CHAPTER 6 DYNAMICAL SYSTEMS OF FIRST-ORDER NEURAL NETWORKS

Abstract. After basic concepts about finite-state dynamical systems such as structural stability and attractors are defined, and prior results are critically reviewed, primitive dynamical neural networks (PDNNs) are remodelled by incorporating spontaneous firing and distinguishing non-neutral invariant sets from spontaneous neutral cycles. These PDNNs are a class of McCulloch and Pitts networks $x(t) = \mathrm{Sgn}(Ex(t-1) - h)$ on $\{-1, 1\}^n$ such that h is the zero vector and all the diagonal elements of the efficacy matrices E are negative. PDNN-definable threshold transformations are characterized in terms of Boolean functions, and the existence of non-neutral minimal attractors are proved for the general dimension n by construction using [ ]-representations of Boolean transformations.

6.1 Finite-state dynamical system (FSDS)

Since a dynamical *neural network* (DNN), which is the subject of this and subsequent chapters, is a *finite-state dynamical system* (FSDS), we first describe its basic concepts in the following by modifying the definitions of well-known concepts about semidynamical systems on a general topological space (Mathematical Society of Japan, 1987, pp. 487-503).

Let X be a finite metric space with an integer-valued distance d. If S is a nonempty subset of X then the ϵ-neighborhood of S, $U_\epsilon S$ for a positive integer ϵ is defined by

$$U_\epsilon S = \{x \mid d(x, S) \leq \epsilon\}.$$

Let φ be a mapping from $X \times \mathbf{Z}_+$ to X. For each $t \in \mathbf{Z}_+$ a transformation $\varphi_t : X \to X$ is defined by $\varphi_t x = \varphi(x, t)$ for every $x \in X$. If φ_t satisfies

(1) $\varphi_s \circ \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbf{Z}_+$,

(2) $\varphi_0 = I_X$ (the identity transformation of X),

then φ is called a *finite-state dynamical system* (FSDS) on the *state space* X, whose points are called *states*. If F is a transformation of X, then F defines a mapping $\varphi : X \times \mathbf{Z}_+ \to X$ by

$$\varphi(x, t) = F^t x, x \in X, t \in \mathbf{Z}_+.$$

Then φ is an FSDS on X such that $\varphi_t = F^t$ and called the FSDS *generated by* F. Let F_v be a transformation of X defined for each point v of an open set U of $\mathbf{R}^m$. If $F = F_v$ for some $v \in U$, then the FSDS generated by F is called a parametrized FSDS with its parameter space U. If $F = F_v$ and $F = F_w$ for every point w of an neighborhood of v for a parametrized FSDS generated by F, then the parametrized FSDS is called *structurally stable*.

If Ψ is a set of sequences of X, then ImΨ, the *image* of Ψ, is $\bigcup_{V \in \Psi} \mathbf{V}$ i.e.the union of the images of the sequences belonging to . A sequence $V = (v_0, v_1, ...)$ is called *cyclic*, if there exists some k such that $a_i = a_j$ for every i and j such that $i = j \bmod k$ and $a_i \neq a_j$ for every i and j such that $i \neq j \bmod k$.

The sequence $(x, Fx, F^2x, ...)$ is called the *orbit* starting at x and denoted by $\mathrm{Orb}_F x$. A cyclic orbit is identified with an element of $\mathrm{CY}(F)$. That is one cyclic orbit obtained from another by shifting the starting point is regarded as the same. For a subset S of X, $\mathrm{Orb}_F S$ is the set of all orbits $\mathrm{Orb}_F x$ such that $x \in S$. The set $\omega_F x$ defined by

$$\omega_F x = \{y \quad | \quad \text{For any } k \in \mathbf{Z}_+, \text{ there exists some } t > k \text{ such that } y = F^t x\}$$

is called the *limit set* of $x; \omega_F S$ is the union of all limit sets $\omega_F x$ such that $x \in S$. Clearly, for any point x, $\omega_F x = \mathbf{C}$ for a cycle $C \in \mathrm{CY}(F)$. This C is called a *limit cycle* of x. A subset S of X is called *invariant* if $FS = S$. Clearly S is an invariant set if and only if $S = \mathrm{Im}\Psi$ for some $\Psi \subseteq \mathrm{CY}(F)$.

