CHAPTER 4 THRESHOLD FUNCTIONS AND TRANSFORMATIONS

ABSTRACT. The first two sections describe basic properties and combinatorial characterization of threshold functions. One-to-one threshold transformations of $\{0,1\}^n$ are self-dual. A self-dual Boolean transformation is a threshold transformation if and only if each of the Boolean functions that represent the transformation in a [ ]-representation introduced in Chapter 2.4 is a threshold function. Then in the last two sections, we construct minimal one-to-one threshold transformations which are circular or skew-circular. As a result, it is found out that each of them is reflective through some isometries of order 2, so that it is isometrically equivalent to a threshold transformation such that its graph consists of 2-cycles and loops, and its inverse is also a threshold transformation.

4.1 Basic properties of threshold functions

A Boolean function $f : \mathbf{Q}^n \to \mathbf{Q} = \{0, 1\}$ is called a *threshold function*, if the sets f and $\neg f$ are separated by a hyperplane in the real n- dimensional space $\mathbf{R}^n$. In other words, f is a threshold function, if there exist a real n-vector $w = (w_1, ..., w_n)$ and a real number θ such that

$$fx = \mathrm{bool}(wx - \theta), \tag{4.1.1}$$

where $wx = w_1 \cdot x_1 + ... + w_n \cdot x_n$, the inner product of w and x, and

$$\mathrm{bool}(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u \le 0. \end{cases}$$

Here w is called the *weight vector* and θ is called the *threshold value* of f. Threshold functions f and g are called *simultaneously realizable*, if both can be defined with the same weight vector. A threshold transformation F of $\mathbf{Q}^n$ is a natural generalization of a threshold function from $\mathbf{Q}^n$ to $\mathbf{Q}$ such that each $p_i F$ is a threshold function.

Since the pioneering work of McCulloch and Pitts (1943), a great number of theoretical and experimental studies (e.g. Arimoto, 1963; Amari, 1972; Goles-Chacc, 1980; Goles & Olivos, 1981; Hopfield, 1982; Goles, Fogelman-Soulie & Pellegrin, 1985; Cottrell, 1988; Blum, 1990; Blessloff & Taylor, 1991 etc.) have been made on neural networks. The information process on neural networks is mathematically summarized as threshold transformations. However, their mathematical properties are yet to be clarified by rigorous analysis. In practical models, important transformations are those whose iterative operations transform a subset whose points share a certain common pattern into a smaller set. Although these transformations are not one- to-one, we deal with one-to-one transformations in the first place, hoping the study will help us with general threshold transformations. In particular our goal in this chapter is to investigate simple threshold transformations that are minimal, one-to-one, and circular.

Various properties of threshold transformations should reflect those of a threshold functions. Therefore, we describe these properties with or without proof in the first two sections of this chapter as basic properties and combinatorial characterizations. Most of these results were obtained in the 1960s and contained in Hu (1965) and Muroga (1971). Some of them are main propositions, and others are minor but used later in this book.

Proposition 4.1.1 Let f be a threshold function: $\mathbf{Q}^n \to \mathbf{Q}$. Then (i) $\neg f$ is a threshold function, (ii) $\bar{\neg} f$ is a threshold function, and (iii) F and $\neg\bar{\neg} f$ are simultaneously realizable.

Proof. We have a real n-vector w and a real number θ such that $fx = 1$ iff $wx - \theta > 0$. Therefore, $fx = 0$ that is $(\neg f)x = 1$ iff $wx - \theta \leq 0$, that is, $-wx + \theta \geq 0$. Since $\mathbf{Q}^n$ is a finite set, there exists $\eta \geq \theta$ such that $-wx + \theta \geq 0$ iff $-wx + \eta > 0$. Therefore, $(\neg f)x = 1$ iff $(-w)x - \eta > 0$, so that $\neg f$ is a threshold function. (ii) $fx = 1$ iff $wx - \theta > 0$ implies that $f(\bar{\neg} x) = 1$ iff $w(\bar{\neg} x) - \theta > 0$, that is, $w(l - x) - \theta > 0$, that is, $(-w)x - (\theta - wl) > 0$, where l is the n-vector such that $l_i = 1$ for every i. Therefore, $f\bar{\neg}$ that is, $\bar{\neg} f$ is a threshold function. (iii) is clear from the above proof of (i) and (ii). □

Proposition 4.1.2 If f is a threshold function: $\mathbf{Q}^n \to \mathbf{Q}$ and p_j is the projection function: $\mathbf{Q}^n \to \mathbf{Q}$, then $p_j \vee f$, $\neg p_j \vee f$, $p_j \cdot f$, and $\neg p_j \cdot f$ are threshold functions: $\mathbf{Q}^n \to \mathbf{Q}$.

Proof. Let f be a threshold function. Then there exists a real n-vector w and a real number θ such that $fx = 1$ iff $wx - \theta > 0$.

Let $\delta > n \cdot \max|w_i| + |\theta|$, and let $v_j = w_j + \delta$, $v_i = w_i$ for every $i \neq j$. Then, if $x_j = 1$, then $vx - \theta = wx - \theta + \delta > 0$; if $x_j = 0$, then $vx - \theta = wx - \theta$. Therefore, $x_j = 1$ or $fx = 1$, if and only if $vx - \theta > 0$, so that $p_j \vee f$ is a threshold function. Further, $\neg p_j \cdot f = \neg(p_j \vee \neg f)$, so that $\neg p_j \cdot f$ is a threshold function.

Let $\eta = (n-1)\max|w_i| - \theta$, and let $v_j = w_j + \eta$, $v_i = w_i$ for every $i \neq j$, and $\zeta = \theta + \eta$. Then, if $x_j = 1$, then $vx - \zeta = wx - \theta$; if $x_j = 0$, then $vx - \zeta = w_1x_1 + .. + w_{j-1}x_{j-1} + w_{j+1}x_{j+1} + .. + w_nx_n - \theta - \eta \leq (n-1)\max|w_i| - \theta - \eta = 0$. Therefore, $x_j = 1$ and $fx = 1$, if and only if $vx - \zeta > 0$, so that $p_j \cdot f$ is a threshold function. Further, $\neg p_j \vee f = \neg(p_j \cdot \neg f)$, so that $\neg p_j \vee f$ is a threshold function. □

Proposition 4.1.3 Let f be a Boolean function: $\mathbf{Q}^n \to \mathbf{Q}$ and p_{n+1} be the projection: $\mathbf{Q}^{\{n+1\}} \to \mathbf{Q}$. If $f \cdot p_{n+1}$ or $f \vee p_{n+1}$ is a threshold function, then f is a threshold function.

Proposition 4.1.4 (Muroga, Toda, & Takasu, 1961; Winder, 1962; Muroga,1971, Theorem 8.1.1.2) If $f : \mathbf{Q}^n \to \mathbf{Q}$ and $g : \mathbf{Q}^n \to \mathbf{Q}$ are simultaneously realizable threshold functions, then $h : \mathbf{Q}^{n+1} \to \mathbf{Q}$ defined by $h = (\neg p_{n+1}) \cdot f \vee p_{n+1} \cdot g$ is a threshold function from $\mathbf{Q}^{n+1}$ to $\mathbf{Q}$.

Proposition 4.1.5 If f is a threshold function, then fT and Tf are threshold functions for any isometry T of $\mathbf{Q}^n$.

Notation For a set of Boolean functions or variables $\{.\}$, let $S_m\{.\}$ denote the disjunction of all conjunctions of m elements of $\{.\}$. For example, $S_2\{p_1, p_2, p_3\} = p_1 \cdot p_2 \vee p_1 \cdot p_3 \vee p_2 \cdot p_3$.

Proposition 4.1.6 If f is a threshold function: $Q^n \to Q$, then

$$g = S_k\{p_{n+1}, .., p_{n+m}\} \vee S_{k-1}\{p_{n+1}, .., p_{n+m}\} \cdot f$$

is a threshold function: $Q^{n+m} \to Q$.

Proof. Let f be a threshold function. Then there exists a real n-vector w and a real number θ such that $fx = 1$ iff $wx - \theta > 0$. In order to prove the proposition, it is sufficient to determine the threshold vector (w, v) such that $v = (\delta, \delta, ..\delta)$ and a threshold value η for g. Then it is sufficient to determine δ and η such that

$$\begin{array}{ll} (1) & wx + k\delta > \eta \text{ for any } x, \\ (2) & wx + (k-1)\delta > \eta \text{ iff } wx > \theta, \\ (3) & wx + (k-2)\delta \leq \eta \text{ for any } x. \end{array}$$

(2) is equivalent to $wx + (k-1)\delta + \theta > \theta + \eta$ iff $wx > \theta$. Therefore, let

$$(k-1)\delta + \theta = \eta. \tag{4.1.2}$$

Then (2) is satisfied. By substituting (4.1.2) for η in (1) and (3), we obtain $\min(wx) + k\delta > (k-1)\delta + \theta$, i.e.

$$\delta > -\min(wx) + \theta, \tag{4.1.3}$$

and $\max(wx) + (k-2)\delta \leq (k-1)\delta + \theta$, i.e.

$$\delta \geq \max(wx) - \theta. \tag{4.1.4}$$

Therefore, let

$$\delta > \max(\sum_i |w_i| + \theta, \sum_i |w_i| - \theta). \tag{4.1.5}$$

Then (4.1.3) and (4.1.4) are satisfied. Thus we determined desired δ and η by (4.1.5) and (4.1.2). □

Proposition 4.1.7 If f is a threshold function: $\mathbf{Q}^N \to \mathbf{Q}$, $\mathbf{M}$ is a proper subset of $\mathbf{N}$, and a is an element of Q^M, then $f|a$ is a threshold function: $\mathbf{Q}^{N\setminus M} \to \mathbf{Q}$.

