 z_2 + \frac{4!}{2!\,2!}z_1^2 z_2^2 + \frac{4!}{1!\,3!}z_1 z_2^3 + z_1^4\right) + \cdots$$
$$= 1 + (z_1 + z_2) + \frac{1}{2!}(z_1 z_2)^2 + \frac{1}{3!}(z_1 + z_2)^3 + \frac{1}{4!}(z_1 + z_2)^4 + \cdots$$
$$= e^{z_1 + z_2}.$$

The coefficient of $z_1^j z_2^k$ in the big expansion above is

$$\frac{1}{j!} \cdot \frac{1}{k!} = \frac{1}{(j+k)!} \cdot \frac{(j+k)!}{j!\,k!}.$$

But, $(j+k)!/(j!\,k!)$ is precisely the coefficient of $z_1^j z_2^k$ in the expansion of $(z_1 + z_2)^{j+k}$.

A3.10 If $x_n = c_1 3^n + c_2 \cdot 1^n$, then $x_{n+1} = 3c_1 3^n + 1 \cdot c_2 \cdot 1^n$ and $x_{n+2} = 9c_1 3^n + 1 \cdot c_2 \cdot 1^n$.

$$\begin{aligned} 4x_{n+1} - 3x_n &= (12c_1 3^n + 4c_2 \cdot 1^n) - (3c_1 3^n + 3c_2 \cdot 1^n) \\ &= 9c_1 3^n + c_2 \cdot 1^n \\ &= x_{n+2}. \end{aligned}$$

Let $m = n + 2$. Since $x_{n+2} = 4x_{n+1} - 3x_n$, $x_m = 4x_{m-1} - 3x_{m-2}$.

A3.11 *a*) If $x_n = k_1(2+3i)^n + k_2(2-3i)^n$,

$$\begin{aligned} x_{n+1} &= (2+3i)k_1(2+3i)^n + (2-3i)k_2(2-3i)^n \\ x_{n+2} &= (2+3i)^2 k_1(2+3i)^n + (2-3i)^2 k_2(2-3i)^n \\ &= (-5+12i)k_1(2+3i)^n + (-5-12i)k_2(2-3i)^n. \\ 4x_{n+1} - 13x_n &= [4(2+3i)-13]k_1(2+3i)^n + [4(2-3i)-13]k_2(2-3i)^n \\ &= (-5+12i)k_1(2+3i)^n + (-5-12i)k_2(2-3i)^n \\ &= x_{n+2}. \end{aligned}$$

b, c) By DeMoivre's formula, $(2 \pm 3i)^n = (5^n \cos n\theta_0 \pm i5^n \sin n\theta_0)$, where $\tan\theta_0 = 3/2$. Now,

$$\begin{aligned} &(c_1 + ic_2)(5^n \cos n\theta_0 + i5^n \sin n\theta_0) + (c_1 - ic_2)(5^n \cos n\theta_0 - i5^n \sin n\theta_0) \\ &= (c_1 5^n \cos n\theta_0 - c_2 5^n \sin n\theta_0) + i(c_2 5^n \cos n\theta_0 + c_1 5^n \sin n\theta_0) \\ &\quad + (c_1 5^n \cos n\theta_0 - c_2 5^n \sin n\theta_0) - i(c_2 5^n \cos n\theta_0 + c_1 5^n \sin n\theta_0) \\ &= 2(c_1 5^n \cos n\theta_0 - c_2 5^n \sin n\theta_0) \\ &= 5^n(C_1 \cos n\theta_0 + C_2 \sin n\theta_0), \end{aligned}$$

where $C_1 = 2c_1$ and $C_2 = -2c_2$.

Appendix 4

A4.1 *a*) $\frac{4}{7}x^7 - \frac{1}{4}x^4$

b) $4x^3 - 4x^{3/2} + 6x^{1/2} - \ln x$

c) $\frac{6}{7}e^{7x}$

d) $\frac{1}{6}e^{3x^2+6x}$

e) $\frac{1}{3}(x^2 + 2x + 4)^{3/2}$

f) $\frac{1}{2}\ln(x^{3/2} + x^{1/2})$.

A4.2 *a*) $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$,

b) $e^{2x}(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4})$.

A4.3 $\displaystyle\sum_{j=0}^{19}\left(\frac{j}{10}\right)^2\left(\frac{1}{10}\right) = \frac{1}{1000}\sum_{j=1}^{19} j^2 = \frac{1}{1000}\cdot\frac{19\cdot 20\cdot 39\cdot}{6} = 2.47,$

using Exercise A1.5.

A4.4 *a*) $\int_1^4 x^{1/2} dx = \frac{2}{3} x^{3/2}\Big|_1^4 = \frac{2}{3} \cdot 8 - \frac{2}{3} \cdot 1 = \frac{14}{3}.$

b) $\int_1^e x \ln x \, dx = \left(\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right)_1^e$

$$= \left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left(\frac{1^2}{2}\ln 1 - \frac{1}{4}\right) = \frac{e^2}{4} + \frac{1}{4}.$$

A4.5 $\int_0^{100} 3q^{-1/2}\, dq = 6q^{1/2}\Big|_0^{100} = 60 =$ total willingness to pay.

Total paid is $100 \cdot \frac{3}{10} = 30$. Consumer surplus is $60 - 30 = 30$.

A4.6 $\int_a^b e^{-\int_a^t r(s)\,ds} P(t)\, dt.$

Appendix 5

A5.1 $P(E^c) + P(E) = P(E^c \cup E) = 1$, by (3) and (2). So $P(E^c) = 1 - P(E)$.

A5.2 $\text{EV} = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \frac{21}{6} = 3.5.$

$\text{Var} = \sum_{i=1}^{6} \frac{1}{6}(i - 3.5)^2 = \frac{2}{6}[(2.5)^2 + (1.5)^2 + (0.5)^2] = \frac{35}{12}.$

$\$2 \cdot 3.5 = \$7.$

A5.3 *a*) $\text{EV} = \frac{1}{36}\cdot 2 + \frac{2}{36}\cdot 3 + \frac{3}{36}\cdot 4 + \frac{4}{36}\cdot 5 + \frac{5}{36}\cdot 6 + \frac{6}{36}\cdot 7 + \frac{5}{36}\cdot 8$

$$+ \frac{4}{36}\cdot 9 + \frac{3}{36}\cdot 10 + \frac{2}{36}\cdot 11 + \frac{1}{36}\cdot 12$$

$$= \left(\frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36}\right)\cdot 14 + \frac{6}{36}\cdot 7$$

$$= \frac{1}{36}(15 \cdot 14 + 6 \cdot 7) = \frac{252}{36} = 7.$$

b) $\text{Var} = 2 \cdot \frac{1}{36}(1 \cdot 5^2 + 2 \cdot 4^2 + 3 \cdot 3^2 + 4 \cdot 2^2 + 5 \cdot 1^2) = \frac{35}{6}$

$$\text{EV} = \frac{6}{36}\cdot 0 + \frac{10}{36}\cdot 1 + \frac{8}{36}\cdot 2 + \frac{6}{36}\cdot 3 + \frac{4}{36}\cdot 4 + \frac{2}{36}\cdot 5 = \frac{70}{36}$$

$$= \frac{35}{18}.$$

A5.4 *a*) $.1(900)^{1/3} + .9(1000)^{1/3} \approx 9.965$.

b) $990^{1/3} \approx 9.967$.

c) $(1000 - x)^{1/3} = 9.9655 \Longrightarrow x = 10.32$.

A5.5 Let f denote the density function for a continuous random variable x running from a to b. Divide $[a, b]$ into n equal segments $[x_{i-1}, x_i]$ each of width $\Delta x = (b - a)/n$.

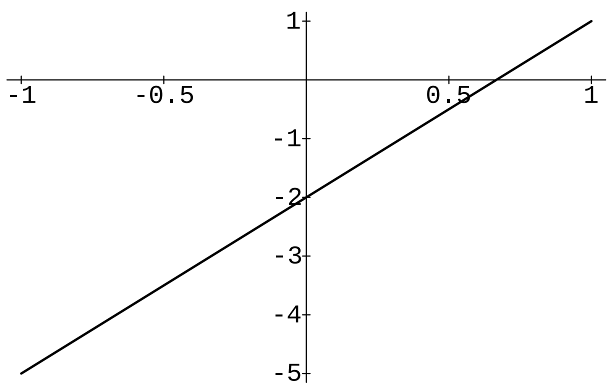
$$\Pr\{x \in [x_{i-1}, x_i]\} = \int_{x_{i-1}}^{x_i} f(x)\,dx \approx f(x_i) \cdot \Delta x.$$

$$\mu \equiv EV \approx \sum_i x_i p(x_i) = \sum_i x_i f(x_i)\Delta x \longrightarrow \int_a^b x f(x)\,dx \quad \text{as} \quad \Delta x \longrightarrow 0.$$

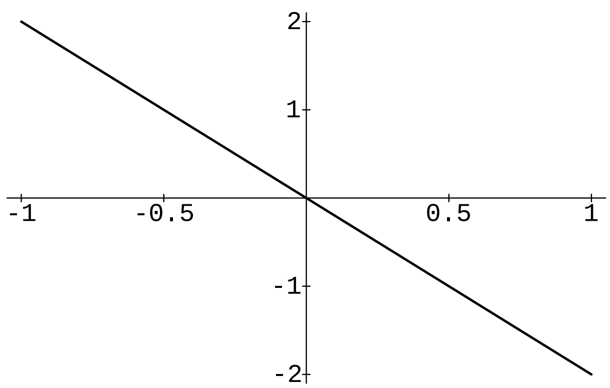
$$\text{Var} \approx \sum_i (x_i - \mu)^2 f(x_i)\Delta x \longrightarrow \int_a^b (x - \mu)^2 f(x)\,dx \quad \text{as} \quad \Delta x \longrightarrow 0.$$