Definition 6.1.1 A subset Φ of $\mathrm{CY}(F)$ is called *attractive* or an *attractor* in the FSDS generated by F, if there exists some ϵ-neighborhood $U_\epsilon(\mathrm{Im}\Phi)$ satisfying

(1) $F(U_\epsilon(\mathrm{Im}\Phi)) \subseteq U_\epsilon(\mathrm{Im}\Phi)$;
(2) $\omega_F(U_\epsilon(\mathrm{Im}\Phi)) = \mathrm{Im}\Phi$.

In particular, if Φ consists of one cycle, the cycle is called attractive. $\mathrm{CY}(F)$ is clearly an attractor. Also, if Ψ and Φ are both attractors, then $\Psi \cup \Phi$ is also an attractor. Therefore, an attractor is called a *minimal attractor*, if no proper subset of it is an attractor. Further, an attractor Φ is called *connected*, if $\Phi = \Psi \cup \Upsilon$, $\Psi \cap \Upsilon = \emptyset$, then $\min_{x \in \mathrm{Im}\Phi, y \in \boldsymbol{\Upsilon}} d(x, y) = 1$; otherwise, called *disconnected*. The *basin* for an attractor Φ is the set of all points x such that $F^k x \in \mathrm{Im}\Phi$ for some k. Two attractors Ψ and Φ are called *separated* if $d(x, y) \geq 2$ for any points x of the basin for Ψ and any point y of the basin for Φ.

Definition 6.1.2 A subset Φ of $\mathrm{CY}(F)$ is called *strong attractor* in the FSDS generated by F, if there exists some ϵ-neighborhood $U_\epsilon(\mathrm{Im}\Phi)$ satisfying

$$F(U_\epsilon(\mathrm{Im}\Phi)) = \mathrm{Im}\Phi.$$

A strong attractor is clearly an attractor.

6.2 A Critical review of prior results

In a nervous system, each neuron exhibits an impulse of one electric state, called *action potential*. Therefore, the state of each neuron can be distinguished by the existence and nonexistence of an action potential. Assuming the nervous system consists of n neurons, we can identify each neuron with an element of $\mathbf{N} = \{1, 2, ..., n\}$. Then the state of the whole nervous system at time t is expressed by a point $x(t) = (x_1(t), x_2(t), ..., x_n(t)) \in \{-1, 1\}^n$.

Let (i, j) denote a synapse, where i and j are integers of $\mathbf{N}$, neuron i being the *postsynaptic* neuron and neuron j being the *presynaptic* neuron. Let E be a real $n \times n$ matrix and h be a real column n-vector, where each element E_{ij} expresses the *efficacy* of the synapse (i, j) and h_i expresses the threshold value for the action potential of neuron i. Then the classical neural network model of McCulloch and Pitts (1943) asserts that the state $x(t+1)$ is defined by the state $x(t)$ and E as follows:

$$\begin{aligned} Fx &= \mathrm{Sgn}(Ex - h), \\ x(t+1) &= F(x(t)), \end{aligned} \tag{6.2.1}$$

where

$$(\mathrm{Sgn}(y))_i = \begin{cases} 1 & \text{if } y_i > 0, \\ -1 & \text{if } y_i \leq 0, \end{cases}$$

Further, $\{-1,1\}^n$ is a finite metric space with the integer-valued *Hamming distance* $d_H(x,y) = |\{i \mid x_i \neq y_i\}|$, where $|S|$ denotes the number of elements of the set S. Therefore, $x(0), x(1), \ldots$ is the orbit starting at $x(0)$, in the FSDS on the state space $\{-1,1\}^n$ generated by the threshold transformation F of $\{-1,1\}^n$.

This discrete-time binary system is now called an artificial neural network and is not regarded as a representation of the activities of biological neurons, the history of studies is partly responsible for this claim. However, a continuous-time and continuous state-model is not manageable for the description of dynamics of a large population of neurons, so that some abstraction to the level of (6.2.1), such as putting the firing mechanism in a black box and aligning action potentials, is more or less inevitable.

A first breakthrough was made in the early 60s by Arimoto (1963).

Theorem 6.2.1 (Arimoto 1963) For any $k \leq 2^n$, there exists a network (6.2.1) having a k-cycle.

