CHAPTER 1 INTRODUCTION

Abstract. This introductory chapter provides basic concepts, devices, and notations that facilitate the developments of the present study on threshold transformations and dynamical systems of neural networks.

1.1 Basic notations

In this chapter we describe basic concepts, devices, and notations that facilitate the developments of the present study. Most of the basic concepts in this chapter are found in introductory chapters of traditional textbooks on "abstract" algebra or in good introductory textbooks on discrete mathematics or combinatorics such as Williamson (1985), Krishnamurthy (1986), and Biggs (1989). We usually deal with finite sets in this book. The number of elements of a finite set X will be denoted by $|X|$. For set operations, the prefix ^{c} denotes the complement of a subset, $\backslash$ denotes the difference of sets:

$$A\backslash B = A \cap B^c = \{x \mid x \in X, x \notin Y\},$$

and $\dot{+}$ denotes the symmetric difference of sets:

$$A\dot{+}B = (A\backslash B) \cup (B\backslash A).$$

Then

$$(A\dot{+}B)\dot{+}C = A\dot{+}(B\dot{+}C).$$

If $q \in X$, then $A\backslash\{q\}$ and $A \cup \{q\}$ are respectively written as $A\backslash q$ and $A \cup q$. A partition of a set S is a family of disjoint subsets A1,..,An such that

$$S = \bigcup_{i=1}^{n} A_i.$$

If F is a function from X to Y, denoted by $F : X \to Y$, and if $A \subseteq X$, then the image of A under F is

$$FA = \{y \mid y = Fx \text{ for some } x \in A\},$$

and the inverse image of A is

$$F^{-1}A = \{x \mid Fx \in A, x \in X\}$$

The inverse image of $a \in Y$ is

$$F^{-1}a = \{x \mid Fx = y, x \in X\}.$$

If $B = FA$, then the expression $A \to_F B$ is also used.

A function $F : X \to Y$ is called an *onto* function or a *surjection* if $FX = Y$. F is called a *one-to-one* function or an *injection* if $Fx = Fx\prime$ for x, $x\prime \in X$ implies $x = x\prime$. F is called a *bijection* if F is both a surjection and an injection. If D and X are any sets, then X^D denotes the set of all functions from D to X. If F is a function from a set X to a set Y, and if G is a function from the set Y to a set Z, then the composition $G \circ F$, often simply written as GF, is the function from X to

Z defined as $(GF)x = G(Fx)$ for every $x \in X$. GF is also called the product of G and F. If $C \subseteq X$, the restriction of a function $F : X \to Y$ to C is denoted by $F|C$.

The set of all integers will be denoted by $\mathbf{Z}$, the set of non-negative integers will be denoted by $\mathbf{Z}_+$, and the set of positive integers will be denoted by $\mathbf{Z}^+$. A sequence $V = (v_0, v_1, ...)$ in X is a function $V : \mathbf{Z}_+ \to X$. The image of V, that is, the set $\{x \mid x = v_i \text{ for some } i\}$, is denoted by $\mathbf{V}$.

A *graph* G consists of a finite set V, whose members are called *vertices*, and a set E of 2-element subsets of V, whose members are called *edges*. V is called a *vertex set*, E is called an *edge set*, and G is expressed as $G = (V, E)$. A directed *graph*, or *digraph* consists of a finite set, whose members are called *vertices*, and a set A of ordered pairs of elements of V, whose members are called *arcs*. V is called a *vertex set*, A is called an *arc set*, and G is expressed as $G = (V, A)$. An arc (x, y) will be expressed by $x \to y$. An arc (x, x) is called a *loop*, and will be expressed by $x\partial$. Let $G = (V, A)$ and $H = (W, B)$ be digraphs. Let F be a bijection from V to W. Assume that $(x, y) \in A$ if and only if $(Fx, Fy) \in B$. Then a function $F\prime$ from A to B can be defined as $F\prime(x, y) = (Fx, Fy)$ for each $(x, y) \in A$. In this case, G and H are called *isomorphic* under the *isomorphism* $(F, F\prime)$ *induced* by F.