Chapter 2 Figures

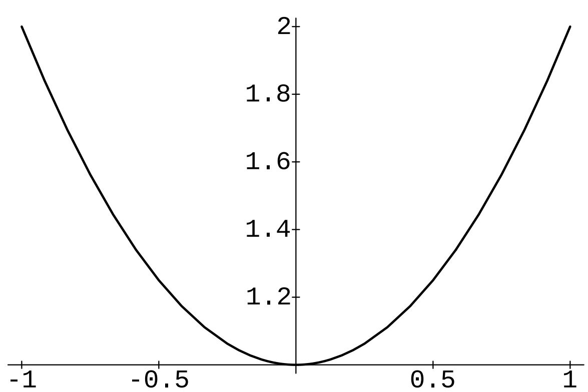
2.1 *i*) $y = 3x - 2$



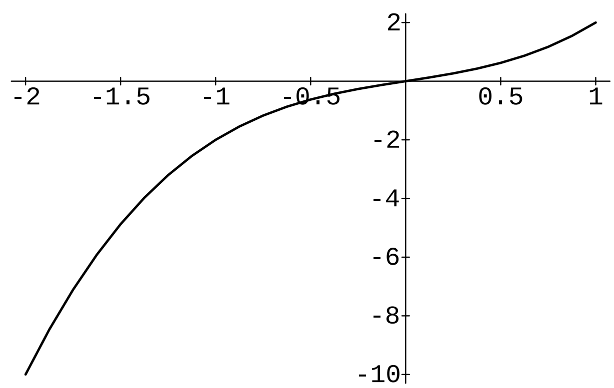
ii) $y = -2x$



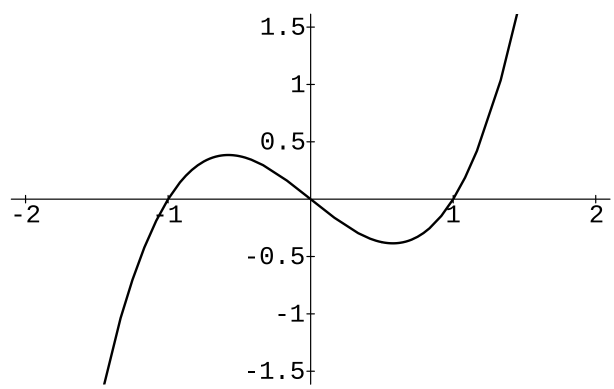
iii) $y = x^2 + 1$



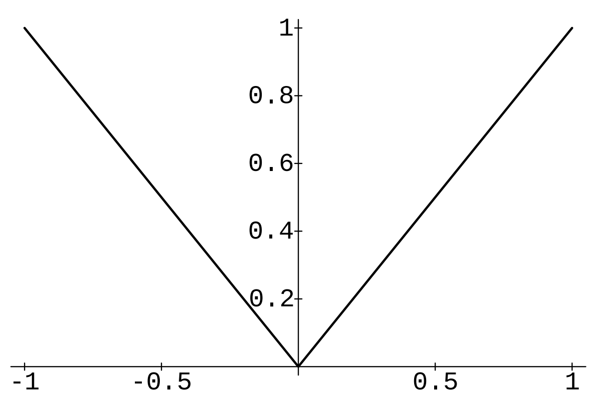
iv) $y = x^3 + x$



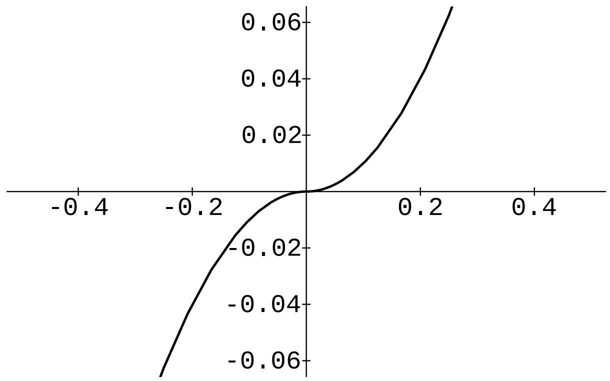
v) $y = x^3 - x$



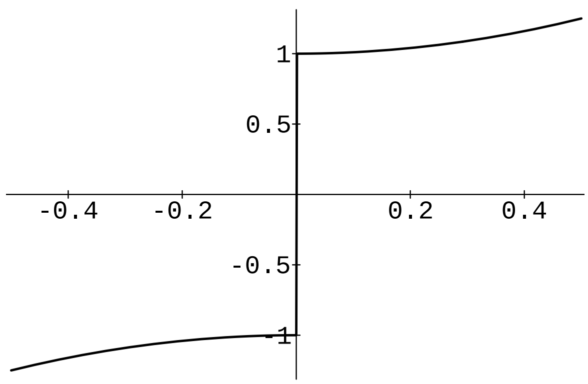
vi) $y = |x|$



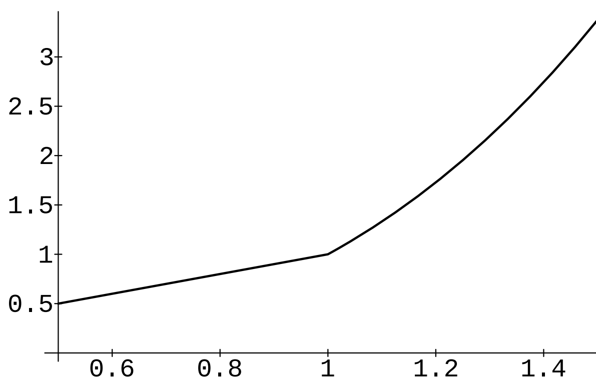
2.16 *a*)


0.06
0.04
0.02
-0.4
-0.2
0.2
0.4
-0.02
-0.04
-0.06

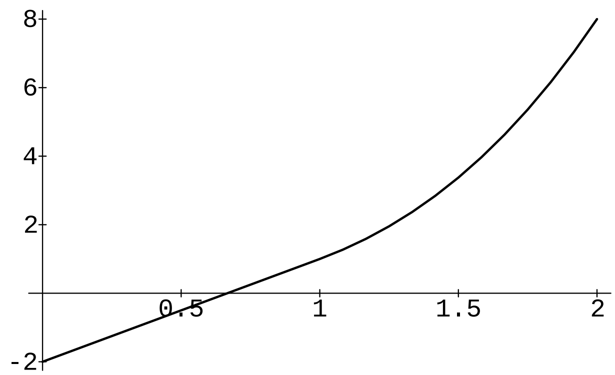
b)


1
0.5
-0.4
-0.2
0.2
0.4
-0.5
-1

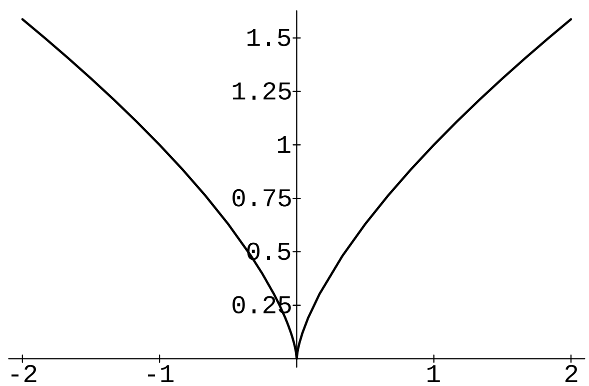
c)


3
2.5
2
1.5
1
0.5
0.6
0.8
1
1.2
1.4

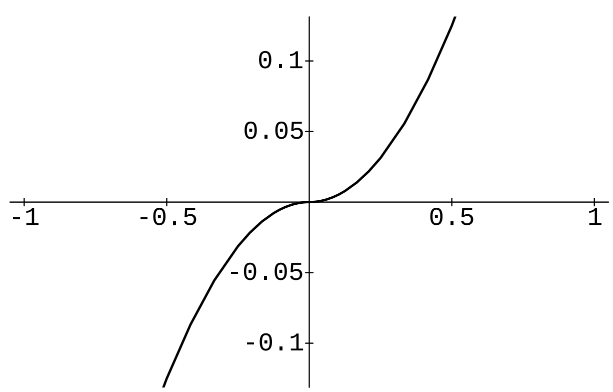
d)



2.18 $f(x) = x^{2/3}$

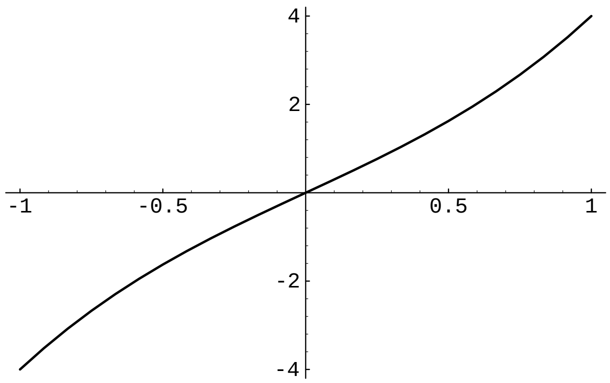


2.19

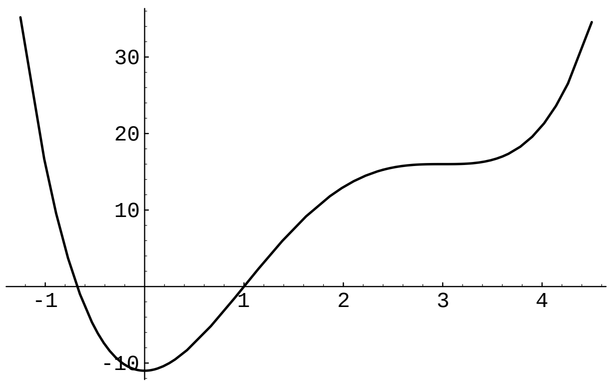


Chapter 3 Figures

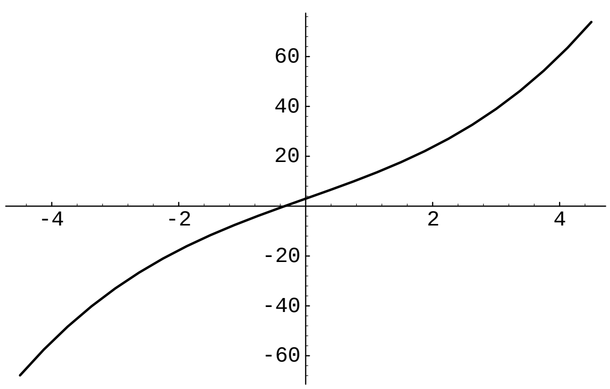
3.1 *a*) $f(x) = x^3 + 3x$



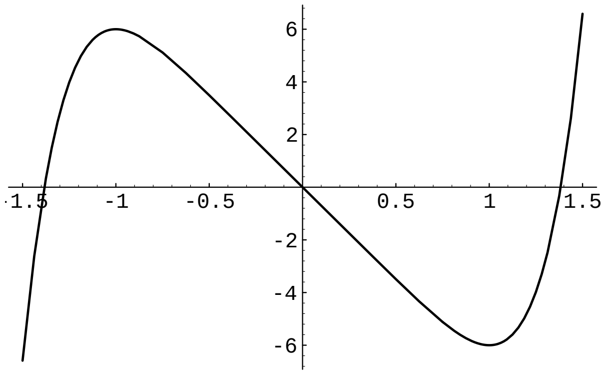
b) $f(x) = x^4 - 8x^3 + 18x^2 - 11$



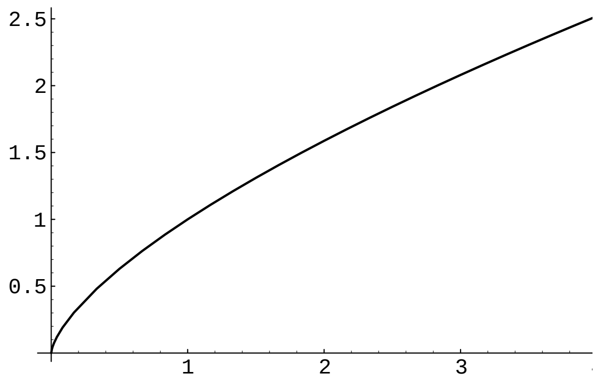
c) $f(x) = \frac{1}{3}x^3 + 9x + 3$



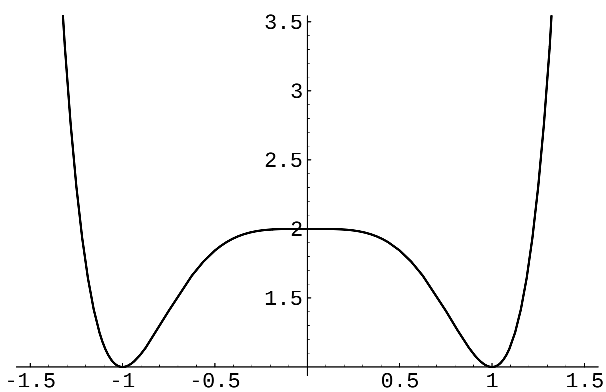
d) $f(x) = x^7 - 7x$



e) $f(x) = x^{2/3}$

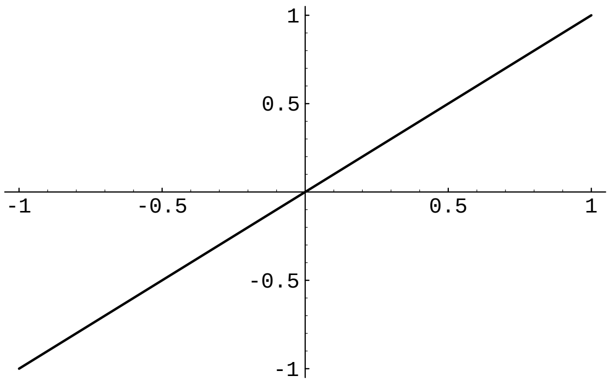


f) $f(x) = 2x^6 - 3x^4 + 2$

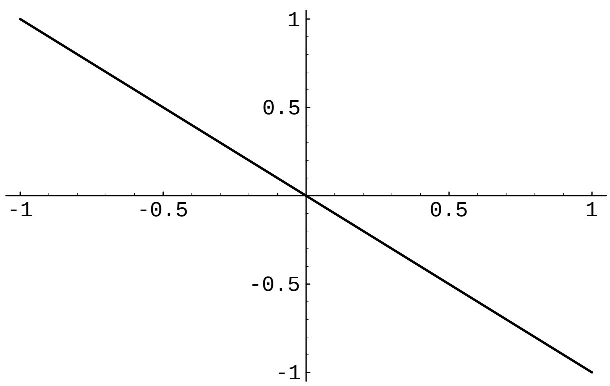


3.4, 3.5 (The same graphs work for each part of each problem.)