Later, the following Goles-Chacc's theorem appeared (Goles-Chacc, 1980; see also Goles & Olivos, 1981; Goles-Chacc et al., 1985, Proposition 2, p. 269).

Theorem 6.2.2 (Goles-Chacc, 1980) If the matrix E in (6.2.1) is symmetric, then any limit cycle is either a loop or a 2-cycle.

Proof. Let the function $\beta : \{-1,1\}^n \times \{-1,1\}^n \to \mathbf{R}$ be defined by

$$\beta(x,y) = -x^T E y + (x^T + y^T)h,$$

and let $\gamma : \mathbf{Z}^+ \to \mathbf{R}$ be defined by

$$\gamma t = \beta(x(t), x(t-1)).$$

Then

$$\gamma(t+1) - \gamma t = -(x^T(t+1) - x^T(t-1))(Ex(t) - h),$$

since E is symmetric. Since $\{-1,1\}^n$ is a finite set, we can assume that $(Ex-h)_i \neq 0$ for every x and every i in (6.2.1) by adjusting h. Therefore, $x_i^T(t+1)(Ex(t)-h)_i > 0$ for every i. Therefore, if $x_i(t+1) \neq x_i(t-1)$ for some i, then either

$$x_i(t+1) > 0, -x_i(t-1) > 0, \text{ and } ((Ex(t)-h))_i > 0,$$

or

$$x_i(t+1) < 0, -x_i(t-1) < 0, \text{ and } ((Ex(t)-h))_i < 0.$$

Therefore, either

$$x(t+1) = x(t-1) \text{ and } \gamma(t+1) - \gamma t = 0$$

or

$$x(t+1) \neq x(t-1) \text{ and } \gamma(t+1) - \gamma t < 0,$$

which proves that $x(t+1) = x(t-1)$ for any point $x(t-1)$ and $x(t+1)$ on any limit cycle. □

However, the condition of the symmetry has no justification in biological nervous systems. Neither Arimoto's theorem nor Goles-Chacc's theorem was universally recognized, and Hopfield (1982) replaced (6.2.1) with n recursive equations by which $(x(t))_i$ is obtained one by one for $i = 1, ..., n$; for example,

$$x_i(t+1) = F_i(x_1(t+1), .., x_{i-1}(t+1), x_i(t), .., x_n(t)).$$

In this serial model, if E is symmetric, $h = o$, where o is the column vector whose every coordinate is 0, and $E_{ii} = 0$ for every i, then any limit cycle is a loop. This result, which is stronger than Goles-Chacc's, was obtained by sacrificing the parallel operation defined by (6.2.1) and therefore a clear departure from the biological root. Note that if a state remains on a loop, then each neuron either continues to fire at every unit time or continues not to fire at all. That is too extreme a case in biological networks. Also, from a technological point, loops represent a very limited amount of information. Instea, non-loop attractors should be the main targets.

Prior results concerning attractors are limited to attractive fixed points (Amari, 1972; Robert, 1986; Cottrell, 1988; Blum, 1990; etc.). First of all, there has been some confusion and limitation about the concept of attractors, so that no standard definition of attractors has been established. For example, in spite of its subtitle, "the world of attractor neural networks," Amit (1989) does not give a definition of "attractor, and takes limit cycles for attractors (P.77). Kamp & Hasler(1990, pp. 6-7) and others define a basin of attraction or radius of attraction for a fixed point without defining an attractor. Earlier, Amari (1972) called a fixed point q stable if there exists $U_\epsilon q, \epsilon \geq 1$, such that $F(U_\epsilon q) = q$. More recently, Cottrell (1988) called a fixed point q a k-attractor if $Fx \in U_{k-1}q$ for any point x such that $d_H(q, x) = k$. These definitions are too limited to be established as standards (see Example 6.3.6 in the next section). In fact, these fixed points are here called strong attractors (Definition 6.1.2). On the other hand, Robert (1986) obtained a necessary and sufficient condition for $U_1 q$ to satisfies (i) and (ii) of our Definition 6.1.1 for a fixed point q (described in Kamp & Hasler, 1990).