Proof. Let f be a threshold function. Then there exists a real n-vector w and a real number θ such that $fx = 1$ iff $wx - \theta > 0$. Therefore, $(f|a)x = f(a, x) = 1$ iff $(P_M w)a + (P_{N\setminus M} w)x - \theta > 0$, i.e.

$$(P_{N\setminus M} w)x - (\theta + (P_M w)a) > 0.$$

Therefore, $f|a$ is a threshold function with the weight vector $P_{N\setminus M} w$ and the threshold value $\theta + (P_M w)a$. □

4.2 Combinatorial properties

A Boolean function f is called m-summable, if there exist $a^{(1)}, ..., a^{(k)}$ and $b^{(1)}, ..., b^{(k)}$ such that $2 \leq k \leq m$, and $a^{(i)} \in f$, $b^{(i)} \in \neg f$ for every i such that $1 \leq i \leq k$, $a^{(i)} \in f^{(i)} \in \neg f$, for every i such that $1 \leq i \leq k$, and

$$\sum_{i=1}^{k} a^{(i)} = \sum_{i=1}^{k} b^{(i)},$$

where the sums are performed in $\mathbf{R}^n$. If f is not *m-summable*, f is called *m-asummable*. If f is m-summable for some m, f is called *summable*. If f is not summable, f is called *asummable*. The following proposition characterizes threshold functions in place of linear separability of f expressed by (4.1.1).

Proposition 4.2.1 (Elgot 1961; Chow 1961) A Boolean function f is a threshold function if and only if f is asummable.

Proof. In the definition of threshold functions in Section 1, the condition that f and $\neg f$ are separated by a hyperplane in the real n-space $\mathbf{R}^n$ can be replaced with the fact that f and $\neg f$ are separated by a hyperplane in the rational n-space. Further, the latter condition is equivalent to the fact that the convex hull of f and the convex hull of $\neg f$ are disjoint. The last condition is equivalent to the fact that f is asummable. □

Corollary 4.2.2 If a Boolean function f is a threshold function, then f is 2-asummable.

Attempts to characterize a threshold function combinatorially in place of linear separability had been made to replace the linear separability conditions (4.1.1), which is described in terms of the weight vector and the threshold value. Unfortunately, there is no such combinatorial expression that describes a necessary and sufficient condition for a Boolean function to be a threshold function. For example, the asummability condition described by Proposition 4.2.1 involves an unlimited number of equations. Therefore, only some necessary conditions for threshold functions are available. We describe here some of these necessary but not sufficient conditions in the following because of their own value, but we use only Corollaries 4.2.2 and 4.2.4 in this book.

Let f be a Boolean function defined on $\mathbf{Q}^n$. If

$$f|q \subseteq f|\bar{\neg} q \text{ or } f|\bar{\neg} q \subseteq f|q$$

for every $q \in \mathbf{Q}^M$ for every $M \subseteq \mathbf{N}$ such that $|M| \leq k$, then f is called *k-monotonic*. In particular, f is called *unate* if $k = 1$, and f is called *completely monotonic* if $k = n$. The following theorem due to Elgot (1961) gives a condition equivalent to 2-asummability. The proof is also found in Muroga (1971, Chapter 3).

Theorem 4.2.3 (Elgot, 1961) A Boolean function of $\mathbf{Q}^n$ is 2-asummable if and only if it is completely monotonic.

Proof. Assume that f is 2-summable. By definition, there exists $x, y \in f$ and $u, v \in \neg f$ such that $x + y = u + v$. Then $x_i = y_i = u_i = v_i$ if and only if $(x+y)_i$ is 0 or 1. Let $K = \{i \mid x_i = y_i = u_i = v_i\}$, $L = \{i \mid i \notin K \text{ and } x_i = u_i\}$, and $M = \{i \mid i \notin K \text{ and } x_i \neq u_i\}$. Then $\mathbf{N}$ is the union mutually disjoint K, L, and M. Further, neither L nor M is empty, since x is either u or v otherwise, contrary to the fact $x \in f$ and $u, v \in \neg f$. Let $P_K x = a$, $P_L x = b$, and $P_M x = c$. Then, $x = (a, b, c)$, $y = (a, \bar{\neg} b, \bar{\neg} c)$, $u = (a, b, \bar{\neg} c)$, $z = (a, \bar{\neg} b, c)$. Considering x and v, $(a, c) \in f|b$ and $(a, c) \in \neg f|\bar{\neg} b = \neg(f|\bar{\neg} b)$, so that $f|b \subseteq f|\bar{\neg} b$ does not hold. Considering y and u, $(a, \bar{\neg} c) \in f|\bar{\neg} b$ and $(a, \bar{\neg} c) \in \neg f|b = \neg(f|b)$, so that $f|\bar{\neg} b \subseteq f|b$. Therefore, f is not completely monotonic.

Conversely, assume that f is not completely monotonic. By definition, there exists a such that neither $f|a \subseteq f|\bar{\neg} a$ nor $f|\bar{\neg} a \subseteq f|a$. Therefore, there exist b

and c such that $(b,a) \in f$, $(b,\bar{\neg}a) \in \neg f$, $(c,\bar{\neg}a) \in f$, and $(c,a) \in \neg f$. Since $(b,a)+(c,\bar{\neg}a)=(b,\bar{\neg}a)+(c,a)$, f is 2-summable. $\square$

Corollary 4.2.4 (Muroga, Toda, & Takasu, 1960) If f is a threshold function defined on $\mathbf{Q}^n$ and $k \in \mathbf{N}$, then either $f|0$ or $f|1$ is a subset of the other, and both are threshold functions, where $0,1 \in \mathbf{Q}^k$.

Another combinatorial condition for 2-asummability was given by Yajima and Ibaraki in terms of prime implicants of the Boolean function:

Theorem 4.2.5 (Yajima & Ibaraki, 1968) A Boolean function f is 2-asummable, if and only if (i) f is unate and (ii) if $u_1 \cdot u_2 \cdot w$ and $v_1 \cdot v_2 \cdot w$ are any prime implicants of f, where terms $u_1 \cdot u_2$ and $v_1 \cdot v_2$ contain no common projection p_i, then there exists a prime *implicant* z of f such that $z \subseteq u_1 \cdot v_1 \cdot w$ or $z \subseteq u_2 \cdot v_2 \cdot w$.

Proposition 4.2.6 If a point q and $\neg q$ belong to a threshold function f from $\mathbf{Q}^n$ to $\mathbf{Q}$, then $|f| \geq 2^{n-1}+1$.

Proof. If both q and $\bar{\neg}q$ belong to f, then the 2-asummability condition for f requires that for any $r \in \mathbf{Q}^n$, either r or $\bar{\neg}r$ or both belong to f. $\square$

The following Theorem (Ueda, 1977, 1979) gives a necessary condition for a Boolean function to be a threshold function in terms of the stabilizer of f. Let f be a Boolean function from $\mathbf{Q}^n$ to $\mathbf{Q}$. Let i and j be elements of $\mathbf{N}$. We define the binary relation $\succeq_f$ by

$$i \succeq_f j \text{ if } i=j \text{ or } f|10 \supseteq f|01,$$

where 10 and 01 are elements of $\mathbf{Q}^{\{i,j\}}$. Further, we define

$$i \sim_f j \text{ if } i \succeq_f j \text{ and } j \succeq_f i.$$

Proposition 4.2.7 (Winder, 1962; Muroga, Toda, & Takasu, 1961) The binary relation $\succeq_f$ is a preorder, i.e. reflective and transitive, and hence the binary relation $\sim_f$ is an equivalence relation on $\mathbf{N}$.

Theorem 4.2.8 (Ueda, 1977, 1979) Let f be 2-monotonic Boolean function defined on $\mathbf{Q}^n$. Then $\mathbf{N}$ is partitioned into sets $M_1, ..., M_k$ for some k such that the stabilizer of f is the direct product of the k symmetric groups $\mathrm{SYM}(M_1), .., \mathrm{SYM}(M_k)$.

Proof. Let a permutation $\sigma \in \mathrm{SYM}(\mathbf{N})$ be an element of the stabilizer of f. Let σ be expressed by the disjoint composition of permutations as $\sigma = \tau \odot .. \odot \upsilon \odot .. \odot \omega$. Let $\tau = (s_1, ..., s_h)$. Since f is 2-monotonic, the preorder $\succeq_f$ is total, i.e. $i \succeq_f j$ or $j \succeq_f i$ for every pair i, j of $\mathbf{N}$, and $\sim_f$ is an equivalence relation on $\mathbf{N}$ by Proposition 4.2.7. Let the set of all equivalence classes determined by $\sim_f$ be $\{M_1, ..., M_k\}$.

If $s_1 \succeq_f s_2$ then $\sigma s_1 \succeq_{\sigma f} \sigma s_2$, so that $s_2 \succeq_f s_3$, since $\sigma f = f$. Therefore, $s_1 \succeq_f s_2 \succeq_f ... \succeq_f s_h \succeq_f s_1$, and hence $s_1 \sim_f s_2 \sim_f ... \sim_f s_h$. Similarly if $s_2 \succeq_f s_1$ then $s_1 \sim_f s_2 \sim_f ... \sim_f s_h$. Therefore, all non-fixed elements on each cycle υ of the disjoint composition are elements of some equivalence class M_j.