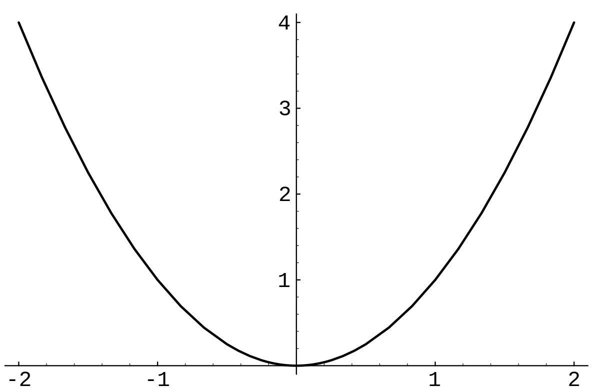
a)



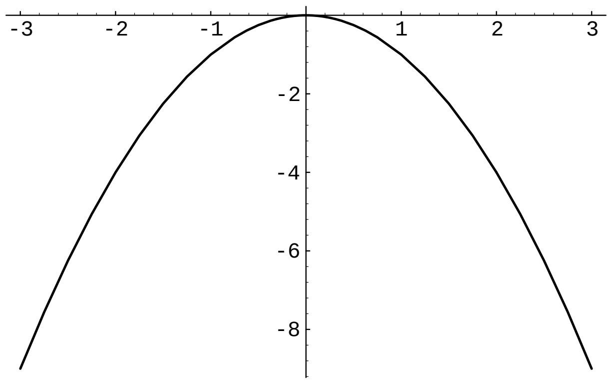
b)



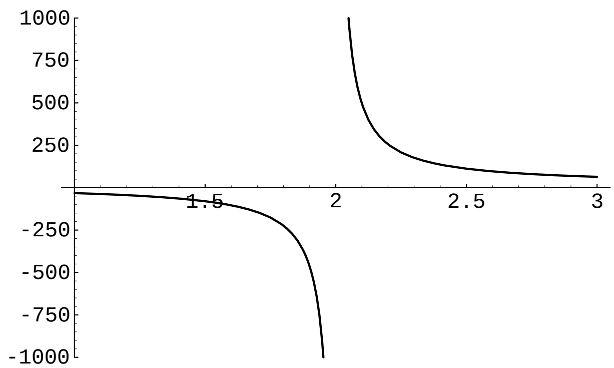
c)



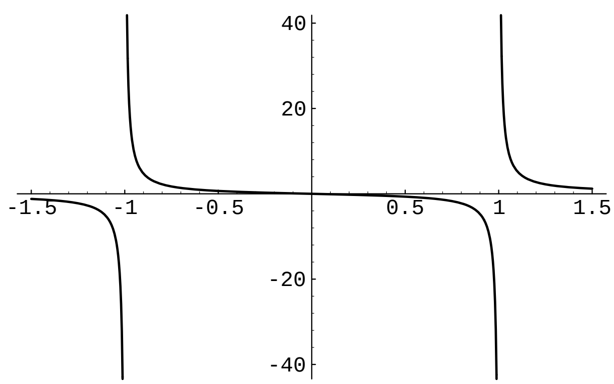
d)



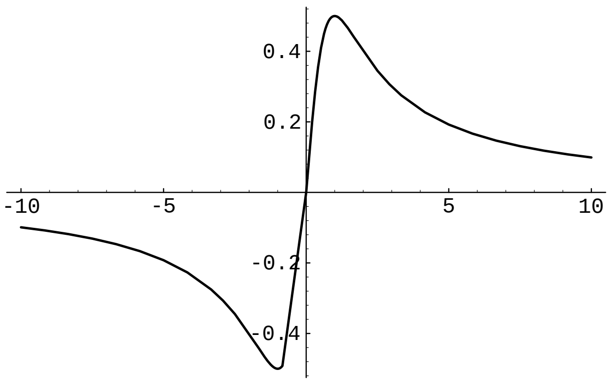
3.6 $f(x) = 16(x + 1)/(x - 2)$



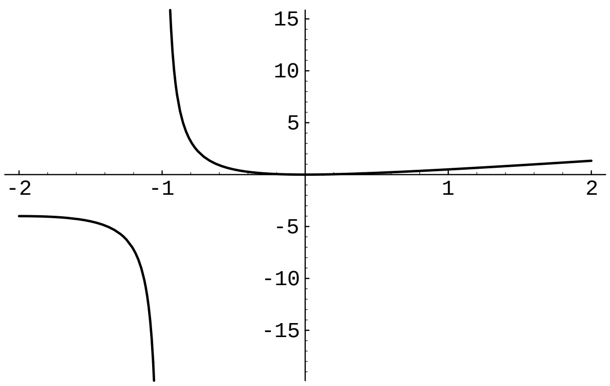
3.7 *a*) $f(x) = x/(x^2 - 1)$



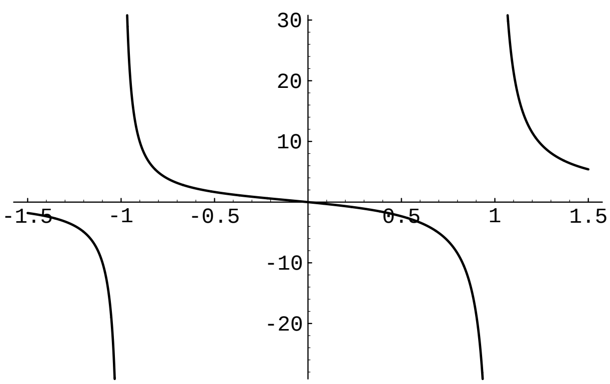
b) $f(x) = x/(x^2 + 1)$



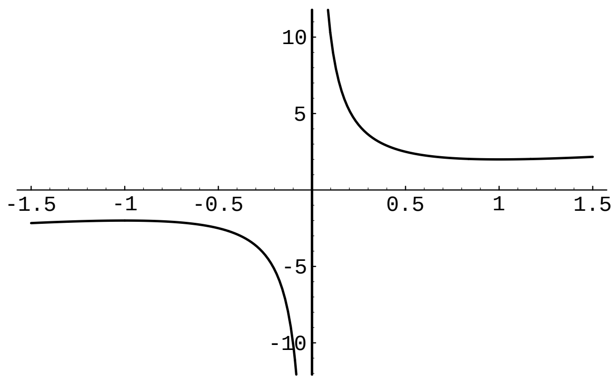
c) $f(x) = x^2/(x + 1)$



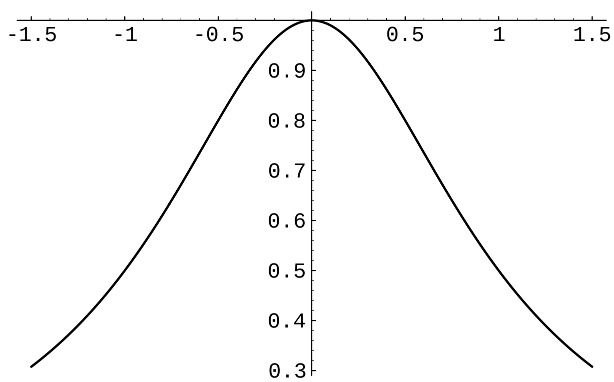
d) $f(x) = (x^2 + 3x)/(x^2 - 1)$



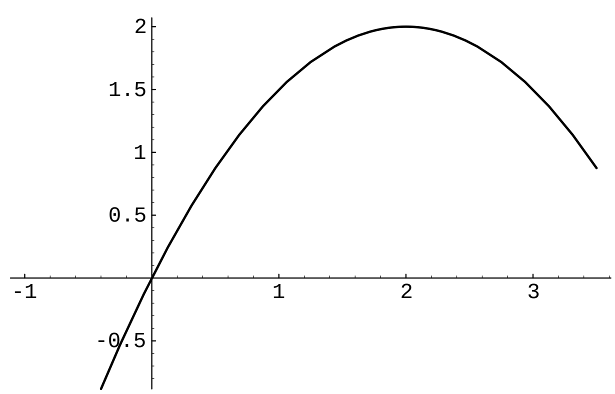
e) $f(x) = (x^2 + 1)/x$



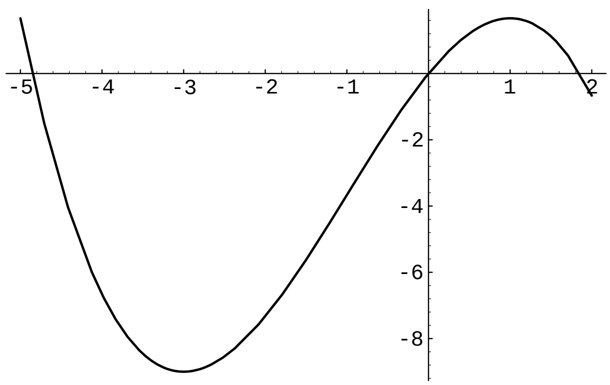
f) $f(x) = 1/(x^2 + 1)$



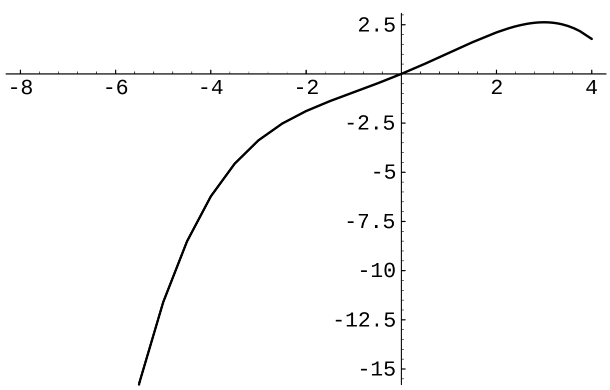
3.8 *a*)



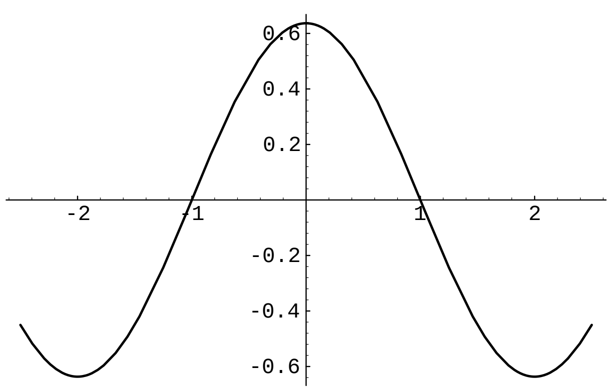
b)



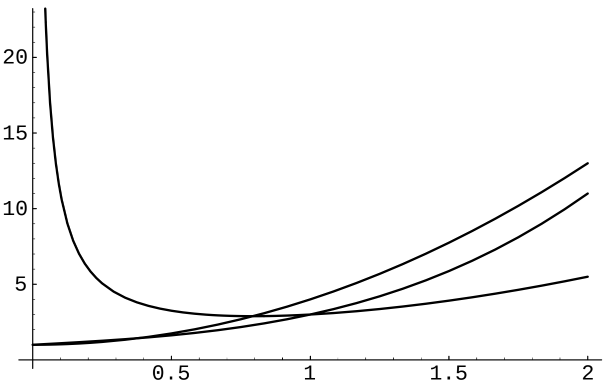
c)



d)