A *walk* in a graph G is a fnite sequence of vertices $V = (v_1, v_2, .., v_k)$ such that $\{v_i, v_{i+1}\}$ is an edge of G for every $1 \le i \le k-1$. Similarly, a *walk* in a digraph G is a sequence of vertices $V = (v_1, v_2, .., v_k)$ such that (v_i, v_{i+1}) is an arc of G for every $1 \le i \le k-1$. In this case, the walk V is expressed as $v_1 \to v_2 \to ... \to v_{k+1}$. If all its vertices are distinct, a walk is called a *path*. A walk $v_1 \to v_2 \to ... \to v_{k+1}$ whose vertices are all distinct except that $v_1 = v_{k+1}$ is called a k- *cycle* or a *cycle of length* k. A cycle of a graph G is called *Hamiltonian*, if it contains all vertices of G.

By a transformation of F of a set X we mean a function from the set to itself. In particular, if F is a bijection, it is called a *one-to-one transformation*. If F is a transformation of a set X, then F defines its digraph,

$$\mathrm{GRAPH}(F) = (X, A),$$

consisting of the vertex set X and the arc set A defined by

$$A = \{(x, y) \mid x \in X, Fx = y\}.$$

If $v_1 \to v_2 \to ... \to v_k \to v_1$ is a cycle of a digraph, then $v_i \to v_{i+1} \to ... \to v_k \to v_1 \to ... \to v_i$ is also a cycle. Therefore, in GRAPH(F), these two cycles are regarded as the same. With this identification in mind, the set of all cycles in GRAPH(F) is denoted by CY(F) (a loop is a 1-cycle).

If F and G are transformations of X and $GF = FG$, then F is called *commutative* with G. A particular case where F and G are commutative is described in the following. If $Fx = x$, that is, (x) is a loop of GRAPH(F), then x is called a *fixed point* of F; if $Fx \ne x$, then x is called a *non-fixed point* of F. If $FA \subseteq A$ for a subset A of X, then the restriction $F|A$ is the transformation of A. We call the set of all non-fixed points of F the *carrier* of F and write Car$F$. If Car$F$ and Car$G$ are disjoint, then a transformation H can be defined by $Hx = Fx$ if $x \in \mathrm{Car}F$, $Hx = Gx$ if $x \in \mathrm{Car}G$, and $Hx = x$ if $x \in (\mathrm{Car}F \cup \mathrm{Car}G)^c$. H is called the sum of F and G and denoted by $F + G$. If $H = F + G$, and if the images $F(\mathrm{Car}F)$ and $G(\mathrm{Car}G)$ are disjoint, then F is commutative with G, and $H = GF = FG$. In this case, H is called the direct sum or more conventionally *disjoint composition* of F and G and denoted by $H = F \odot G$.

If X and Y are sets, then the *Cartesian product* $X \times Y$ is the set of ordered pairs (x, y), $x \in X$ and $y \in Y$. If F is a transformation of X and G is a transformation of Y, then the *direct product* $F \times G$ is the transformation of $X \times Y$ defined by $(F \times G)(x, y) = (Fx, Gy)$ for every $(x, y) \in X \times Y$. The Cartesian products of n sets and the direct product of n transformations are similarly defined.

Let X be a set and d be a function from $X \times X$ to the set of non-negative real numbers satisfy the following conditions:

$$\begin{aligned} d(a,b) &= 0 \text{ if and only if } a = b, \\ d(a,b) &= d(b,a) \text{ for every } a, b, \\ d(a,b) &\leq d(a,c) + d(c,b) \text{ for every } a, b, c. \end{aligned}$$

Then, d is called a *distance* and X is called a *metric space*. If X is a finite set, then X is called a *finite metric space*. The distance between a point x and a non-empty subset A of X is defined by

$$d(x, A) = \min\{d(x, y) \mid y \in A\},$$

where min denotes the minimum element. The distance between non- empty subsets A and B is defined by

$$d(A, B) = \min\{d(x, B) \mid x \in A\}.$$

1.2 Permutations

Let m be a positive integer. Integers a and b are called *congruent modulo* m, and written as $a \equiv b \bmod m$, if $a - b$ is divisible by m. Further, if $0 \leq b < m$ here, b is the remainder obtained by dividing a by m and denoted by $a\%m$. The relation $\equiv \bmod m$ is an equivalence relation on $\mathbf{Z}$. The set of all equivalence classes with respect to this equivalence relation is called the *residue class ring with m elements*. We use the two expressions $\mathbf{Z}_m = \{0, 1, 2, .., m-1\}$ and $\mathbf{N}_m = \{1, 2, .., m\}$ for the residue class ring, where each equivalence class containing a representative integer a is expressed by the same integer a. Since we mostly use the second expression $\mathbf{N}_m$, its algebraic structure is described in the following. If a and b are elements of $\mathbf{N}_m$, then $a + b$ is the element c of $\mathbf{N}_m$ such that $c \equiv a + b \bmod m$, and $a \cdot b$ is the element c of $\mathbf{N}_m$ such that $c \equiv a \cdot b \bmod m$. Then $\mathbf{N}_m$ is a ring with m as the zero element. For example, $-a = n - a$ for any $a = 1, 2, .., m-1$, and $-m = m$. Further, we assume the order relation $1 < 2 < ... < m-1 < m$ in $\mathbf{N}_m$. Hereafter, we denote $\mathbf{N}_n$ simply by $\mathbf{N}$.