Now let $i \sim_f j$, $i \neq j$. Then by the definition of $\sim_f$, $f|10 = f|01$. Therefore, $(i,j)f = f$ for the interchange $(i,j) \in \mathrm{SYM}(\mathbf{N})$. Therefore, the stabilizer of f is the direct product of $\mathrm{SYM}(M_1), ..., \mathrm{SYM}(M_k)$. $\square$

4.3 Basic properties of transformations

A Boolean transformation F of $\mathbf{Q}^n$ is called a threshold transformation, if $F_i = p_i F$ is a threshold function for every i. That is, there exists an $n \times n$ real matrix W and real column n-vector h such that

$$Fx = \mathrm{Bool}(Wx - h), \tag{4.3.1}$$

where

$$(\mathrm{Bool}(x))_i = \begin{cases} 1 & \text{if } x_i > 0, \\ 0 & \text{if } x_i \leq 0, \end{cases}$$

for every i. Here W is called the *weight matrix* and h is called the *threshold vector* of F. Now, we consider the set $\{-1, 1\}$ in place of $\mathbf{Q} = \{0, 1\}$. As described in Chapter 2.1, a transformation F of $\mathbf{Q}^n$ and a transformation G of $\{-1, 1\}^n$ is equivalent, if $G = \mathrm{Bool}^{-1} \circ F \circ \mathrm{Bool}$ or $F = \mathrm{Sgn}^{-1} \circ G \circ \mathrm{Sgn}$, where

$$(\mathrm{Sgn}(y))_i = \begin{cases} 1 & \text{if } y_i > 0, \\ -1 & \text{if } y_i \leq 0, \end{cases}$$

that is, the following diagram is commutative.

$$\begin{array}{ccc} \mathbf{Q}^n & \xrightarrow{F} & \mathbf{Q}^n \\ \mathrm{Sgn} \downarrow & & \downarrow \mathrm{Sgn} \\ \{-1, 1\}^n & \overrightarrow{G} & \{-1, 1\}^n. \end{array}$$

If Bool and Sgn are respectively restricted to $\{-1, 1\}^n$ and $\mathbf{Q}^n$, then

$$\mathrm{Bool}(x) = (1/2)(x + l) \text{ and } \mathrm{Sgn}(y) = 2y - l,$$

where l is the column n-vector whose every coordinate is 1. Therefore, if F is a threshold transformation defined by (4.3.1), then

$$\begin{aligned} Gy &= \mathrm{Sgn}(\mathrm{Bool}(W(1/2(y + l)) - h)) \\ &= \mathrm{Sgn}(W(1/2(y + l)) - h) \\ &= \mathrm{Sgn}(Wy - (2h - Wl)). \end{aligned}$$

Generally, H is said to be a *threshold transformation* of $\{-1, 1\}^n$ if there is a real $n \times n$ matrix V and a real n-vector r such that

$$Hy = \mathrm{Sgn}(Vy - r). \tag{4.3.2}$$

Here V is called the *weight matrix* and s is called the *threshold vector* of H. Thus we showed that, in the above commutative diagram, if F is a threshold transformation, then G is a threshold transformation defined with the same weight matrix W. Conversely, let H be defined by (4.3.2). Then the corresponding equivalent transformation J is

$$\begin{aligned} Jx &= \mathrm{Bool}(\mathrm{Sgn}(V(2x - l) - r)) \\ &= \mathrm{Bool}(\mathrm{Sgn}(Vx - (1/2)(Vl + r))) \\ &= \mathrm{Bool}(Vx - (1/2)(Vl + r)). \end{aligned}$$

Therefore J is a threshold transformation defined by the same weight matrix V. An advantage of using $\{-1, 1\}^n$ for threshold functions and transformations in place of $\mathbf{Q}^n$ is apparent in the following propositions and theorem.

Proposition 4.3.1 Let H be a threshold transformation of $\{-1, 1\}^n$ defined by (4.3.2). If the threshold vector r is the zero vector, and if there is no point y

in $\{-1,1\}^n$ such that $(Vy)_i = 0$ for some i, then H is self-dual. Conversely, if H is self-dual, then $Hy = \mathrm{Sgn}(Vy)$, and there is no point of $\{-1,1\}^n$ such that $(Vy)_i = 0$ for some i.

Proof. Let $Hy = \mathrm{Sgn}(Vy)$ for every y. Then $(H-)y = H(-y) = \mathrm{Sgn}(V(-y)) = \mathrm{Sgn}(-(Vy))$ for every y. If there is no point y of $\{-1,1\}^n$ such that $(Vy)_i = 0$ for some, then $\mathrm{Sgn}(-(Vy)) = -\mathrm{Sgn}(Vy) = -Hy$, so that $H- = -H$, that is H is self-dual.

Conversely, let H be self-dual and $Hy = \mathrm{Sgn}(Vy - r)$ for every y. Then by slightly changing r, we can obtain $r\prime$ such that $Hy = \mathrm{Sgn}(Vy - r\prime)$ and there is no point y such that $(Vy - r\prime)_i = 0$ for some i. Then, $-\mathrm{Sgn}(V(-y) - r\prime) = \mathrm{Sgn}(-(V(-y) - r\prime)) = \mathrm{Sgn}(Vy + r\prime)$ and $-\mathrm{Sgn}(V(-y) - r\prime) = \mathrm{Sgn}(Vy - r\prime)$, so that $\mathrm{Sgn}(Vy + r\prime) = \mathrm{Sgn}(Vy - r\prime)$ for every y. If $r\prime_i \neq 0$, then $(Vy)_i > |r\prime_i|$ or $(Vy)_i < -|r\prime_i|$, so that $\mathrm{Sgn}((Vy)_i - r\prime_i) = \mathrm{Sgn}((Vy)_i)$ and there is no y such that $(Vy)_i = 0$. Therefore, $Hy = \mathrm{Sgn}(Vy)$, and there is no y such that $(Vy)_i = 0$ for some i. □

The following Proposition 4.3.2 (i) shows that the set of the Boolean isometries is a subset of the one-to-one threshold transformations. As defined in Chapter 2.2, transformations F and G of $\mathbf{Q}^n$ are called *isometrically equivalent* if there exist isometries S and T of $\mathbf{Q}^n$ such that $G = SFT$. The following Proposition 4.3.2 (ii) shows that if F is a threshold transformation and G is isometrically equivalent to F, then G is also a threshold transformation. The set of all Boolean isometries is the subgroup of the transformation group consisting of all one-to-one self-dual transformations. The following Theorem 4.3.3 shows that the set of the one-to-one threshold transformations is a subset of the set of the one-to-one self-dual transformations.

Proposition 4.3.2 (i) Orthogonal transformations are threshold transformations. (ii) If F is a threshold transformation and T is an orthogonal transformation, then FT and TF are threshold transformations. (iii) If F is an threshold transformation, then F is orthogonally equivalent to a minimal threshold transformation.

Proof. (i) If $T \in O\{-1,1\}^n$ then there exists an orthogonal matrix A such that $Ty = Ay = \mathrm{Sgn}(Ay)$ for every y.

(ii) Let G be a threshold transformation of $\{-1,1\}^n$ defined $Gy = \mathrm{Sgn}(Vy - r)$, and let $T \in O\{-1,1\}^n$ defined by $Ty = Ay$, where V and A are $n \times n$ real matrix, and r is a real n-vector. Then

$$\begin{aligned} (GT)y &= \mathrm{Sgn}(V(Ay) - r) = \mathrm{Sgn}((VA)y - r), \\ (TG)y &= A\mathrm{Sgn}(Vy - r) = A\mathrm{Sgn}(Vy - r\prime), \end{aligned}$$

where $r\prime$ is obtained by modifying r so that there is no point $y \in \{-1,1\}^n$ such that $(Vy - r\prime)_i = 0$ for some i. Then

$$(TG)y = \mathrm{Sgn}(A(Vy - r\prime)) = \mathrm{Sgn}((AV)y - Ar\prime).$$

(iii) is clear from (ii) and the definition of minimal transformations. □

Theorem 4.3.3 One-to-one threshold transformations are self-dual.

Proof. Let $F = (F_1, ..., F_n)$, where $F_i = p_i F$, be a one-to-one threshold transformation of $\mathbf{Q}^n$; then $|\neg F_i| = |F_i|$. Therefore, if a point q and its complement $\bar{\neg} q$ belong to F_i, then another point r and its complement $\bar{\neg} r$ must be in $\neg F_i$. This contradicts the 2-asummability of F_i, because $(q + \bar{\neg} q)/2 = (r + \bar{\neg} r)/2$ in $\mathbf{R}^n$. Therefore, F_i is self-dual, so that F is self-dual. □

Proposition 4.3.4 Let a self-dual Boolean function $F_i : \mathbf{Q}^n \to \mathbf{Q}$ be defined as

$$F_i = p_i \cdot g_i \vee \neg p_i \cdot (\neg \bar{\neg} g_i).$$

for a Boolean function $g_i : \mathbf{Q}^{\mathbf{N} \setminus i} \to \mathbf{Q}$. Then F_i is a threshold function if and only if g_i is a threshold function.

Proof. If g_i is a threshold function, then g_i and $\neg \bar{\neg} g_i$ are simultaneously realizable threshold functions by Proposition 4.1.1 (iii). Therefore, F_i is also a threshold function according to Proposition 4.1.4. If F_i is a threshold function, clearly g_i is also a threshold function. □

Corollary 4.3.5 A self-dual Boolean transformation F represented as $[f_1, ..., f_n]$ is a threshold transformation, if and only if f_i is a threshold function for every i.

The above corollary implies that $\langle f \rangle$ is a threshold transformation if and only if f is a threshold function, and $\langle \langle f \rangle \rangle$ is a threshold transformation if and only if f is a threshold function. Finally, the following two propositions are clear.