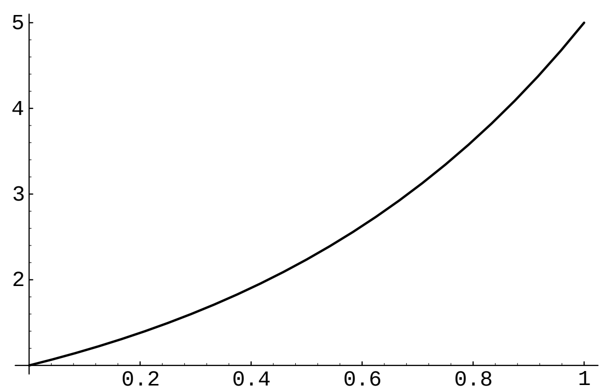


3.15

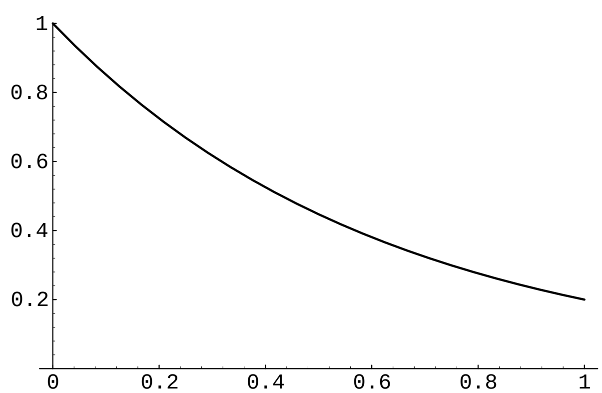


Chapter 5 Figures

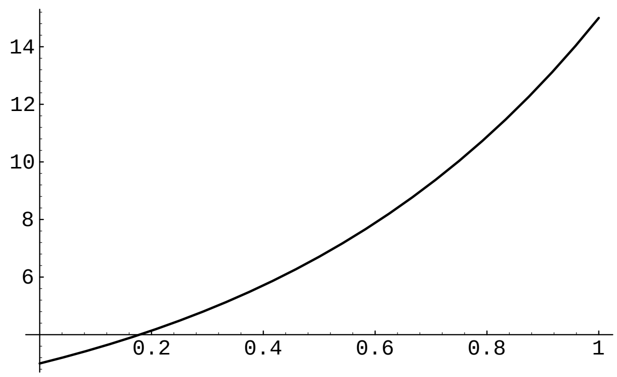
5.2 *a*)



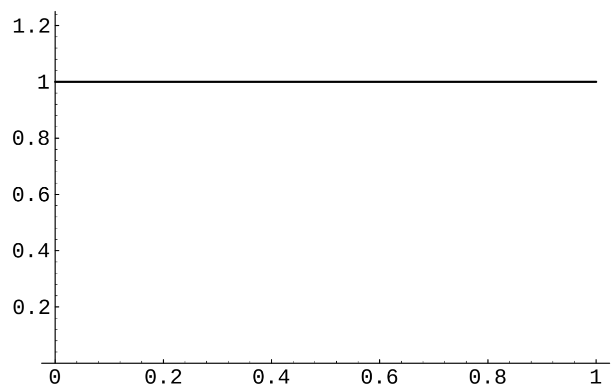
b)



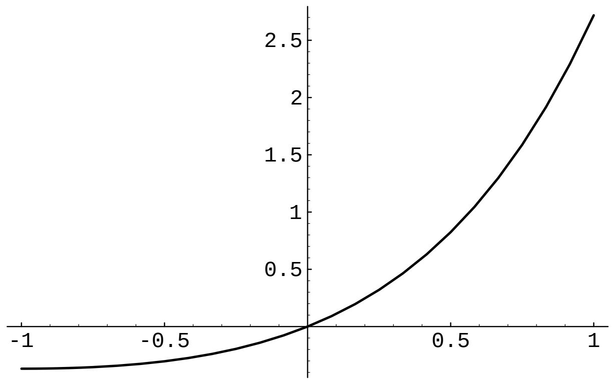
c)



d)



5.9 *a*)



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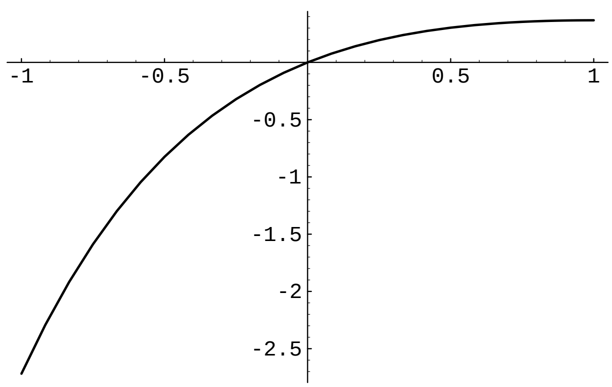
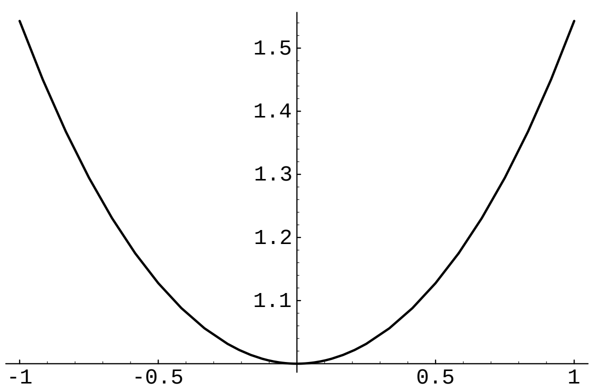
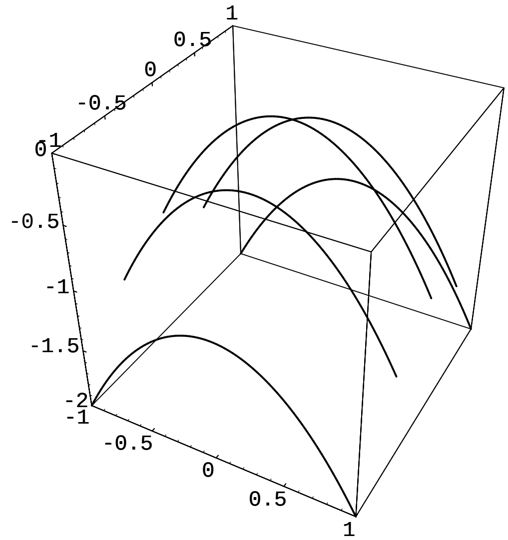


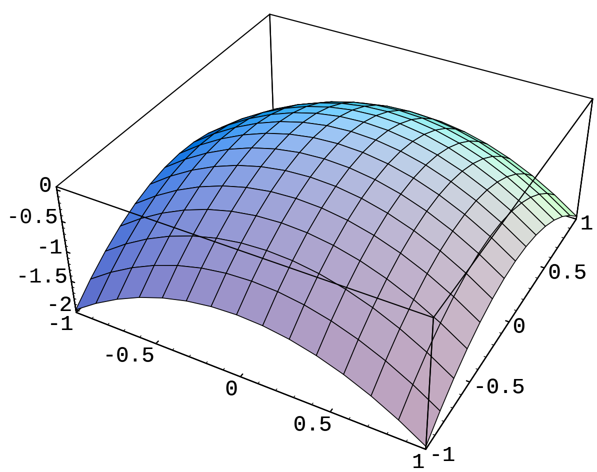
c)



Chapter 13 Figures

13.1 *a*)





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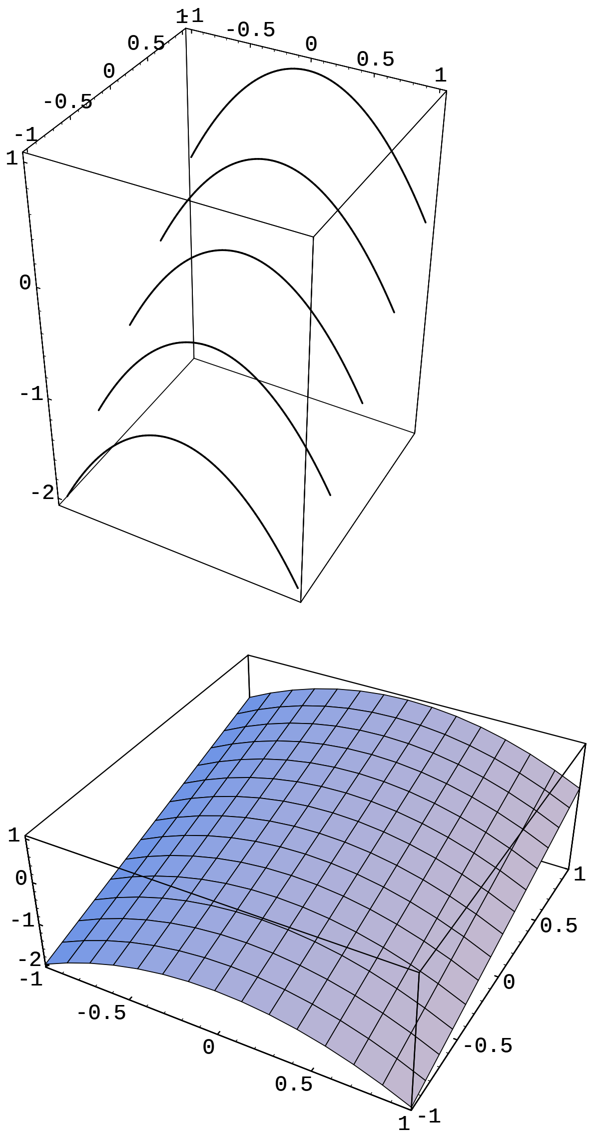
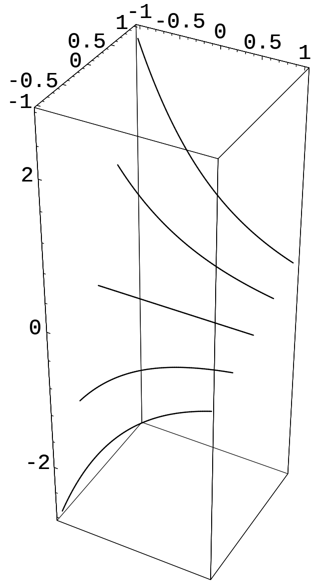
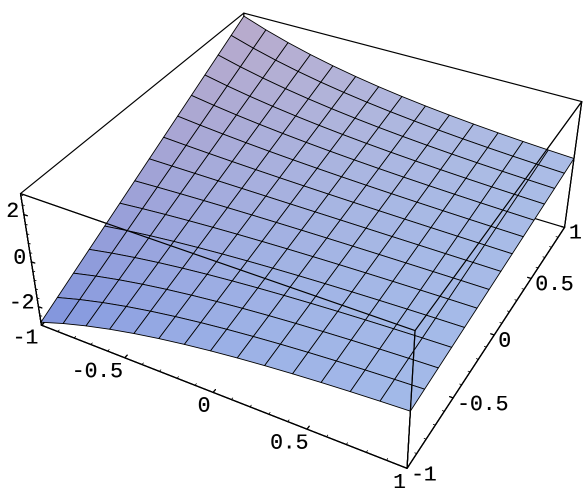
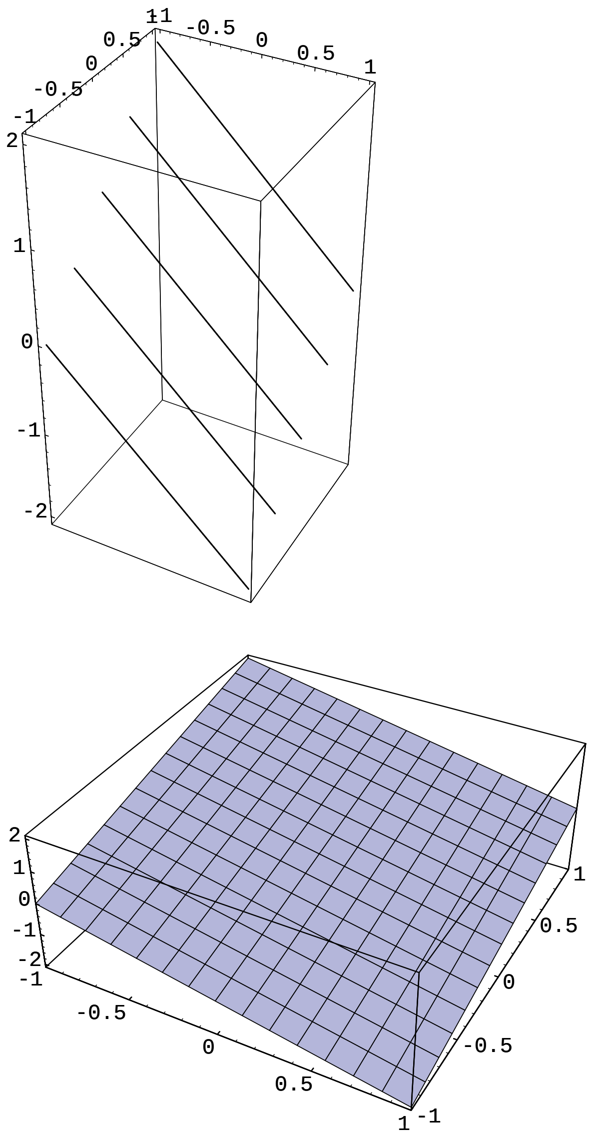


c)