Lemma 1.2.1 The equation

$$s \cdot x = 1 \tag{1.2.1}$$

in $\mathbf{N}$ has a unique solution x, if and only if s and n are relatively prime.

Proof. For example, see Theorem 10.2 of Ore (1988, p. 238). □

The unique solution x of (1.2.1) is called the inverse of s and denoted by s^{-1} or $1/s$. Let $\mathbf{U_n}$ denote the set of all elements x of $\mathbf{N}$ such that x and n are relatively prime. As shown by Lemma 1.2.1, $\mathbf{U_n}$ is the set of all *invertible* elements of $\mathbf{N}$, that is, elements having multiplicative inverses, and forms a multiplicative abelian group.

If $\mathbf{G}$ is a group, then $|\mathbf{G}|$ is called the *order* of $\mathbf{G}$. For the order $\varphi(n)$ of $\mathbf{U_n}$, φ is known as Euler's function. If $\mathbf{H}$ is a subgroup of $\mathbf{G}$ generated by elements τ,...,ω

of **G**, then **H** is denoted by $\langle \tau, .., \omega \rangle$. If σ and τ are elements of a group **G** and if there exists an element γ of **G** such that $\tau = \gamma^{-1}\sigma\gamma$, then τ is called a *conjugate* of σ, and σ and τ are said to be *conjugate*.

If h is a function from a group **G** to a group **H** such that $h(\sigma\tau) = (h\sigma)(h\tau)$ for all elements σ and τ of **G**, then h is called a *homomorphism* from **G** to **H**. The *direct product* of n groups $\mathbf{G_1}$,..,$\mathbf{G_n}$ is the Cartesian product $\mathbf{G_1} \times ... \times \mathbf{G_n}$ with the multiplication defined by $(g_1, ..., g_n)(g\prime_1, ..., g\prime_n) = (g_1 g\prime_1, ..., g_n g\prime_n)$. If X is a finite set, then the set of all one-to-one transformations, which are called *permutations* of X, is a group with the composition of transformations as its binary operation. This group is called the *symmetric group* on X and denoted by SYM(X). The order of SYM(X) is $|X|!$. The identity element of SYM(X) is the identity permutation ι of X. A one-to-one transformation τ of *length* m of a finite set X is called a *cyclic permutation*, if GRAPH(τ) consists of one m-cycle and loops. In particular, a cyclic permutation σ of length m of X can be expressed by

$$\sigma = (s_1, s_2, .., s_m),$$

if $\sigma s_i = s_{i+1}$ for every $1 \leq i \leq m-1$, $\sigma s_m = s_1$, and $\sigma j = j$ for every $j \in X$ not belonging to $\{s_1, s_2, .., s_m\}$. In particular, the cyclic permutation

$$\rho = (1, 2, .., n)$$

is defined by $\rho i = i + 1$ for every $i \in \mathbf{N}$. Then we have the following elementary theorem on permutations.

Proposition 1.2.2 If τ is not the identity permutation, then τ is a disjoint composition of cyclic permutations, each of length at least 2.

Consider a linear function τ :

$$\tau i = a \cdot i + b \tag{1.2.2}$$

on **N**. For τ to be a permutation, it is necessary and sufficient that a is invertible. We call (a, b) the *coefficients*, a the *slope*, and b the *segment* of the *linear permutation* τ. In this case, it is verified that

$$\tau\rho = \rho^a\tau, \tag{1.2.3}$$

$$\rho\tau = \tau\rho^{a^{-1}} \tag{1.2.4}$$

for the cyclic permutation $\rho = (1, 2, .., n)$.