Proposition 4.3.6 Let G be a threshold transformation of $\mathbf{Q}^{n+1}$. If the transformations H and L of $\mathbf{Q}^n$ are defined by $H_i(x_1, ..., x_n) = G_i(x_1, ..., x_n, 1)$ and $L_i(x_1, ..., x_n) = G_i(x_1, ..., x_n, 0)$ for $i = 1, .., n$, then H and L are threshold transformations.

Proof. There exists a real $(n+1) \times (n+1)$ matrix E and a real $(n+1)$-vector h such that

$$Gx = \mathrm{Bool}(Ex - h).$$

Let $h_i\prime = h_i - E_{in+1}$ and $E\prime_{ij} = E_{ij}$ for $1 \leq i, j \leq n$. Then clearly

$$Hx = \mathrm{Bool}(E\prime x - h\prime).$$

Therefore, H is a threshold transformation. □

Proposition 4.3.7 Let L and M be disjoint subsets of $\mathbf{N}$. If F is a threshold transformation of $\mathbf{Q}^L$, and if G is a threshold transformation of $\mathbf{Q}^M$. Then the direct product of F and G is a threshold transformation.

Proof. The proof is clear from the definitions of threshold transformations and the direct product. □

Proposition 4.3.8 If $|f_k| \leq 2^{n-2}$ for $F = [f_1, ..., f_n]$, then

$$(f_k|1) \subseteq \neg \bar{\neg} (f_k|1).$$

Proof. Assume that $(f_k|1) \subseteq \neg \bar{\neg} (f_k|1)$, i.e. $\neg (f_k|1) \supseteq \bar{\neg} (f_k|1)$ does not hold. Then there exists some x such that $P_{\mathbf{N} \setminus k} x \in \bar{\neg} (f_k|1)$ and $P_{\mathbf{N} \setminus k} x \in (f_k|1)$. Therefore, by Proposition 4.2.6, $|f| = \bar{\neg} (f_k|1) \geq 2^{n-1} + 1$. □

Corollary 4.3.9 If $F = [f_1, ..., f_n]$ is a self- dual minimal threshold transformation, then

$$(f_i|1) \subseteq \neg \bar{\neg} (f_i|1) \text{for every } i.$$

Proof. Assume that $(f_k|1) \subseteq \neg\bar{\neg}(f_k|1)$ does not hold for some k. Then by Proposition 4.3.8, $|f_k| \geq 2^{n-2}+1$. Then $|\bar{\neg}(f_k|1)| = |f_k| \geq 2^{n-2}+1$. Therefore, $|\bar{\neg}(f_k|1)| < |f_k|$. Hence, F is not minimal, because $\mathrm{Var}(k^- F) < \mathrm{Var}F$ by Proposition 2.4.3. □

4.4 Circular one-to-one transformations

Let $F = \langle f \rangle$ for $\mathbf{Q}^n$ be a circular threshold transformation, then F is isometrically equivalent to a circular minimal self-dual transformation $\langle g \rangle$ by Theorem 2.5.2 of Chapter 2. Further $\langle g \rangle$ is a threshold transformation by Proposition 4.3.2 (ii) of the present chapter. Therefore, in this section we construct minimal, circular, one-to-one threshold transformations.

There are two naive approaches for obtaining a one-to-one threshold transformations. One is to construct a one-to-one transformation first and modify it to a threshold transformation. The other is to construct a threshold transformation first and modify it to a one-to-one transformation.

We exclude from here compressible transformations, which are described in Chapter 5.2. In the follwing Examples, $F = \langle f \rangle$ is a transformation of $\mathbf{Q}^n$.

Notation In this section, $\mu \in \mathrm{SYM}(\mathbf{N})$ denotes the linear permutation of coefficients $(-1, 2)$, that is,

$$\mu = (2, n)(3, n-1) \cdots (i, n-i+2) \cdots ([(n-1)/2+1, n+1-[((n-1)/2]).$$

Example 4.4.1 n is odd ($n = 2m+1$).

$$f = p_1 \cdots p_{m+1} \cdot \neg p_{m+2} \cdots \neg p_{2m+1}.$$

GRAPH(F) consists of one $2n$-cycle and loops. E.g.

$$\begin{aligned} 11100 \quad &\rightarrow \quad 01100 \rightarrow 01110 \rightarrow 00110 \rightarrow ... \\ &\rightarrow \quad 11000 \rightarrow 11100. \end{aligned}$$

For $n = 3$, F is isometrically equivalent to the compressible transformation $\langle p_1 \cdot p_2 \cdot p_3 \rangle$. F is uniquely minimal for $n \geq 5$.

As a generalization of Example 4.4.1, we now determine a condition for the transformation $F = \langle f \rangle$ of $\mathbf{Q}^n$ for $n = 2m+1$ such that

$$f = p_1 \cdot q_2 \cdot ... \cdot q_n, \quad \text{where } q_i = p_i \text{ or } \neg p_i \text{ for every } i$$

to be one-to-one.

Let $f = \{x\}$, a one-element set, where $x = (1, x_2, ..., x_n)$. We assume $Fx = (0, x_2, ..., x_n)$. In order to determine a condition for $F(\mathrm{Car}F) = \mathrm{Car}F$, suppose $Fx = \bar{\neg}\rho n^{h-1}x$ for $0 < h-1 < n$. Then

$$p_1 x = 1, \quad 1^- x = \bar{\neg}\rho^{h-1}x, \tag{4.4.1}$$

that is,

$$(0, x_2, ..., x_n) = (\neg x_{n+2-1+}, .., \neg x_n, 0, \neg x_2, .., \neg x_{n+2-(h+1)}),$$

that is,

$$\neg x_1 = 0 = \neg x_{n+2-h}, x_2 = \neg x_{n+2-(h-1)}, .., x_{h-1} = \neg x_n, x_h = 0, x_{h+1} = \neg x_2, .., x_n = \neg x_{n+2-(h+1)},$$

that is,

$$\begin{aligned} x_h = 0, x_{h+(h-1)} = \neg x_h, x_{h+2(h-1)} = \neg x_{h+(h-1)}, ..., \\ x_{n+2-h} = \neg x_{n+2-(h+(h-1))}, \neg x_1 = 0 = \neg x_{n+2-h}, \end{aligned} \tag{4.4.2}$$

If $h-1$ is relatively prime with n, the system of equations (4.4.2) sequentially and uniquely determines x_i from x_h to x_1 by step $h-1$ of their subscripts. Further, the values of x_i change $n-1$ times as the values of x_i are determined from x_h to x_1. Since $n-1$ is even, x_i is consistently determined for every i.

Example 4.4.2 $n = 7$ and $h-1 = 2$. The equation $1^- x = \bar{\neg}(\rho)^2 x$ for $x = (1, x_2, .., x_7)$ is

$$\neg x_1 = 0 = \neg x_6, x_2 = \neg x_7, x_3 = 0, x_4 = \neg x_2, x_5 = \neg x_3, x_6 = \neg x_4, x_7 = \neg x_5,$$

that is,

$$x_3 = 0, x_5 = \neg x_3, x_7 = \neg x_5, x_2 = \neg x_7, x_4 = \neg x_2, x_6 = \neg x_4, \neg x_1 = 0 = x_6.$$

The solution is $x = 1100110$.

Now assume $0 < h-1 < n$ is relatively prime with n, and let x be the solution of (4.4.1). Then

$$\begin{aligned}(\bar{\neg}\rho^{h-1})^2 x &= (\bar{\neg}\rho^{h-1})(1^- x)\\ &= (\neg x_{n-h+2}, .., \neg x_n, 1, \neg x_2, .., \neg x_{n-h+1})\\ &= h^- \bar{\neg}(\rho^{h-1} x)\\ &= h^- 1^- x.\end{aligned}$$

In general,

$$(\bar{\neg}\rho^{h-1})^i x = (1 + (i-1)(h-1))^- (1 + (i-2)(h-1))^- ...(1 + (h-1))^- 1^- x \quad \text{for any positive integer } i. \tag{4.4.3}$$

Therefore, $(\bar{\neg}\rho^{h-1})^i x$ for $i = 1, .., 2n-1$ are all different from x.

Let $f = \{x\}$ and $F = \langle f \rangle$. Then

$$Fx = 1^- x = \bar{\neg}\rho^{h-1} x,$$

and F is one-to-one with one $2n$-cycle.

To test whether F is reflective or not, suppose F is reflective through μ. Then $\mu F x = \bar{\neg} x$ by Proposition 3.4.2. That is,

$$(0, x_n, x_{n-1}, .., x_2) = (0, \neg x_2, \neg x_3, .., x_n),$$

so that $x_i = \neg x_{n+2-i}$, for every $i \geq 2$, that is,$x = \bar{\neg} 1^- \mu x$.

we transform the system of equations (4.4.1) by replacing x with $\bar{\neg} 1^- \mu x$. Then we have

$$p_1(\bar{\neg} 1^- \mu x) = 1, \quad 1^-(\bar{\neg} 1^- \mu x) = \bar{\neg}\rho^{h-1}(\bar{\neg} 1^- \mu x). \tag{4.4.4}$$

By (4.4.2), (4.4.4) is equivalent to

$$\neg x_{n+2-h} = 0, \neg x_{n+2-(h+h-1)} = x_{n+2-h}, \neg x_{n+2-(h+2(h-1))} = x_{n+2-(h+(h-1))}, \cdots, \neg x_h = x_{h+(h-1)}, \neg x_1 = 0 = x_h. \tag{4.4.5}$$

Clearly, (4.4.5) is equivalent to (4.4.2). That is, the system of equations (4.4.1) is invariant under the transformation $\bar{\neg} 1^- \mu$, so that the solution x of (4.4.1) is also invariant under $\bar{\neg} 1^- \mu$, i.e. $\bar{\neg} 1^- \mu x = x$.