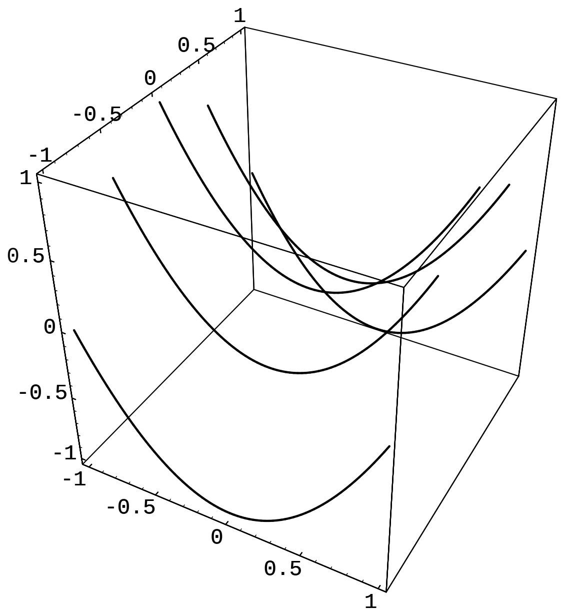


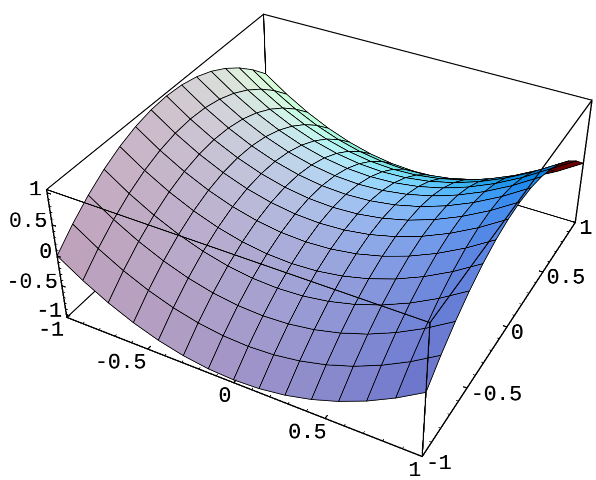


d)

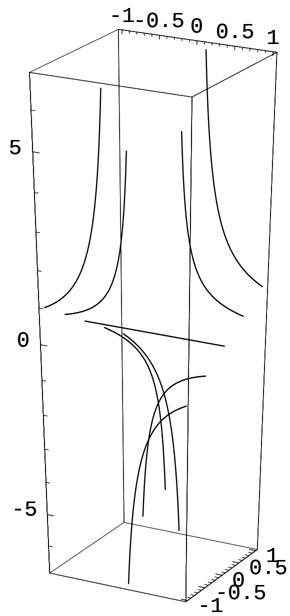


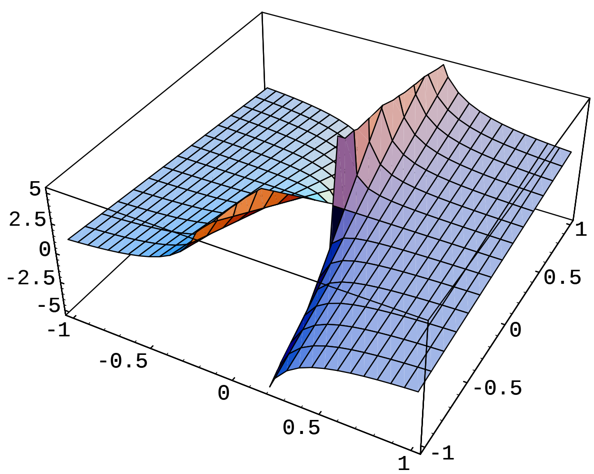
e)



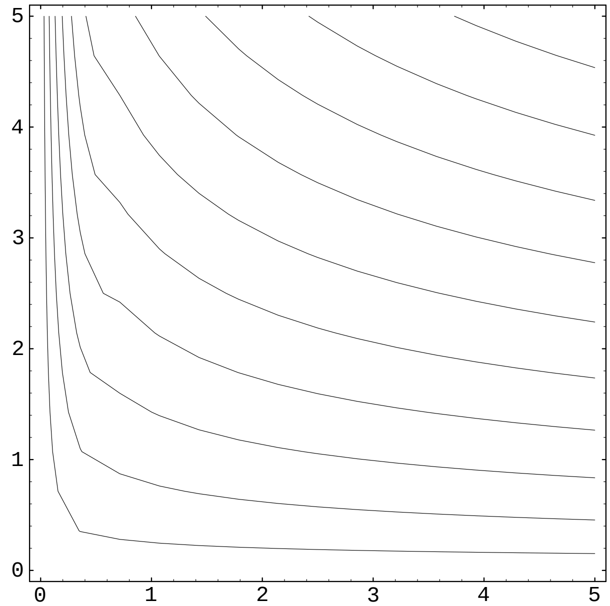


f)

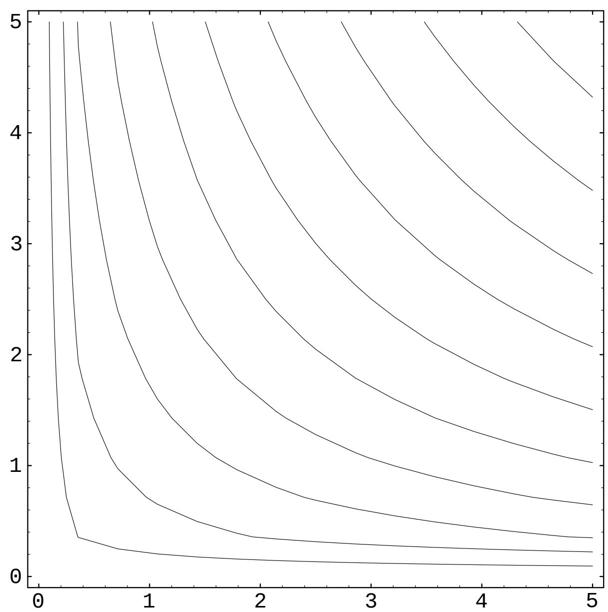




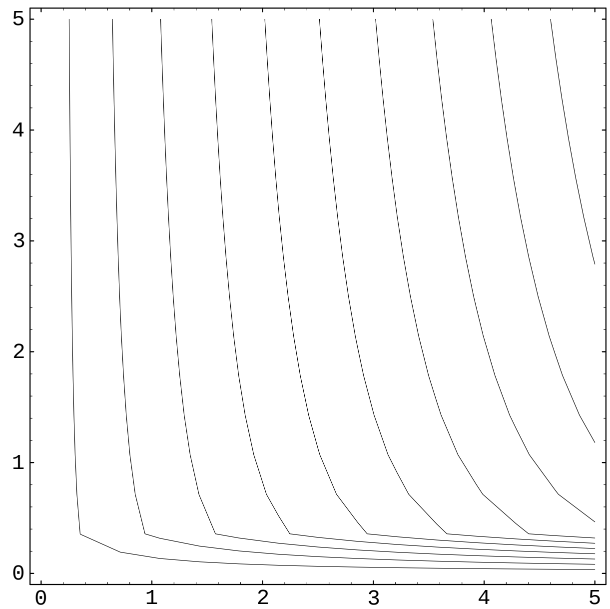
13.7 *a*)



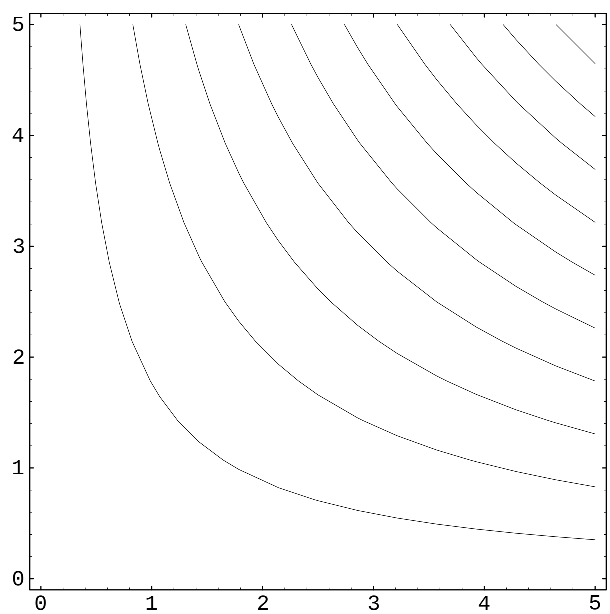
b)



c)

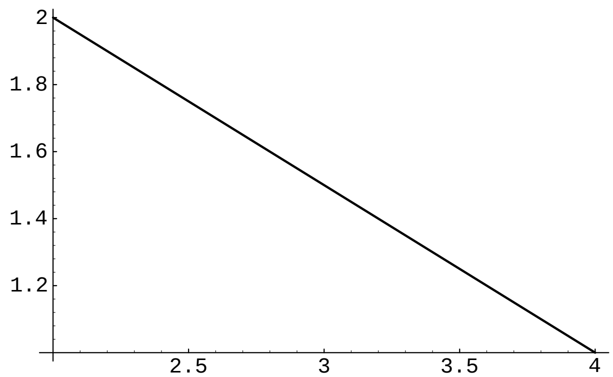


d)

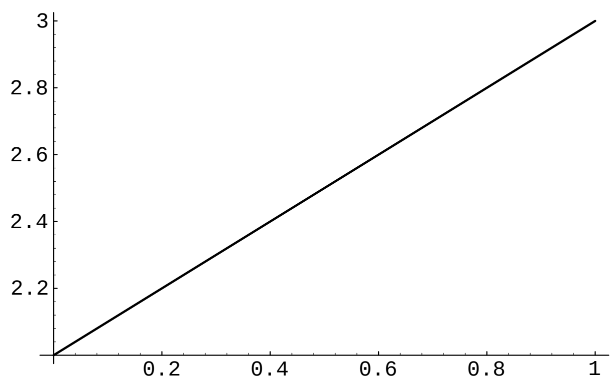


13.9 The last two plots are different views of the curve in part *d*. The first is from a "generic" position, and the last is from the top, looking down on the *xy*-plane.