According to Definition 6.1.1, the set of all cycles $\mathrm{CY}(F)$ is an attractor (a loop is a 1-cycle). Therefore, if the transformation F of (6.2.1) is one-to-one, e.g., if E is the identity matrix I and $h = o$, then we obtain a minimal attractor Φ such that $\mathrm{Im}\Phi = \{-1, 1\}^n$. If $E = I$ and $h_i = -2$ for every i, F has the unique loop (11..1), so that there exists an F having an attractive loop for every n. Further, if $E_{ij} = -1$ for every i, j and $h = o$ for $n \geq 3$, then $11..1 \leftrightarrow -1-1..-1$ is an attractive 2-cycle. Example 6.3.4 (b) in the next section shows such an attractor for $n = 2$. Therefore, there exists an F having an attractive 2-cycle for every $n \geq 2$. However, beyond these results and those obtained by direct products, it is not clear whether there exist other attractors. Nevertheless, the enhanced Arimoto theorem (Theorem 5.5.2 of Chapter 5) implies the following Arimoto-Ueda theorem. The proof is clear by letting $\epsilon = n$ in Definition 6.1.1.

Theorem 6.2.3 For any $k \leq 2^n$ there exists a McCulloch and Pitts network (6.2.1) that has an attractive unique k-cycle.

The network of the enhanced Arimoto theorem has a powerful convergence property, since any state converges to a unique cycle regardless of the initial state. However, in the real world, it seems often rather desirable that the convergence depends on the initial state.

In relation to this dependence on the initial state, one feature that has not been incorporated in prior studies is the elementary but essential fact that *in many neurons the postsynaptic potentials merely modify spontaneous firing that occurs without any synaptic input* (Kalat, 1995, p. 63). As a result, prior studies have failed to define what the spontaneous or neutral activities of a single neuron or a population of neurons are. As a result, we have not been able to distinguish neurons' significant activities from their insignificant activities. Particularly, we have not been able to sort out a great number of loops or 2-cycles often appearing in the network (6.2.1).

6.3 Dynamical neural networks (DNNs)

In the current situations described in the last section, we remodel and characterize dynamical systems of neural networks that are amiable to global analysis of population dynamics and still retain some of the essential features of biological networks. The remodelled networks are not new ones but a restricted class of the McCulloch and Pitts model (6.2.1). I call them *primitive dynamical neural networks* (*PDNNs*). They are dynamic, because they are not only dynamical systems but also each neuron performs neutral spontaneous firing at rate 1/2 in their prototype. They are primitive, because they are generated by single threshold transformations like the McCulloch and Pitts model and also because the threshold vectors are zeros.

Our main objective is to prove the existence of previously unknown non-loop attractors. Let us start with the following assumption.

Assumption 6.3.1 The periodic firing of the action potentials with period 2 of any neuron is neutral.

Assumption 6.3.1 claims that the periodic state transition $... \to -1 \to 1 \to -1 \to 1 \to -1 \to ...$ of any neuron is neutral or indifferent whether it is disconnected or connected with other neurons. In general, any 2-cycle $q \leftrightarrow -q$ in $\{-1, 1\}^n$ represents a neutral activity. I call it a neutral cycle and any other cycles significant. In this case, the generating transformation F can be defined by

$$E = -I, \ h = o,$$

in (6.2.1), where I is the $n \times n$ identity matrix, and o is the column n-vector whose every coordinate is 0. Here every neuron fires spontaneously without any input from other neurons, and any state is on a neutral cycle. Now I make the following definition.

Definition 6.3.2 A *primitive dynamical neural network* (PDNN) is a parametrized FSDS on $\{-1, 1\}^n$ with parameters $v = (v_{12}, .., v_{1n}, v_{21}, v_{23}, .., v_{2n}, .., v_{n1}, .., v_{nn-1}) \in \mathbf{R}^{n(n-1)}$ and generated by the threshold transformation F defined by

$$Fx = \mathrm{Sgn}(Ex), \tag{6.3.1}$$

where $E = E_v$ is an $n \times n$ real matrix such that $E_{ii} = -1$ for every i and $E_{ij} = v_{ij}$ for every other i and j. The n is called the *dimension* of the DNN.