Now let $F = \langle f \rangle$ for $f = \{x\}$. Then

$$\mu F x = (0, x_n, x_{n-1}, .., x_2), \quad \text{while } \bar{\neg} x = (0, \neg x_2, \neg x_3, ...\neg x_n).$$

. Therefore,

$$\mu F x = \bar{\neg} x,$$

since $\bar{\neg} 1^- \mu x = x$. Also

$$\lambda F(f \cup \bar{\neg} f)^c \cap \bar{\neg} f = \emptyset,$$

since λF is one-to-one. Therefore, by Proposition 3.4.2, F is reflective through any linear permutation of order 2. Thus we obtained the following theorem.

Theorem 4.4.3 Assume $0 < h - 1 < n$ is relatively prime with odd n. Then there exists a one-to-one reflective transformation $F = \langle f \rangle$ of $\mathbf{Q}^n$ such that

$$\begin{aligned} f &= p_1 \cdot \alpha_2 q_2 \cdot \ldots \cdot \alpha_n q_n, \quad \text{where } \alpha_i = I_{\mathbf{Q}} \text{ or } \neg \text{ for every } i, \\ F &= \bar{\neg} \rho^{h-1} \quad \text{on Car} F. \end{aligned}$$

In this case, f is uniquely determined by (4.4.2). F has one $2n$-cycle.

Note that $F = \bar{\neg}\rho^{h-1}$ on Car$F$ does not gurantee F to be reflective, although any isometry is reflective. For example, let $f = \{1110010, 1001011, 1000110\}$ and $F = \langle f \rangle$. Then $F = \bar{\neg}\rho^2$ on Car$F$, but F is not reflective.

Example 4.4.4 For Example 4.4.2, we have $F = \langle f \rangle$, $f = 1 \cdot 2 \cdot \neg 3 \cdot \neg 4 \cdot 5 \cdot 6 \cdot \neg 7$. The 14-cycle of F is

$$\begin{array}{ccccccccccccc} 1100110 & \rightarrow & 0100110 & \rightarrow & 0110110 & \rightarrow & 0110010 & \rightarrow & 0110011 & \rightarrow & 0010011 & \rightarrow & 0011011 \\ \uparrow & & & & & & & & & & & & \downarrow \\ 1100100 & \leftarrow & 1101100 & \leftarrow & 1001100 & \leftarrow & 1001101 & \leftarrow & 1001001 & \leftarrow & 1011001 & \leftarrow & 0011001 \end{array}$$

Example 4.4.5 Let $F = \langle f \rangle$ be a transformation described in Theorem 4.4.3 and let $F^2 = \langle g \rangle$. Let $f = \{x\}$. Then $F^2 x = \{1, h\}^- x$ by (4.4.3), so that

$$g = \{x, \bar{\neg}\rho^{-(h-1)} x\}.$$

On the other hand, $1^- x = \bar{\neg}\rho^{(h-1)} x$, so that

$$\begin{aligned} \bar{\neg}\rho^{-(h-1)} 1^- x &= x, \\ (\rho^{-(h-1)} 1)^- \bar{\neg}\rho^{-(h-1)} x &= x, \\ \bar{\neg}\rho^{-(h-1)} x &= (\rho^{-(h-1)} 1)^- x. \end{aligned}$$

Therefore,

$$g = \{x, (\rho^{-(h-1)} 1)^- x\}.$$

Therfore, F^2 is a threshold transformation. Similarly, F^3 is athreshold transformation.

Example 4.4.6 n is even ($n = 2m$, $n \geq 4$). We start from a one-to-one transformation $F = \langle f \rangle$ defined by

$$f = p_1 \cdots p_{m+1} \cdot \neg p_{m+2} \cdots \neg p_{2m} \vee p_1 \cdots p_{m-1} \cdot \neg p_m \cdots \neg p_{2m}.$$

Then we add

$$g = p_1 \cdots p_m \cdot \neg p_{m+1} \cdots \neg p_{2m} \vee p_1 \cdots p_{m-1} \cdot \neg p_m \cdots \neg p_{2m}.$$

to get a threshold transformation

$$H = \langle f \rangle \odot \langle g \rangle = \langle h \rangle.$$

Then
$$h = p_1 \cdot .. \cdot p_{m-1} \cdot (p_m \vee \neg p_{m+1}) \cdot \neg p_{m+2} \cdot .. \cdot \neg p_{2m}.$$
GRAPH(H) consists of three n-cycles and loops. E.g.
$$\begin{aligned} 1100 &\to 0110 \to 0011 \to 1001 \to 1100, \\ 1110 &\to 0111 \to 1011 \to 1101 \to 1110, \\ 1000 &\to 0100 \to 0010 \to 0001 \to 1000. \end{aligned}$$
For $n = 4$, H is not minimal and isometrically equivalent to the compressible transformation $\langle p_1 \cdot p_2 \cdot p_3 \cdot p_4 \rangle$. H is uniquely minimal for $n \geq 6$.

In the following, we construct one-to-one circular transformations of small dimensions, and then modify them to general dimensions if possible. In $F = \langle f \rangle$, f is expressed with skipped p in these examples.

Example 4.4.7 Start with 110010 and add 110100 to create a one-to-one transformation. Then, to make it a threshold transformation, add 110110 to obtain
$$f = 1 \cdot 2 \cdot \neg 3 \cdot (4 \vee 5) \cdot \neg 6$$
GRAPH(F) consists of three 6-cycles and loops. The flow graph is
$$[1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot \neg 5 \cdot \neg 6]\partial, \quad [1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot 5 \cdot \neg 6]\partial.$$
f can be generalized to
$$\begin{aligned} f &= 1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot (\neg 5 \vee \neg 6) \cdot 7 \cdot \neg 8, \\ f &= 1 \cdot 2 \neg 3 \cdot 4 \cdot \neg 5 \cdot (6 \vee 7) \cdot \neg 8 \cdot 9 \cdot \neg 10, \\ f &= 1 \cdot 2 \neg 3 \cdot 4 \cdot \neg 5 \cdot 6 \cdot (\neg 7 \vee \neg 8) \cdot 9 \cdot \neg 10 \cdot 11 \cdot \neg 12, \end{aligned}$$
and so on. In general, GRAPH(F) consists of three $2m$-cycles and loops.

f can also be generalized to
$$\begin{aligned} f = \;& 1 \cdot .. \cdot m \cdot \neg(m+1) \cdot .. \cdot \neg(2m-1) \\ & \cdot (2m \vee (3m-1)) \cdot ((2m+1) \cdot .. \cdot 3m-2) \cdot (\neg(3m) \cdot .. \cdot \neg(4m-2). \end{aligned}$$
GRAPH(F) consists of three $(4m-2)$-cycles and loops.

Example 4.4.8 Start with 111010. Add 101000, 101110, and 100010 to create a one-to-one transformation having the cycle $\mathrm{Orb}_\rho 111010$. Then add 101010 and 111000 and get a one-to-one threshold transformation F defined by
$$f = 1 \cdot 3 \cdot \neg 4 \cdot \neg 6 \vee 1 \cdot (3 \vee \neg 4) \cdot \neg 2 \cdot 5 \cdot \neg 6.$$
GRAPH(F) consists of three 6-cycles, one 2-cycle and loops. A flow graph is
$$[1 \cdot 2 \cdot 3 \cdot \neg 4 \cdot 5 \cdot \neg 6]\partial, \quad [1 \cdot 2 \cdot 3 \cdot \neg 4 \neg 5 \neg 6]\partial, \quad [1 \cdot \neg 2 \cdot 3 \cdot \neg 4 \cdot 5 \cdot \neg 6]\partial.$$

Example 4.4.9
$$f = 1 \cdot 2 \cdot \neg 3 \cdot \neg 6 \vee 1 \cdot 2 \cdot 4 \cdot \neg 5 \cdot \neg 6 \vee 1 \cdot \neg 3 \cdot 4 \cdot \neg 5 \cdot \neg 6.$$
GRAPH(F) consists of four 6-cycles, two 3-cycles and loops. A flow graph is
$$[1 \cdot 2 \cdot 3 \cdot 4 \cdot \neg 5 \cdot \neg 6]\partial, \quad [1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot 5 \cdot \neg 6]\partial, \quad [1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot \neg 5 \cdot \neg 6]\partial.$$

Example 4.4.10
$$f = 1 \cdot 2 \cdot \neg 3 \cdot \neg 6 \vee 1 \cdot 2 \cdot \neg 5 \cdot \neg 6 \vee 1 \cdot \neg 3 \cdot 4 \cdot \neg 5 \cdot \neg 6.$$

GRAPH(F) consists of five 6-cycles, two 3-cycles and loops. A flow graph is

$$[1\cdot 2\cdot 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 5\cdot \neg 6]\partial,$$
$$[1\cdot 2\cdot \neg 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot \neg 3\cdot 4\cdot 5\cdot \neg 6]\partial.$$

Example 4.4.11

$$f = 1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 5 \vee 1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 6 \vee 1\cdot 2\cdot \neg 4\cdot \neg 5\cdot \neg 6 \vee 1\cdot 3\cdot \neg 4\cdot \neg 5\cdot \neg 6.$$