The cyclic permutation ρ itself is a linear permutation of coefficients $(1, 1)$. If τ is a linear permutation of coefficients (a, b) then τ^{-1} is a linear permutation of coefficients $(a^{-1}, -a^{-1}b)$. For example, let $\tau \in$ SYM($\mathbf{N}$) be a linear permutation of slope $-1 = n-1$. If the coefficients of τ are $(-1, t)$, then τ^{-1} is a linear permutation of coefficients $(1/-1, -t/-1) = (-1, t)$. Therefore, $\tau^{-1} = \tau$. In particular, if the coefficients of τ are $(-1, 1)$, then τ is

$$\lambda = (1, n)(2, n-1) \cdots ([n/2], n - [n/2] + 1),$$

where $[x]$ denotes the greatest integer equal to or less than x. If σ and τ are linear permutations of respective coefficients (a, b) and (c, d), then $\sigma\tau$ is a linear permutation of coefficients $(ac, ad + b)$. If τ is a linear permutation of slope a, then $\tau\rho$ and $\rho\tau$ are also linear permutations of slope a. Since the coefficients of ρ are $(1, 1)$, the coefficients of $\tau\rho$ and $\rho\tau$ are respectively $(a, a + b)$ and $(a, b + 1)$, if the coefficients of τ are (a, b). In summary, we have

Proposition 1.2.3 Let LIN($\mathbf{N}$) denote the set of all linear permutations of $\mathbf{N}$. Then LIN($\mathbf{N}$) is a subgroup of the symmetric group SYM($\mathbf{N}$), and the function h from LIN($\mathbf{N}$) to $\mathbf{U_n}$ that associates each τ of LIN($\mathbf{N}$) with its slope is a homomorphism.

1.3 Pólya actions

Let $\mathbf{G}$ be a group and X be a set. Then it is said that $\mathbf{G}$ *acts on* X, or $\mathbf{G}$ is a *transformation group* for X, if there is a homomorphism H from $\mathbf{G}$ to SYM(X). In this case, H is called an *action* of $\mathbf{G}$ on X or a *representation* of $\mathbf{G}$ on X. The action H is omitted from expressions in the following, when it is clear, so that $(H\tau)x$ is simply written as τx for an element τ of $\mathbf{G}$ and an element x of X. If $\mathbf{G}$ acts on X and x is an element of X, then the subgroup $\mathbf{G}_x$ defined by

$$\mathbf{G}_x = \{g \in \mathbf{G} \mid gx = x\}$$

is called the *stabilizer* of x.

Example 1.3.1 If F is a one-to-one transformation of a set X, then the set $\mathbf{G} = \{F^i \mid i \in \mathbf{Z}\}$ is a subgroup of SYM(X) generated by F. $\mathbf{G}$ is a transformation group for X since $H : \mathbf{G} \rightarrow \mathrm{SYM}(X)$ defined by $H(F^i) = F^i$ is clearly a homomorphism. The order of $\mathbf{G}$ is called the *order* of F.

If a group $\mathbf{G}$ acts on a set D, and X is a set, then the *Pólya action* H of $\mathbf{G}$ on X^D is defined by

$$((H\tau)f)d = f(\tau^{-1}d)$$

for every element τ of $\mathbf{G}$, every element f of X^D, and every element d of D. If σ and τ are elements of $\mathbf{G}$, then

$$\begin{aligned} ((H(\sigma\tau))f)d &= f((\sigma\tau)^{-1}d) = f((\tau^{-1}\sigma^{-1})d) \\ &= f(\tau^{-1}(\sigma^{-1}d)) = ((H\tau)f)(\sigma^{-1}d) = ((H\sigma)((H\tau)f))d \\ &= (((H\sigma)(H\tau))f)d. \end{aligned}$$

That is,

$$H(\sigma\tau) = (H\sigma)(H\tau).$$

Therefore, the Pólya action is an homomorphism and hence in fact an action.

Let $\mathbf{Q} = \{0, 1\}$. Then $\mathbf{Q}^{\mathbf{N}}$ is the set of all binary strings of length n. Let $x = (x_1, x_2, .., x_n) \in \mathbf{Q}^mathbfN$. The period of x is the minimum element k of $\mathbf{N}$ such that $x_{i+k} = x_i$ for every $i \in \mathbf{N}$. The density of x denoted by $d(x)$ is the number of 1s in x, i.e.

$$d(x) = |\{i \mid x_i = 1\}|.$$

The Pólya action H of SYM($\mathbf{N}$) on $\mathbf{Q}^{\mathbf{N}}$ associates a permutation τ of $\mathbf{N}$ with a permutation of coordinates of $\mathbf{Q}^{\mathbf{N}}$ by

$$(H\tau)(x_1, x_2, .., x_n) = (x_{\tau^{-1}1}, x_{\tau^{-1}2}, .., x_{\tau^{-1}n}).$$