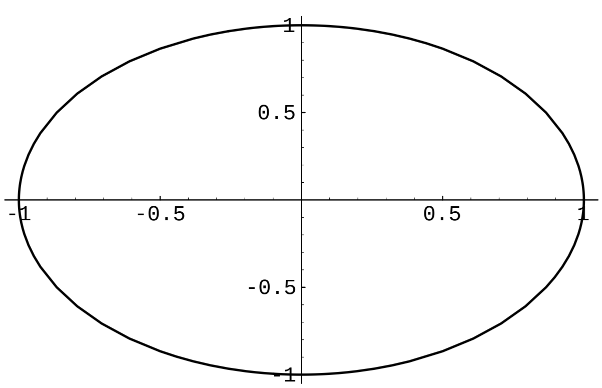
a)



b)



c)



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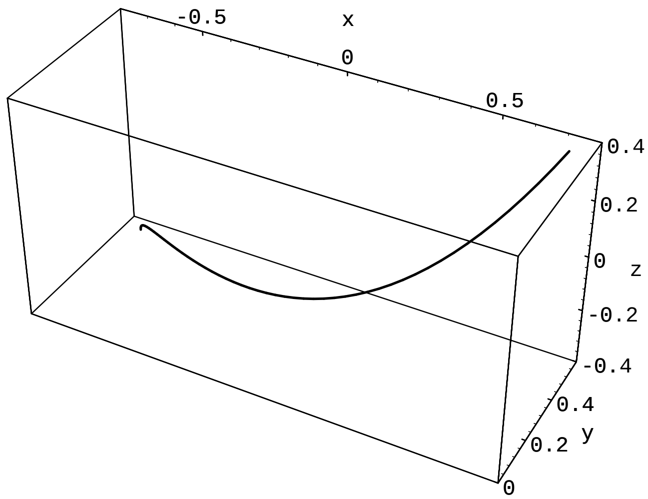

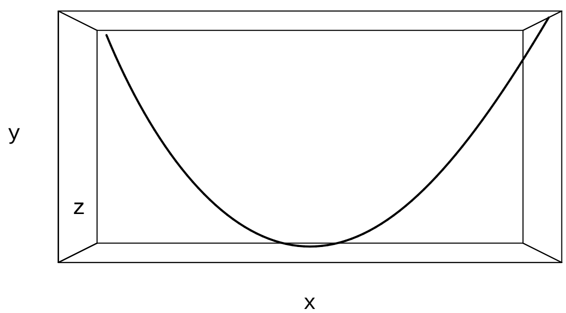

Chapter 18 Figures

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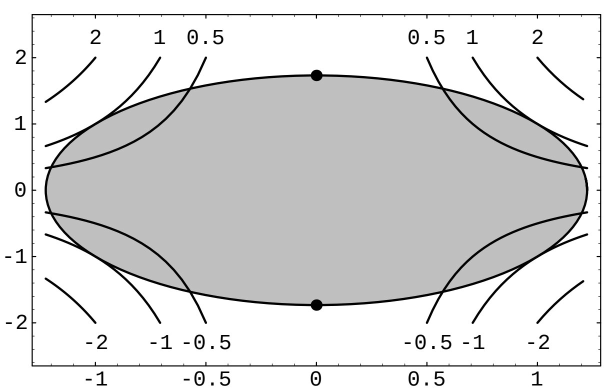
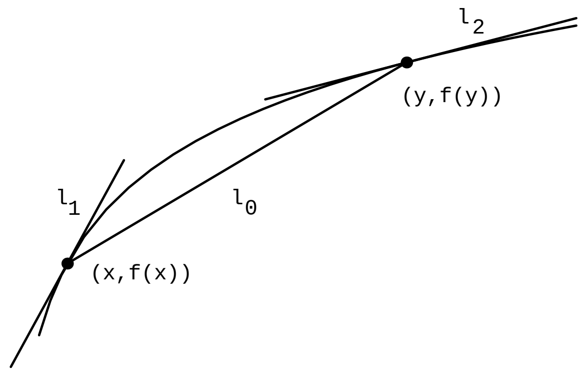


Chapter 21 Figures

21.6



Chapter 22 Figures

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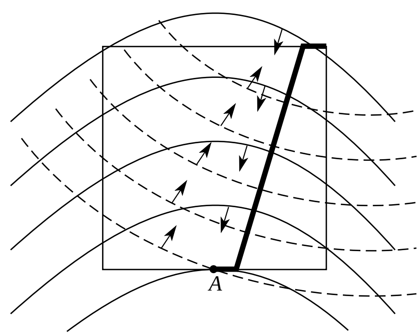

Chapter 24 Figures

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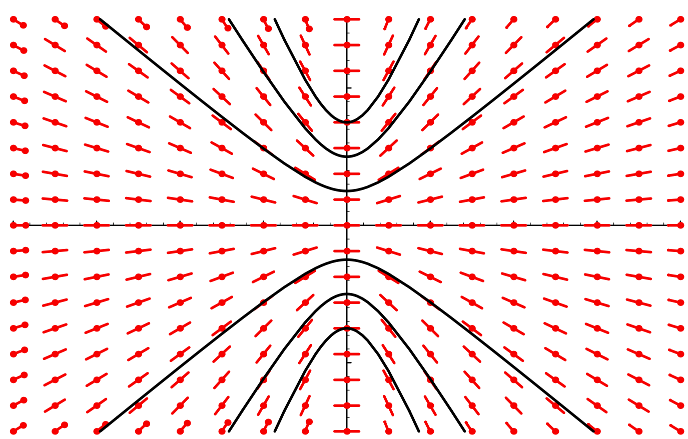

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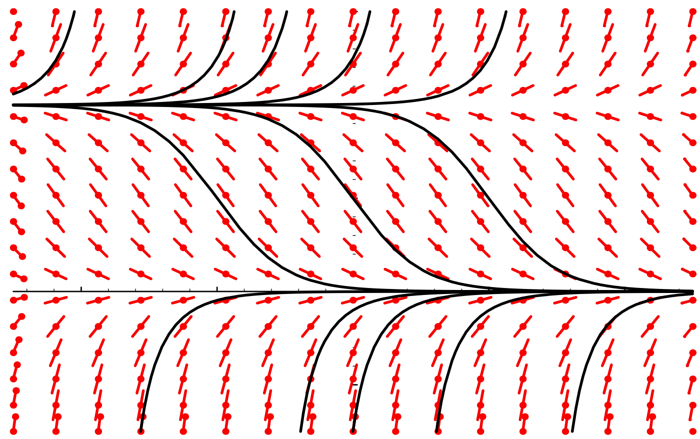

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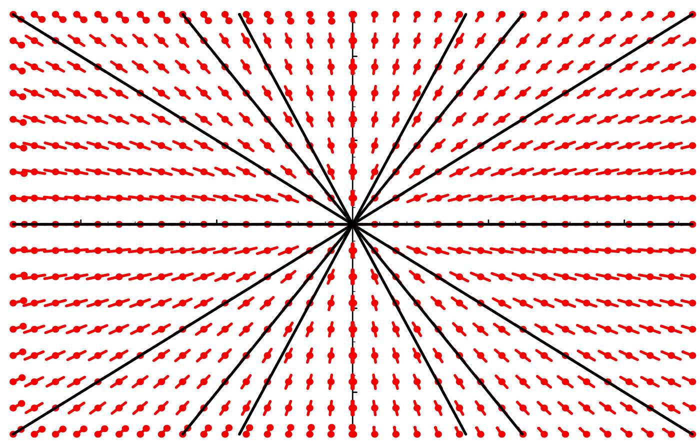

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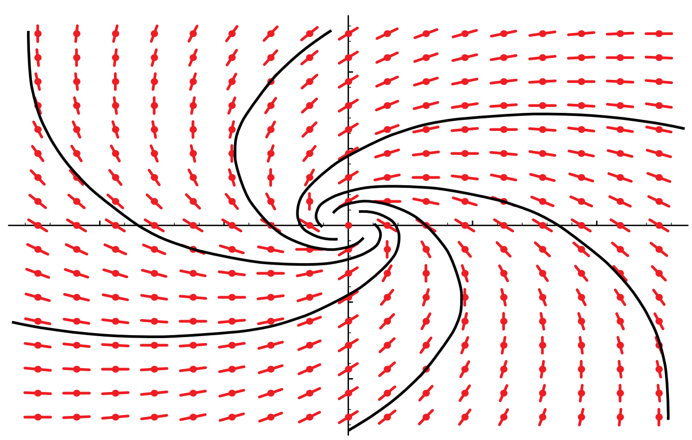

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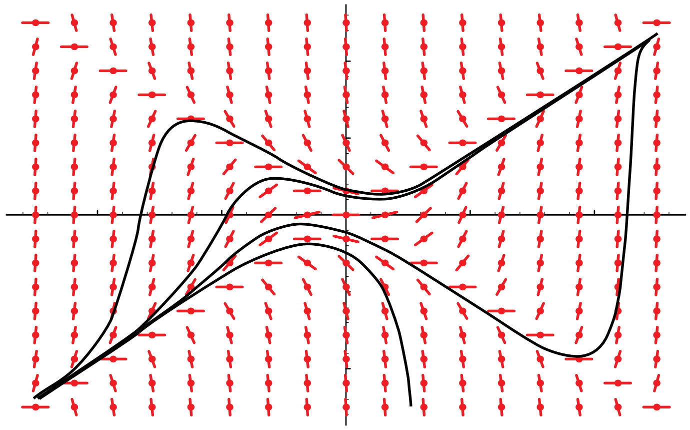

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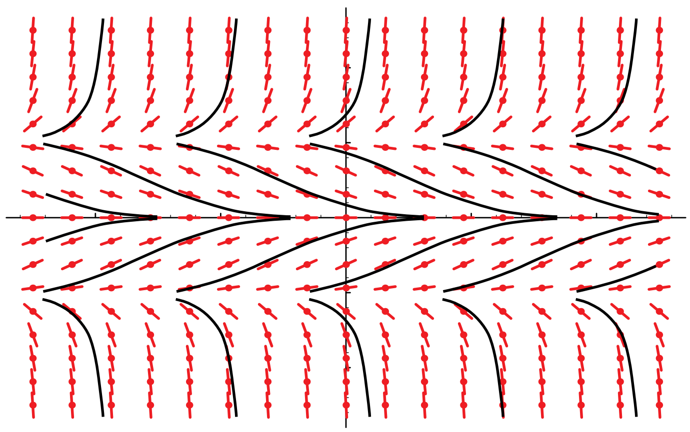

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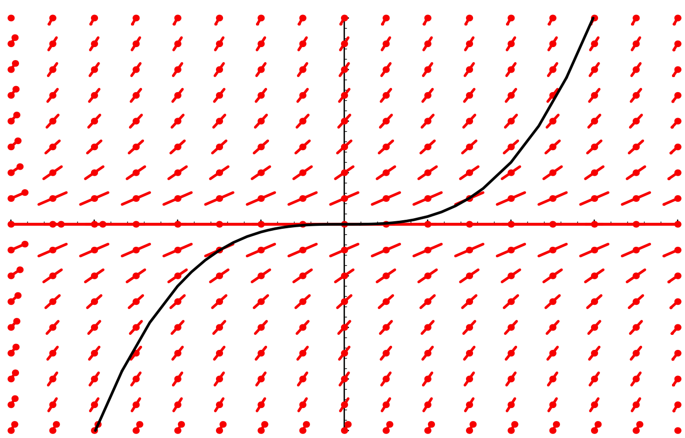
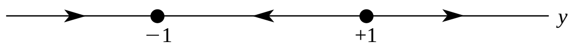
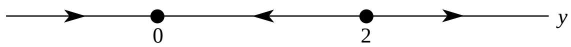
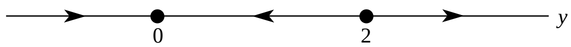
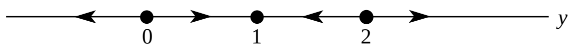
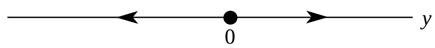