GRAPH(F) consists of six 4-cycles, one 6-cycle and loops. A flow graph is

$$[1\cdot 2\cdot 3\cdot 4\cdot \neg 5\cdot \neg 6] \leftrightarrow [1\cdot 2\cdot 3\cdot \neg 4\cdot 5\cdot \neg 6],\quad [1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 5\cdot \neg 6]\partial.$$

Example 4.4.12

$$f = 1\cdot 2\cdot \neg 5\cdot \neg 6 \vee 1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 6.$$

GRAPH(F) consists of three 6-cycles, two 12-cycle and loops. A flow graph is

$$\begin{array}{ccc} [1\cdot 2\cdot 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial, & & [1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 5\cdot \neg 6]\partial, \\ [1\cdot 2\cdot 3\cdot \neg 4\cdot 5\cdot \neg 6] & \leftrightarrow & [1\cdot 2\cdot \neg 3\cdot 4\cdot \neg 5\cdot \neg 6]. \end{array}$$

Example 4.4.13

$$f = 1\cdot 2\cdot \neg 3\cdot \neg 6 \vee 1\cdot 2\cdot 4\cdot \neg 5\cdot \neg 6.$$

GRAPH(F) consists of five 6-cycles and loops. The flow graph is

$$[1\cdot 2\cdot 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot \neg 3\cdot 4\cdot 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot \neg 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial.$$

Example 4.4.14

$$f = 1\cdot 2\cdot \neg 3\cdot \neg 6 \vee 1\cdot 2\cdot \neg 5\cdot \neg 6.$$

GRAPH(F) consists of six 6-cycles and loops. A flow graph is

$$[1\cdot 2\cdot 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot 3\cdot \neg 4\cdot \neg 5\cdot \neg 6]\partial,$$
$$[1\cdot 2\cdot \neg 3\cdot 4\cdot 5\cdot \neg 6]\partial,\quad [1\cdot 2\cdot \neg 3\cdot 4\cdot \neg 5\cdot \neg 6]\partial.$$

Example 4.4.15 $n \geq 4$.

$$f = p_1 \cdot p_2 \cdot S_{n-3}\{p_3, ..., p_n\}.$$

If n is even, GRAPH(F) consists of two n-cycles, one 2-cycle and loops. E.g.

$$\begin{array}{lcl} 1110 & \to & 0010 \to 1011 \to 1000 \to 1110, \\ 1101 & \to & 0100 \to 0111 \to 0001 \to 1101, \\ 1111 & \to & 0000 \to 1111. \end{array}$$

If n is odd, GRAPH(F) consists of one $2n$-cycle, one 2-cycle and loops. E.g.

$$\begin{array}{lcl} 11110 & \to & 00010 \to 11011 \to 01000 \\ & \to & 01111 \to 00001 \to 11101 \to 00100 \\ 10111 & \to & 10000 \to 11110, \\ 11111 & \to & 00000 \to 11111. \end{array}$$

Example 4.4.16 $n = 4m+2$, $m \geq 1$.

$$\begin{array}{rcl} f & = & p_1 \cdots p_{m+1} \cdot \neg p_{m+2} \cdots \neg p_{2m+1} \cdot p_{2m+3} \cdots p_{3m+1} \cdot \neg p_{3m+3} \\ & & \cdots \neg p_{4m+2} \cdot (p_{2m+2} \vee p_{3m+2}). \end{array}$$

GRAPH(F) consists of three n-cycles and loops. E.g.

$$\begin{aligned}
110100 &\rightarrow 010110 \rightarrow 010011 \rightarrow 011001\\
&\rightarrow 001101 \rightarrow 100101 \rightarrow 110100,\\
101001 &\rightarrow 101100 \rightarrow 100110 \rightarrow 110010\\
&\rightarrow 011010 \rightarrow 001011 \rightarrow 101001,\\
110110 &\rightarrow 010010 \rightarrow 011011 \rightarrow 001001\\
&\rightarrow 101101 \rightarrow 100100 \rightarrow 110110.
\end{aligned}$$

Example 4.4.17 f is defined recursively as follows, where $f = f^{(n)}$ for $\mathbf{Q}^n$:

$$\begin{aligned}
f^{(4)} &= p_1 \cdot p_2 \cdot \neg p_3 \cdot \neg p_4,\\
f^{(5)} &= p_1 \cdot p_2 \cdot \neg p_4 \cdot \neg p_5.\\
f^{(n)} &= p_1 \cdot p_2 \cdot \neg p_{n-1} \cdot \neg p_n \vee f^{(n-2)} \cdot \neg p_n.
\end{aligned}$$

The proof that f is a one-to-one threshold transformation is described in the following.

Proposition 4.4.18 f of Example 4.4.17 is a threshold function.

Proof. $f^{(n)}$ can be expressed as $f^{(n)} = p_1 \cdot p_2 \cdot \neg p_n \cdot (\neg p_{n-1} \vee f^{(n-2)})$. If $f^{(n-2)}$ is a threshold function, then $f^{(n)}$ is a threshold function by Proposition 4.1.2. $\square$

Lemma 4.4.19 A point $x = (x_1, ..., x_n)$ belongs to $f^{(n)}$ if and only if (i) $x_1 = x_2 = 1$, $x_n = 0$, and (ii) there exist consecutive 0s in x, and there exist no consecutive 1s after the last consecutive 0s and before x_1.

Proof. This characterization of f follows from its recursive definition above. $\square$

Lemma 4.4.20 $\mu F f \subseteq \bar{\neg} f$.

Proof. Let $x \in f$. (i) If $x = 111x_3 \cdots x_{n-1}0$, then $Fx = 011(Fx)_4 \cdots (Fx)_{n-1}0$, so that $\mu Fx \in \bar{\neg} f$ by Lemma 4.4.19. Similarly, (ii) If $x = 1100x_5 \cdots x_{n-1}0$, then $Fx = 0110(Fx)_5 \cdots (Fx)_{n-1}0$, so that $\mu Fx \in \bar{\neg} f$. (iii) If $x = 110101 \cdots 0100x_{2m+1} \cdots x_{n-1}0$ then $Fx = 0101 \cdots 0110(Fx)_{2m+1} \cdots (Fx)_{n-1}0$, so that $\mu Fx \in \bar{\neg} f$. (iv) If $x = 110101 \cdots 011x_{2m} \cdots x_{n-1}0$, then $Fx = 010101 \cdots 011(Fx)_{2m} \cdots (Fx)_{n-1}0$, so that $\mu Fx \in \bar{\neg} f$. $\square$

Lemma 4.4.21 $\mu F(f \vee \bar{\neg} f)^c \cap \bar{\neg} f = \emptyset$.

Proof. Let $x \notin f \cup \bar{\neg} f$. If $x_1 = 1$, then $(\mu Fx)_1 = 1$, so that $\mu Fx \notin \bar{\neg} f$. Let x$_1 = 0$. Then $(Fx)_1 = 0$, since $x \notin \bar{\neg} f$. (i) Let $(x_1, x_2) = 00$. Then $((Fx)_1, (Fx)_2) = 00$, so that $(\mu Fx)_n = 0$, and hence $\mu Fx \notin \bar{\neg} f$. (ii) Let $(x_1, x_2) = 01$. (iia) If $x \in \rho f$ then $(Fx)_2 = 0$, so that $(\mu Fx)_n = 0$, and hence $\mu Fx \notin \bar{\neg} f$. Let $x \notin \rho f$. (iib) Let $x_3 = 0$. If $x_n = 1$, then $(Fx)_n = 1$, so that $(\mu Fx)_2 = 1$, and hence $\mu Fx \notin \bar{\neg} f$. Let $x_n = 0$. Let the first same consecutive elements be $(x_i, x_{i+1}) = 00$, i.e. $x = 010 \cdots 10100x_{i+2} \cdots 0$. Then $Fx = 010 \cdots 10100(Fx)_{i+2} \cdots$, so that $\mu Fx = 0(Fx)_n \cdots 00101 \cdots 01 \notin \bar{\neg} f$. Let the first same consecutive elements be $(x_i, x_{i+1}) = 11$. Then $x = 010 \cdots 1011x_{i+2} \cdots 0$. Then $Fx = 010 \cdots 1001(Fx)_{i+2} \cdots$, so that $\mu Fx \notin \bar{\neg} f$. (iic) Let $x_3 = 1$. Since $x = 011x_4 \cdots \notin \rho f$, $x_n = 1$. Therefore, $Fx = 01(Fx)_3 \cdots 1$, so that $\mu Fx = 01(\mu Fx)_3 \cdots \notin \bar{\neg} f$. $\square$

Proposition 4.4.22 F is reflective through μ.

Proof. Apply the results of Lemmas 4.4.20 and 4.4.21 to Proposition 3.4.2 of Chapter 3. □

From Propositions 4.4.18 and 4.4.22 it follows that

Proposition 4.4.23 The transformation of Example 4.4.17 is a one-to-one threshold transformation.

Proposition 4.4.24 Any transformation F in this section is reflective through any linear permutation of slope -1. In particular, F^{-1} is isometrically similar to F and a threshold transformation.

Proof. Let $F = \langle f \rangle$. We have already proved that F of Example 4.4.17 is reflective through any linear permutation of slope -1. It is easily confirmed that $\mu F x \in \bar{\neg} f$ for every $x \in f$ and $\mu F x \notin \bar{\neg} f$ for every $x \notin f \vee \bar{\neg} f$ for the other examples. Therefore, by Proposition 3.4.2 of Chapter 3, F is reflective through any linear permutation of slope -1. □

Corollary 4.4.25 Any transformation in this section is isometrically equivalent to threshold transformations whose graphs consist of 2-cycles and loops.