$\mathbf{Q}^{\mathbf{N}}$ is simply denoted by $\mathbf{Q}^n$ hereafter. For example, let ρ be the cyclic permutation $(1, 2, .., n)$. The transformation ρ defined by the Pólya action on $\mathbf{Q}^n$ is the *right rotation* of coordinates of $\mathbf{Q}^n$, that is,

$$\rho(x_1, x_2, .., x_n) = (x_n, x_1, .., x_{n-1})$$

for every $x = (x_1, x_2, .., x_n) \in \mathbf{Q}^n$.

If a group $\mathbf{G}$ acts on a set X, then an equivalence relation $\sim_{\mathbf{G}}$ on X can be defined by $x \sim_{\mathbf{G}} y$ if there is an element $\tau \in \mathbf{G}$ such that $\tau x = y$. Each equivalence class with respect to the equivalence relation $\sim_{\mathbf{G}}$ is called an orbit of $\mathbf{G}$ acting on X. The orbit containing an element x of X is denoted by $\mathrm{Orb}_{\mathbf{G}} x$. The union $\bigcup_{x \in S} \mathrm{Orb}_{\mathbf{G}} x$ of the orbits for a subset $S \subseteq X$ is denoted by $\mathrm{Orb}_{\mathbf{G}} S$. Then, the equivalence relation $\sim_{\mathbf{G}}$ is extended to the set of all non-empty subsets of X, that is, $A \sim_{\mathbf{G}} B$ if $\mathrm{Orb}_{\mathbf{G}} A = \mathrm{Orb}_{\mathbf{G}} B$.

1.4 Boolean functions

From now on, $\mathbf{Q} = \{0, 1\}$ is not simply a two-point set, but regarded as the *minimal Boolean algebra* with the unary operation $\neg$ called *complementation* or *negation* and the binary operations $\vee$ called *disjunction* or *OR* and $\cdot$ called *conjunction* or *AND* such that

$$\begin{aligned} \neg 0 &= 1, \quad \neg 1 = 0, \\ 0 \vee 0 &= 0, \quad 0 \vee 1 = 1 \vee 0 = 1, \quad 1 \vee 1 = 1, \\ 0 \cdot 0 &= 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1. \end{aligned}$$

Further, the binary relation $(=)$ can be introduced by defining

$$x(=)y = x \cdot y \vee \neg x \cdot \neg y.$$

Let $L, M \subseteq \mathbf{N}$. If $x \in \mathbf{Q}^M$, then the value of x at $i \in M$ is denoted by x_i. A function from $\mathbf{Q}^M$ to $\mathbf{Q}^L$ is called a *Boolean function.* A function from $\mathbf{Q}^M$ to itself is called a *Boolean transformation.* If $A \subseteq \mathbf{Q}^M$, then 1_A denotes the characteristic function for A, i.e. $1_A x = 1$ if and only if $x \in A$. Let $L \subseteq M \subseteq \mathbf{N}$. Then the *projection function* $P_L : \mathbf{Q}^M \to \mathbf{Q}^L$ is defined by

$$(P_L x)_i = x_i \text{ for every } i \in L \text{ for every } x \in \mathbf{Q}^M.$$

If $j \in M$, then the projection function $p_j : \mathbf{Q}^M \to \mathbf{Q}$ is defined by

$$p_j x = x_j \text{ for every } x \in \mathbf{Q}^M.$$

Let L and M be disjoint subsets of $\mathbf{N}$. Then $\mathbf{Q}^{L \cup M}$ can be identified with the Cartesian product $\mathbf{Q}^L \times \mathbf{Q}^M$ by identifying $x \in \mathbf{Q}^{L \cup M}$ with $(P_L x, P_M x) \in \mathbf{Q}^L \times \mathbf{Q}^M$, where $P_L : \mathbf{Q}^{L \cup M} \to \mathbf{Q}^L$ and $P_M : \mathbf{Q}^{L \cup M} \to \mathbf{Q}^M$ are the projection functions defined above.