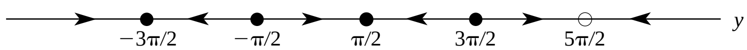
24.24 

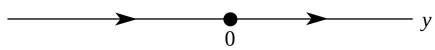
24.25 *a*) 

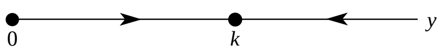
b) 

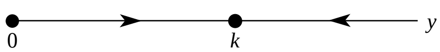
c) 

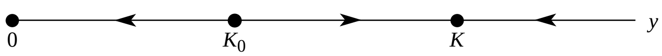
d) 

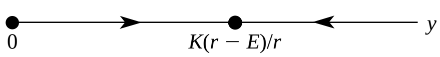
e) 

f) 

24.26 *a*) 

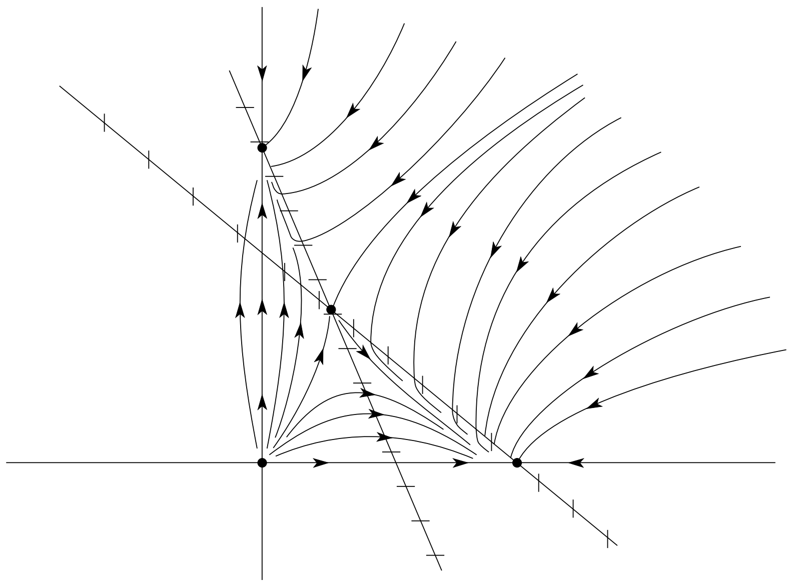
b) 

c) 

d) 

Chapter 25 Figures

25.19



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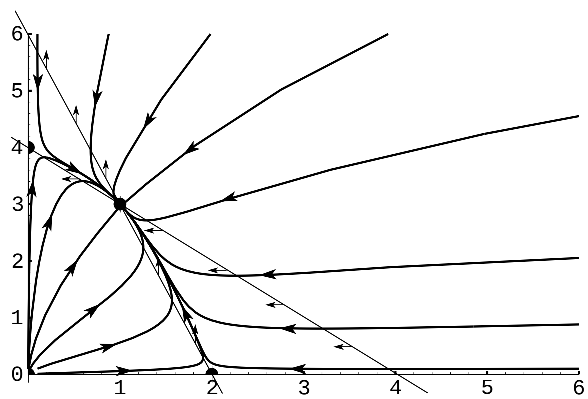

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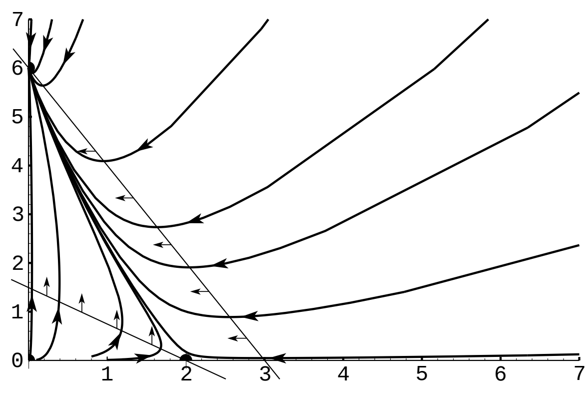

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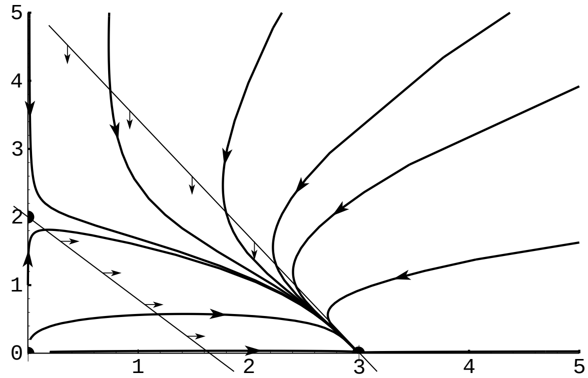

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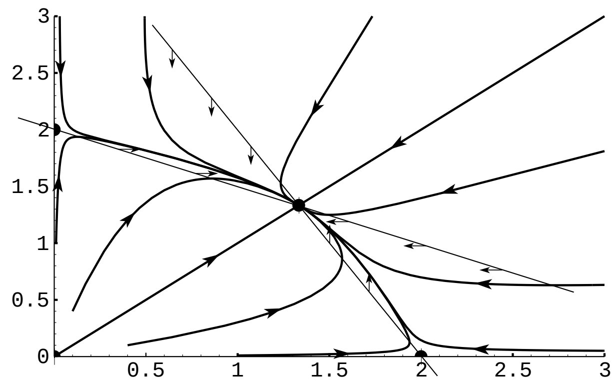

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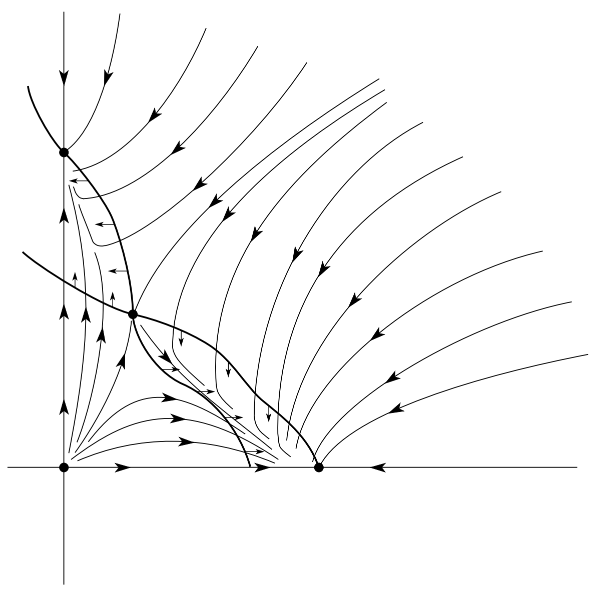

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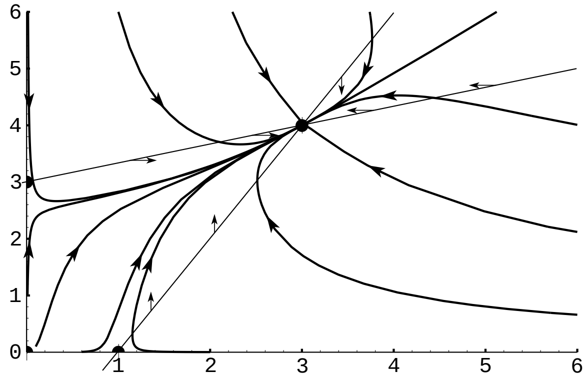

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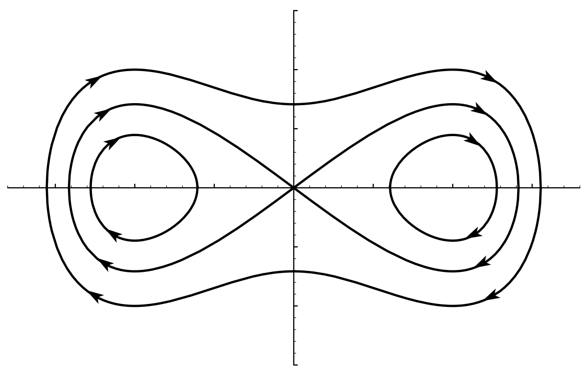

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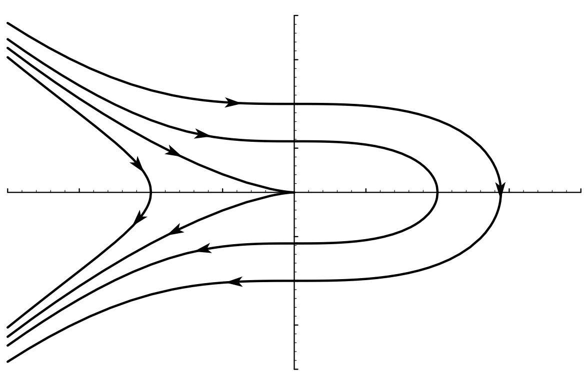

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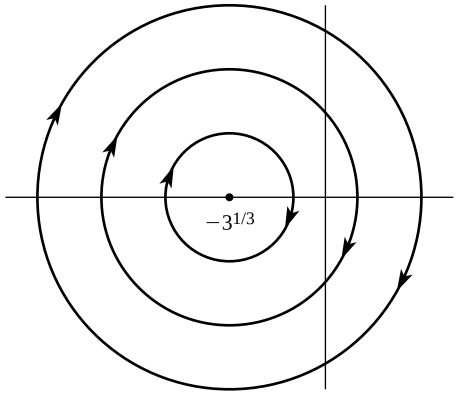

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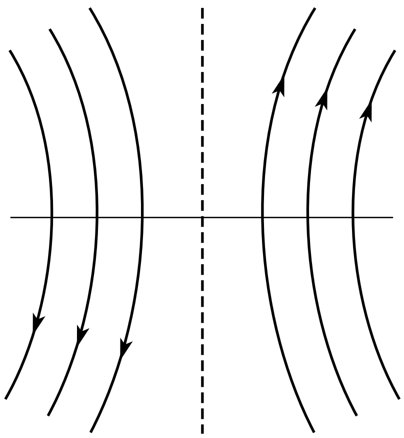

Chapter 28 Figures

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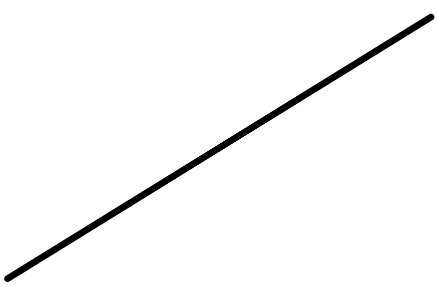

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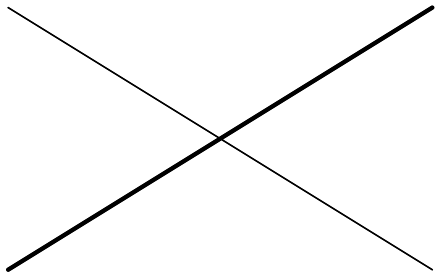

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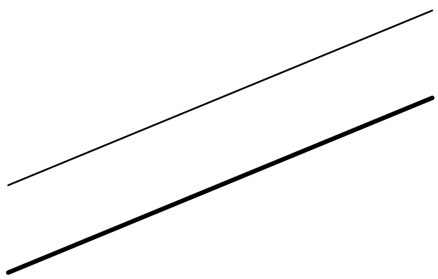

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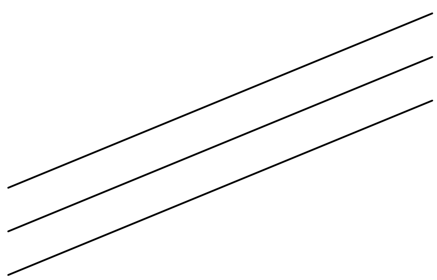