4.5 Skew-circular one-to-one transformations

In this section, we construct minimal, skew-circular, one-to-one threshold transformations. All transformations $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$ in the examples of this section are threshold transformations, since f are threshold functions. They are also mutually isometrically non-equivalent and uniquely minimal.

Notation Let λ and $\mu \in \mathrm{SYM}(\mathbf{N})$ denote the linear permutations of coefficients $(-1, 1)$ and $(-1, 2)$ respectively, that is,

$$\begin{aligned}
\lambda &= (1, n) \cdot (2, n-1) \cdots (i, n-i+1) \cdots ([n/2], n-[n/2]+1), \\
\mu &= (2, n) \cdot (3, n-1) \cdots (i, n-i+2) \cdots ([(n-1)/2]+1, n+1-[(n-1)/2]).
\end{aligned}$$

Example 4.5.1 $F = \langle\langle f \rangle\rangle$, where $f = p_1 \cdots p_i \cdots p_n$, is a one-to-one threshold transformation, which is reflective through λ. Further, $\mathrm{CS}(F) = \{(1, 2n), (2^n - 2n, 1)\}$.

As a generalization of Example 4.5.1, we now determine a condition for the transformation $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$ such that $f = p_1 \cdot q_2 \cdot \ldots \cdot q_n$, where $q_i = p_i$ or $\neg p_i$ for every i, to be one-to-one.

Let $f = \{x\}$, a one-element set, where $x = (1, x_2, .., x_n)$. We assume $Fx = (0, x_2, .., x_n)$. In order to determine a condition for $F(\mathrm{Car}F) = \mathrm{Car}F$, suppose $Fx = (\rho n^-)^{h-1} x$ for $0 < h-1 < 2n$. Then

$$p_1 x = 1, \quad 1^- x = (\rho n^-)^{h-1} x. \tag{4.5.1}$$

If $0 < h-1 < n$, then

$$(0, x_2, .., x_n) = (\neg x_{n+2-h}, .., \neg x_n, 1, x_2, .., x_{n+2-(h+1)}),$$

that is,

$$\neg x_1 = 0 = \neg x_{n+2-h}, x_2 = \neg x_{n+2-(h-1)}, ..., x_{h-1} = \neg x_n,$$
$$x_h = 1, x_{h+1} = x_2, ..., x_n = x_{n+2-(h+1)},$$

that is,

$$\begin{aligned} x_h = 1, x_{h+(h-1)} = \alpha_h x_h,\ x_{h+2(h-1)} = \alpha_{h+(h-1)} x_{h+(h-1)}, .., \\ x_{n+2-h} = \alpha_{n+2-(h+(h-1))} x_{n+2-(h+(h-1))},\ \neg x_1 = 0 = \neg x_{n+2-h}, \end{aligned} \tag{4.5.2}$$

where

$$\alpha_i = \begin{cases} \neg & \text{for } n+2-(h-1) \le i \le n, \\ I_{\mathbf{Q}} \quad \text{(identity)} & \text{for } 2 \le i \le n+2-(h+1). \end{cases}$$

In particular, the number of i such that $\alpha_i = \neg$ is $h-2$.

If $n < h-1 < 2n$, then for $h' = h-n$,

$$(0, x_2, ..., x_n) = (x_{n+2-h'}, .., x_n, 0, \neg x_2, .., \neg x_{n+2-(h'+1)}),$$

that is,

$$\begin{aligned} x_{h'} = 0,\ x_{h'+(h'-1)} = \alpha'_{h'} x_{h'},\ x_{h'+2(h'-1)} = \alpha'_{h'+(h'-1)} x_{h'+(h'-1)}, .., \\ x_{n+2-h'} = \alpha'_{n+2-(h'+(h'-1))} x_{n+2-(h'+(h'-1))},\ \neg x_1 = 0 = x_{n'+2-h'}, \end{aligned} \tag{4.5.2$'$}$$

where

$$\alpha'_i = \begin{cases} I_{\mathbf{Q}} & \text{for } n+2-(h'-1) \le i \le n, \\ \neg & \text{for } 2 \le i \le n+2-(h'+1), \end{cases}$$

particularly the number of i such that $\alpha_i = \neg$ is $n-h'$.

If $0 < h-1 < n$ is relatively prime with n, the system of equations (4.5.2) sequentially and uniquely determines x_i from $x_h = 1$ to $\neg x_1 = 0$ by step $h-1$ of their subscripts. Further, the values of x_i change $h-1$ times as the values of x_i are determined from x_h to x_1. Therefore, if $h-1$ is relatively prime with $2n$, then x_i is consistently determined for every i.

Similarly, if $0 < h'-1 < n$, $h'-1$ is relatively prime with n, and $n-h'$ is even (i.e. $n < h-1 < 2n$ and $h-1$ is relatively prime with $2n$), then (4.5.2)' is uniquely solved.

Consequently (4.5.1) is uniquely solved if $0 < h < 2n$ and h is relatively prime with $2n$.

Example 4.5.2 Let $n = 7$ and $h-1 = 9$. The equation $1^- x = (\rho 7^-)^9 = \bar{\neg}(\rho 7^-)^2 x$ for $x = (1, x_2, .., x_7)$ is

$$\neg x_1 = 0 = x_6, x_2 = x_7, x_3 = 0, x_4 = \neg x_2,$$
$$x_5 = \neg x_3, x_6 = \neg x_4, x_7 = \neg x_5,$$

that is,

$$x_3 = 0, x_5 = \neg x_3, x_7 = \neg x_5, x_2 = x_7,$$
$$x_4 = \neg x_2, x_6 = \neg x_4, \neg x_1 = 0 = x_6.$$

The solution is $x = 1001100$.

Now assume $0 < h-1 < 2n$ and $h-1$ is relatively prime with $2n$, and let x be the solution of (4.5.1). Then

$$\begin{aligned} ((\rho n^-)^{h-1})^2 x &= (\rho n^-)^{h-1}(1^- x) \\ &= (\neg x_{n-h+2}, .., \neg x_n, 0, x_2, .., x_{n-h+1}) \\ &= h^-((\rho n^-)^{h-1} x) \\ &= h^- 1^- x. \end{aligned}$$

In general,

$$(\rho n^-)^{i(h-1)} x = (1+(i-1)(h-1))^-(1+(i-2)(h-1))^- ... (1+(h-1))^- 1^- x \quad \text{for every positive integer } i. \tag{4.5.3}$$

Therefore, $(\rho n^-)^{i(h-1)}x$ for $i = 1, .., 2n-1$ are all different from x and $(\rho n^-)^{n(h-1)}x = \bar{\neg} x$. Let $f = \{x\}$ and $F = \langle\langle f \rangle\rangle$. Then

$$Fx = 1^- x = (\rho n^-)^{h-1} x,$$

and F is one-to-one with one $2n$-cycle.

Now we transform the system of equations (4.5.1) by replacing x with μx. Then we have

$$p_1(\mu x) = 1, \quad 1^- \mu x) = \bar{\neg}(\rho n^-)^{h-1}(\mu x). \tag{4.5.5}$$

If $0 < h-1 < n$, then by (4.5.2), (4.5.5) is equivalent to

$$x_{n+2-h} = 1, x_{n+2-(h+h-1)} = \epsilon_h x_{n+2-h}, x_{n+2-(h+2(h-1))} = \epsilon_{h+(h-1)} x_{n+2-(h+(h-1))}, ..., \\ x_h = \epsilon_{n+2-(h+(h-1))} x_{h+(h-1)}, x_1 = 0 = \neg x_h. \tag{4.5.6}$$

If $2 \le i \le n+2-(h+1)$, then

$$n+2-(h-1)-(n+2)+(h+1) \le n+2-(h-1)-i \le n+2-(h-1)-2,$$

so that $2 \le n+2-(h+1)$. Therefore, if $i+j = n+2-(h-1)$, then $\epsilon_i = \epsilon_j$ by (4.5.3). Therefore, (4.5.6) is equivalent to (4.5.2). That is, the system of equations (4.5.1) is invariant under μ, so that the solution x of (4.5.1) is also invariant under μ, i.e. $\mu x = x$. The same is true for $n < h < 2n$.

Now let $F = \langle\langle f \rangle\rangle$ for $f = \{x\}$. Then

$$\lambda F x = (x_n, x_{n-1}, .., x_2, 0), \quad \text{while } \bar{\neg}(\rho n^-)^{n-1} x = (x_2, x_3, ... x_n, 0).$$

Therefore,

$$\lambda F x = \bar{\neg}(\rho n^-)^{n-1} x,$$

since $\mu x = x$. Also

$$\lambda F(f \cup \bar{\neg} f)^c \cap \bar{\neg}(\rho n^-)^{n-1} f = \emptyset,$$

since λF is one-to-one. Therefore, by Proposition 3.4.5, F is reflective through $(\rho n^-)^i \lambda$ for every i. Thus we obtained the following theorem.

Theorem 4.5.3 Assume $0 < h-1 < 2n$ is relatively prime with $2n$, Then there exists a one-to-one reflective transformation $F = \langle\langle f \rangle\rangle$ of $\mathbf{Q}^n$ such that

$$\begin{aligned} f &= p_1 \cdot \alpha_2 q_2 \cdot ... \cdot \alpha_n q_n, \quad \text{where } \alpha_i = I_{\mathbf{Q}} \text{ or } \neg \text{ for every } i, \\ F &= (\rho n^-)^{h-1} \quad \text{on Car}F. \end{aligned}$$

In this case, f is uniquely determined by (4.5.2) or (4.5.2)'. F has one $2n$-cycle.