Hereafter, $\mathbf{Q}^{\mathbf{N}}$ is simply denoted by $\mathbf{Q}^n$. Let f and g be Boolean functions from $\mathbf{Q}^n$ to $\mathbf{Q}$. Then the *conjunction* of f and g denoted by $f \cdot g$ is the function: $\mathbf{Q}^n \to \mathbf{Q}$ defined by

$$(f \cdot g)x = (fx) \cdot (gx)$$

for every $x \in \mathbf{Q}^n$. The *disjunction* of f and g denoted by $f \vee g$ is the function: $\mathbf{Q}^n \to \mathbf{Q}$ defined by

$$(f \vee g)x = (fx) \vee (gx)$$

for every $x \in \mathbf{Q}^n$. Also a unary operation $\neg$ called *complementation* is defined by

$$(\neg f)x = \neg(fx)$$

for every $x \in \mathbf{Q}^n$. Further, $f(=)g$ is defined by

$$f(=)g = f \cdot g \vee \neg f \cdot \neg g.$$

In this book, we refer to the set f for a Boolean function $f : \mathbf{Q}^n \to \mathbf{Q}$ meaning the set $f^{-1}1$, i.e. the inverse image of 1. Therefore, $x \in f$ means $fx = 1$. Also, the set $\neg f$ is the set $f^{-1}0$. The formula $f \subseteq g$ for Boolean functions f and g is clear by this identification of a Boolean function with a set. Also, $|f|$ is the number of elements of $f^{-1}1$. Further,

$$\begin{aligned} f \cdot g &= f \cap g, \\ f \vee g &= f \cup g, \\ \neg f &= f^c. \end{aligned}$$

Therefore, the corresponding laws such as the associative and commutative laws for set operations $\cap$, $\cup$, and ^{c} hold for $\cdot$,$\vee$, and $\neg$ respectively. In particular, we will frequently use the distributive laws:

$$\begin{aligned} f \cdot (g \vee h) &= f \cdot g \vee f \cdot h, \\ f \vee (g \cdot h) &= (f \vee g) \cdot (f \vee h), \end{aligned}$$

and De Morgan's law:

$$\begin{aligned} \neg(f \vee g) &= (\neg f) \cdot (\neg g), \\ \neg(f \cdot g) &= \neg f \vee \neg g. \end{aligned}$$

A conjunction $g = f_1 \cdot f_2 \cdots f_m$ of Boolean functions $f_i : \mathbf{Q}^n \to \mathbf{Q}$ is called a *term* of *degree* m, if there exists an injection $\varphi : \mathbf{N}_m \to \mathbf{N}$ such that $f_i = p_{\varphi i}$ or $\neg p_{\varphi i}$ for each i. A term g is called an *implicant* of a Boolean function f if $g \subseteq f$. An implicant g of f is called a *prime implicant* if $h = g$ for any implicant h of f such that $g \subseteq h$. The disjunction of terms are called a disjunctive form. A Boolean function f can be represented by the disjunctive form that is the disjunction of all its prime implicants. An irredundant disjunctive form of a Boolean function f is a disjunctive form that represents f such that the removal of any one of its terms does not represents f. In this book, Boolean functions are usually expressed by an irredundant disjunctive form.

Let L and M be disjoint subsets of $\mathbf{N}$, and f and g be respectively Boolean functions from $\mathbf{Q}^L$ to $\mathbf{Q}$ and $\mathbf{Q}^M$ to $\mathbf{Q}$. Then the product $f \cdot g$ is the function from $\mathbf{Q}^{L \cup M}$ to $\mathbf{Q}$ defined by

$$f \cdot g = (f \circ P_L) \cdot (g \circ P_M).$$

Let a an element of $\mathbf{Q}^M$, where M is a proper subset of $\mathbf{N}$, and $f : \mathbf{Q}^{\mathbf{N}} \to \mathbf{Q}$. Then $f|a$ is the function from $\mathbf{Q}^{\mathbf{N} \setminus \mathbf{M}}$ to $\mathbf{Q}$ defined by

$$(f|a)x = f(x, a),$$

where (a, x) is an element of $\mathbf{Q}^{\mathbf{N}}$ defined by

$$P_M(a, x) = a \quad \text{and} \quad P_{N \setminus M}(a, x) = x.$$

Clearly $\neg f|a = \neg(f|a)$. f can be expressed as

$$f = p_i \cdot (f|1) \vee \neg p_i \cdot (f|0),$$

where $1, 0 \in \mathbf{Q}^{\{i\}}$. Conversely, if $f = p_i \cdot g \vee \neg p_i \cdot h$ for $g, h : \mathbf{Q}^{\mathbf{N} \setminus i} \to \mathbf{Q}$, then $g = f|1$ and $h = f|0$.