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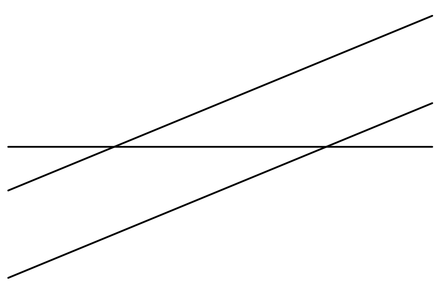

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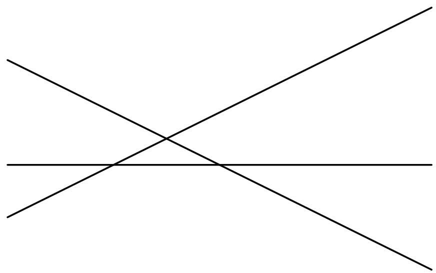

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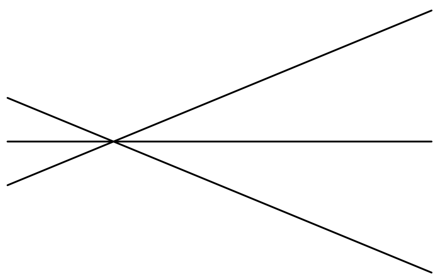

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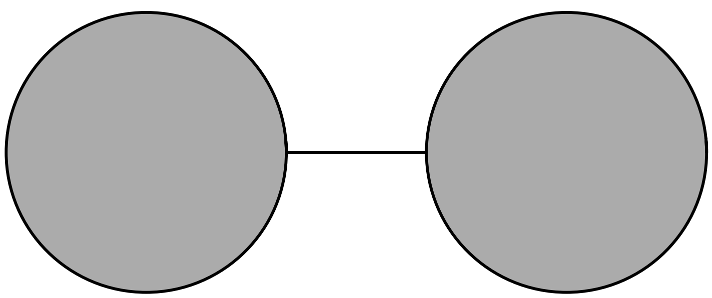
29.9



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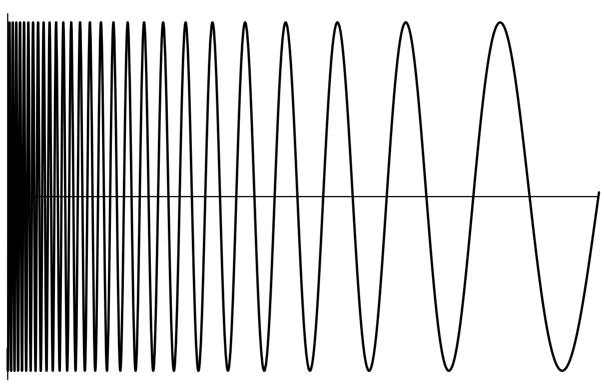

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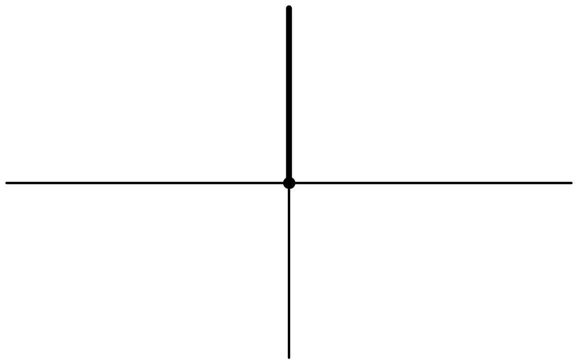

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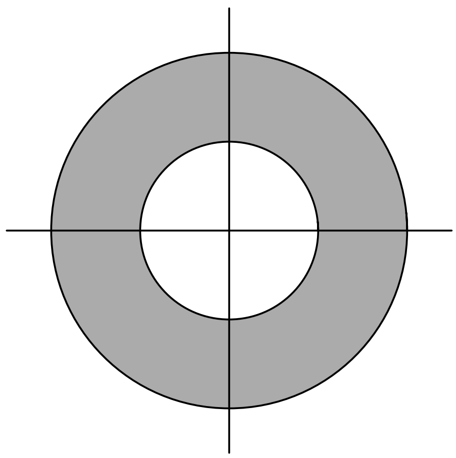

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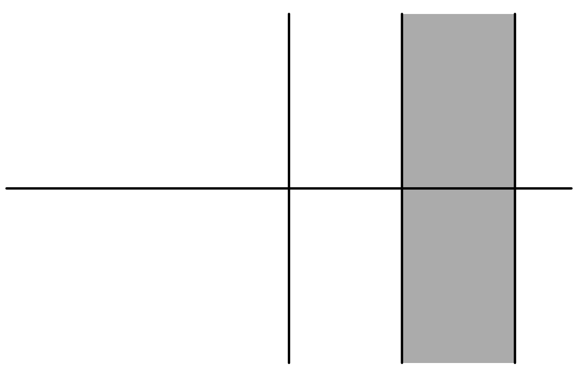

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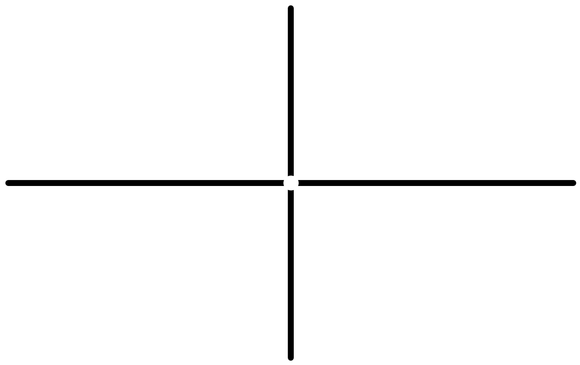

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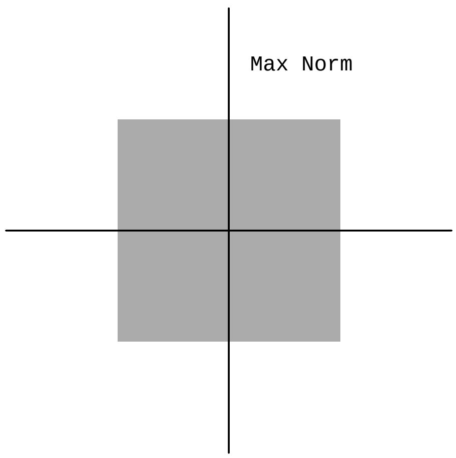

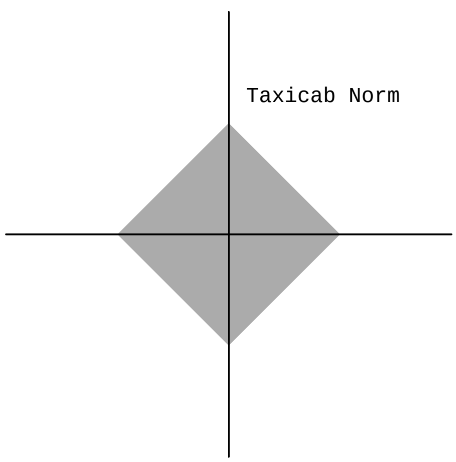

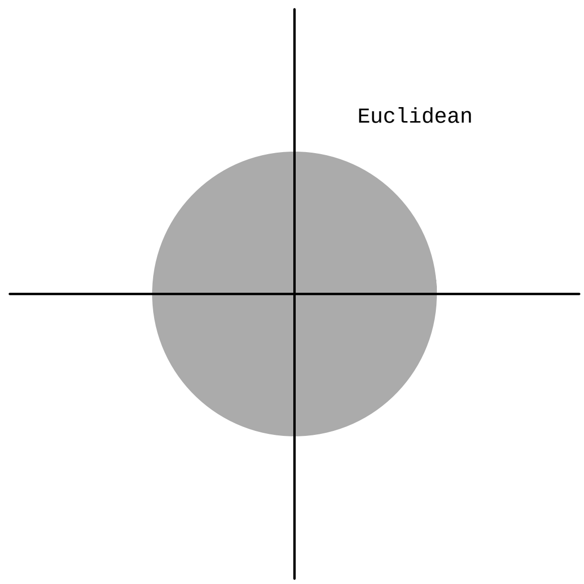
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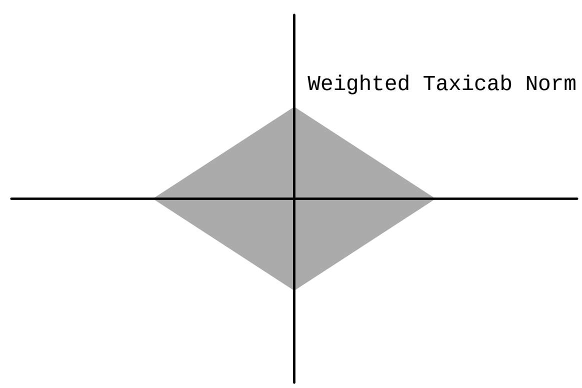
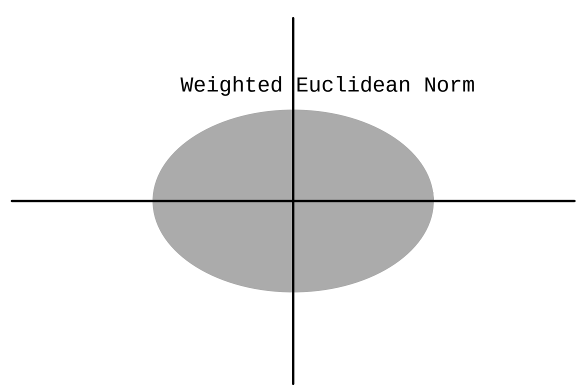
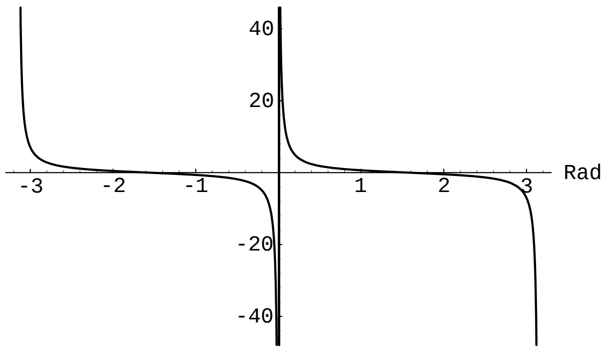

Weighted Taxicab Norm

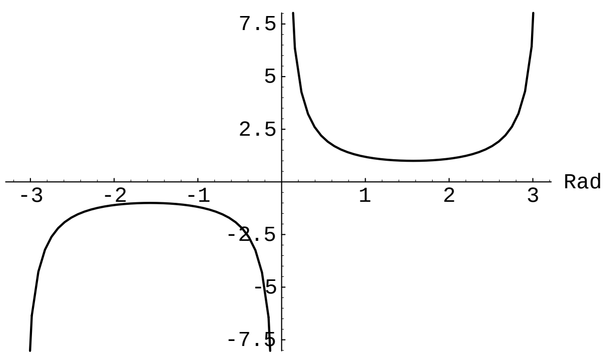

Weighted Euclidean Norm

Appendix 2 Figures

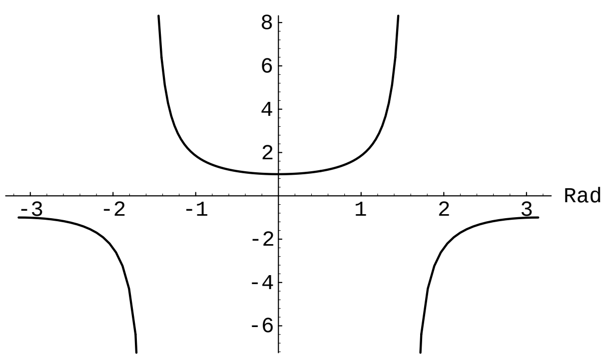
A2.2 Graph of the cotangent function on the range $[-\pi, \pi]$



Graph of the cosecant function on the range $[-\pi, \pi]$



Graph of the secant function on the range $[-\pi, \pi]$



A2.8