Example 4.5.4 For Example 4.5.2, we have $F = \langle\langle f \rangle\rangle$, $f = 1 \cdot \neg 2 \cdot \neg 3 \cdot 4 \cdot 5 \cdot \neg 6 \cdot \neg 7$. The 14-cycle of F is

$$\begin{gathered}1001100 \to 0001100 \to 0011100 \to 0011000 \to 0011001 \to 0111001\\ \to 0110001 \to 0110011 \to 1110011 \to 1100011 \to 1100111\\ \to 1100110 \to 1000110 \to 1001110 \to 1001100.\end{gathered}$$

Example 4.5.5 Consider F of Example 4.5.1. Then $F^2 = \langle\langle p_1 \cdots \cdot p_{n-1} \rangle\rangle$ is a one-to-one threshold transformation. F^2 is reflective through λ, since F is reflective through λ. $\mathrm{CS}(F^2) = \{(2, n), (2^n - 2n, 1)\}$. Further, $F^3 = \langle\langle p_1 \cdots p_{n-2} \cdot (p_{n-1} \vee \neg p_{n-1}) \rangle\rangle$ is also an incompressible and inexpansible threshold transformation that is reflective through λ.

Example 4.5.6 Let $F = \langle\langle f \rangle\rangle$ be the self-dual transformation of $\mathbf{Q}^n$ defined by

$$f = p_1 \cdot p_2 \cdot S_{n-4}\{p_3, p_4, ..., p_{n-1}\} \cdot p_n.$$

The proof of the reflectiveness and hence one-to-one of F is given in the following.

Lemma 4.5.7 Let F of Example 4.5.6 be $[f_1, ..., f_n]$. If $i \neq j$, then $f_i \cap (f_j \vee \bar{\neg} f_j) = \emptyset$.

Proof. $f_i = (\rho n^-)^{i-1} f$ by definition. First, $f_1 \cap f_i = \emptyset$ for every $i \neq 1$ by the following reasons: $x \in f_2$ implies $x_1 = 0$, while $x \in f_1$ implies $x_1 = 1$; $x \in f_3$ implies $x_2 = 0$, while $x \in f_1$ implies $x_2 = 1$. If $i \geq 4$, then $x \in f_i$ implies that the density of x is less than $n - 2$, while the density of any $x \in f_1$ is $n - 1$ or n. Next, $f_i = (\rho n^-)^{i-1} f_1$ and $f_j = (\rho n^-)^{i-1} f_{j-i+1}$, and $(\rho n^-)^{i-1}$ is one-to-one, so that $f_i \cap f_j = \emptyset$ for every $i \neq j$. Similarly, we can show that $f_i \cap \bar{\neg} f_j = \emptyset$ for every $i \neq j$. $\square$

Lemma 4.5.8 $\lambda(\mathrm{Car}F) \subseteq \mathrm{Car}F$ in Example 4.5.6.

Proof. Let f_i be decomposed as $f_i = g_i \vee h_i$, where

$$\begin{aligned} g_1 &= p_1 \cdot p_2 \cdot p_3 \cdots p_{n-2} \cdot \neg p_{n-1} \cdot p_n, \\ h_1 &= p_1 \cdot p_2 \cdot S_{n-5}\{p_3, p_4, ..., p_{n-2}\} \cdot p_{n-1} \cdot p_n. \\ g_i &= (\rho n^-)^{i-1} g_1 \text{ for every } i, \\ h_i &= (\rho n^-)^{i-1} h_1 \text{ for every } i. \end{aligned}$$

Then it can be shown that $\lambda g_i = \bar{\neg} g_{n-i+4}$ or g_{n-i+4}, and $\lambda h_i = \bar{\neg} h_{n-i+2}$ or h_{n-i+2} for every i. For example, suppose $4 \leq i \leq n-1$. Then

$$\begin{aligned} g_i &= p_i \cdot p_{i+1} \cdots p_n \cdot \neg p_1 \cdots \neg p_{i-3} \cdot p_{i-2} \cdot \neg p_{i-1}. \\ \lambda g_i &= p_{n-i+1} \cdot p_{n-i} \cdots p_1 \cdot \neg p_n \cdot \neg p_{n-1} \cdots \neg p_{n-i+4} \cdot p_{n-i+3} \cdot \neg p_{n-i+2} \\ &= \neg p_{n-i+4} \cdot \neg p_{n-i+5} \cdots \neg p_n \cdot p_1 \cdots p_{n-i+1} \cdot \neg p_{n-i+2} \cdot p_{n-i+3} \\ &= \bar{\neg} g_{n-i+4}. \\ h_i &= p_i \cdot p_{i+1} \cdot S_{n-5}\{p_{i+2}, .., p_n, \neg p_1, .., \neg p_{i-3}\} \cdot \neg p_{i-2} \cdot \neg p_{i-1}. \\ \lambda h_i &= p_{n-i+1} \cdot p_{n-i} \cdot S_{n-5}\{p_{n-i-1}, .., p_1, \neg p_n, .., \neg p_{n-i+4}\} \cdot \neg p_{n-i+3} \cdot \neg p_{n-i+2} \\ &= \neg p_{n-i+2} \cdot \neg p_{n-i+3} \cdot S_{n-5}\{\neg p_{n-i+4}, .., \neg p_n, p_1, .., p_{n-i-1}\} \cdot p_{n-i} \cdot p_{n-i+1} \\ &= \bar{\neg} h_{n-i+2}. \end{aligned}$$

Therefore, $\lambda(\mathrm{Car}F) \subseteq \mathrm{Car}F$. $\square$

Proposition 4.5.9 F in Example 4.5.6 is a one-to-one threshold transformation, which is reflective through λ.

Proof. F is clearly a threshold transformation. To prove that F is reflective through λ, we will show that (i) $\lambda F f_i \subseteq \bar{\neg} f_{\lambda i}$ for every i and (ii) $\lambda F(f_i \cup \bar{\neg} f_i)^c \cap \bar{\neg} f_{\lambda i} = \emptyset$ for every i. (i) Let $x \in f_i$ for some i. Then if $2 \leq i \leq n-1$, then

$$x_i \cdot x_{i+1} \cdot S_{n-4}\{x_{i+2}, .., x_n, \neg x_1, .., \neg x_{i-2}\} \cdot \neg x_{i-1} = 1.$$

$$\begin{aligned} \bar{\neg} f_{\lambda i} &= \bar{\neg} f_{n-i+1} \\ &= \neg p_{n-i+1} \cdot \neg p_{n-i+2} \cdot S_{n-4}\{\neg p_{n-i+3}, .., \neg p_n, p_1, .., p_{n-i-1}\} \cdot p_{n-i}. \end{aligned}$$

By Lemma 4.5.7,

$$\lambda F x = (x_n, .., x_{i+1}, \neg x_i, x_{i-1}, .., x_1).$$

Therefore,

$$f_{\lambda i} \bar{\neg}(\lambda F x) = x_i \cdot \neg x_{i-1} \cdot S_{n-4}\{\neg x_{i-2}, .., \neg x_1, x_n, .., x_{i+2}\} \cdot x_{i+1} = 1,$$

that is, $\lambda F x \in \bar{\neg} f_{\lambda i}$. Similarly, if $x \in f_i$ for $i = 1$ or n, then $\lambda F x \in \bar{\neg} f_{\lambda i}$. (ii) Let $x \notin f_i \cup \bar{\neg} f_i$. (iia) If $x \in f_j \cup \bar{\neg} f_j$ for some $j \neq i$ then $\lambda F x \in f_{\lambda j} \cup \bar{\neg} f_{\lambda j}$ by (ii), so that $\lambda F x \notin \bar{\neg} f_{\lambda i}$ by Lemma 4.5.7. (iib) Let $x \notin f_j \cup \bar{\neg} f_j$ for every j. Then x is a fixed point of F, so that $\lambda F x = \lambda x$. Suppose $\lambda F x \in \bar{\neg} f_k$ for some k. Then $\lambda x \in \bar{\neg} f_k$, so that $x = \lambda(\lambda x) \in \bar{\neg} f_l$ for some l by Lemma 4.5.8, which is a contradiction. Consequently, by Proposition 3.3.2 of Chapter 3, F is reflective through λ. $\square$

Example 4.5.10 Let $n = 5$ and $F = \langle\langle 1 \cdot 3 \cdot 4 \cdot (\neg 2 \vee \neg 5)\rangle\rangle$. GRAPH($F$) consists of three 10-cycles and two loops. The flow graph is

$$[1 \cdot 2 \cdot 3 \cdot 4 \cdot 5] \leftrightarrow [1 \cdot \neg 2 \cdot 3 \cdot 4 \cdot 5], \quad [1 \cdot 2 \cdot \neg 3 \cdot 4 \cdot 5]\partial.$$

F is reflective through λ.

Example 4.5.11 The following transformations $\langle\langle f^{(i)}\rangle\rangle$ of $\mathbf{Q}^i$ are expected to be one-to-one and reflective through λ.

$$\begin{aligned} f^{(4)} &= 1 \cdot 2 \cdot (3 \vee 4), \quad f^{(6)} = 1 \cdot 2 \cdot (3 \vee 4 \cdot (5 \vee 6)), ... \\ f^{(5)} &= 1 \cdot 2 \cdot (3 \vee 4 \cdot 5), \quad f^{(7)} = 1 \cdot 2 \cdot (3 \vee 4 \cdot (5 \vee 6 \cdot 7)), ... \end{aligned}$$

Open Question All one-to-one minimal threshold transformations we have so far are reflective through some isometries of order 2 as well as all isometries. Whether it is true for all minimal one-to-one threshold transformations is an open question.