Thus in PDNNs, the prototype DNN generated by $-I$ is modified by further synaptic connections. However, the threshold vector h remains o, and the synaptic efficacy Eii between each neuron itself remains -1. Here, the assumption $h = o$ is a strong constraint. However, if $h = o$, then $E_{ii} = -1$ for every i is equivalent to $E_{ii} < 0$ for every i. As a result, such an extreme case as $E = I$ and $h = o$ is effectively excluded from PDNNs. In a supposed biological system from which the current model is abstracted, each neuron is self-oscillatory. Therefore, a network in which a neuron is not self-oscillatory but performs "self-sustained" firing (central pattern generation) by means of post synaptic rebound and inhibitory input from surrounding neurons is not covered here (see Coombes & Doole, 1996). The time period between $t = i$ and $t = i+1$ is assumed here to be the sum of the time for the action potential and the absolute refractory period. Then the relative refractory period and spontaneous release of inhibitory chemical transmitters contribute to the negativity of diagonal elements.

If a neuron is spontaneously firing in this model, then the firing rate per unit time is 1/2. On the other hand, the maximum firing rate is 1 and the minimum firing rate 0. Therefore, the firing rate of any neuron cannot exceed 2 times the spontaneous firing rate. Alternatively, we can construct a DNN of spontaneous firing rate 1/3 or less. For example, the periodic state transition, $\cdots \to -1 \to -1 \to 1 \to -1 \to -1 \to 1 \to \cdots$, is neutral in a DNN of spontaneous firing rate 1/3. A state $x(t+1)$ depends on $x(t)$ and $x(t-1)$ in this DNN. Therefore, a DNN of spontaneous firing rate 1/3 or less can incorporate *temporal summation* in its postsynaptic potential, while the postsynaptic potential $(Ex(t))_i$ of a PDNN of spontaneous firing rate 1/2 is only *spatial summation*. Therefore, such an alternative model far better reflects a real nervous system. Further, any real nervous system is not autonomous. The dynamics of the system depends on information that changes at every unit time and that is input from the outside of the system, from neurons of other nervous systems and/or from external stimulus. Still further, the rigid synchronization (alignment) of firing for all neurons is unrealistic. However, mathematical analysis will be extremely difficult in such DNN models that possibly simulate real nervous systems. Therefore, as a first step, we have to limit our subject to an extremely simplified model with spontaneous firing rate 1/2 and does not claim any immediate application to real nervous systems except a preparation for the analysis of such DNN models.

If transformations F and G of $\{-1, 1\}^n$ are orthogonally similar, then the DNNs generated by F and G are called *orthogonally similar*. A transformation that generates a PDNN is not necessarily self-dual. However, we have the following proposition.

Proposition 6.3.3 The PDNN defined by Definition 6.3.2 is *structurally stable*, if and only if F is self-dual.

Proof. Assume that the PDNN defined by Definition 6.3.2 is structurally stable. Then $(F-)x = F(-x) = \mathrm{Sgn}(E(-x)) = \mathrm{Sgn}(-(Ex))$ for every x. Also there is no i such that $(Ex)_i = 0$ for some x; therefore $\mathrm{Sgn}(-(Ex)) = -\mathrm{Sgn}(Ex) = -Fx$, so that $F- = -F$, that is F is self-dual. Conversely, assume F is self-dual. Then there is no i such that $(Ex)_i = 0$ for some x. Therefore there exists a neighborhood $U_\epsilon v$ such that if $w \in U_\epsilon v$, then $\mathrm{Sgn}(E_w x) = \mathrm{Sgn}(Ex)$ for every x. Therefore the PDNN is structurally stable. $\square$

The set of all significant cycles in GRAPH(F) is denoted by SCY(F). We are concerned with an attractor that is a subset of SCY(F), since any neural activities that are significant and less sensitive to disturbance such as failures in synchronization are described with such a significant attractor. Further, a desirable attractor would be minimal and connected. The following examples illustrate a variety in the present PDNN model.

Example 6.3.4 Complete picture for dimension 2. Orthogonally non-similar transformations are described below.

$$E = \begin{bmatrix} -1 & \epsilon \\ \delta & -1 \end{bmatrix},$$

(1) $\epsilon, \delta < -1$:

$$11 \leftrightarrow -1-1 \qquad \text{(Neutral cycle)},$$

$$1-1\partial, -11\partial \qquad \text{(Non-attractive loops)}.$$

SCY$(F) = \{(1-1), (-11)\}$ is not attractive.

(2) $\epsilon < -1, -1 < \delta < 1$:

$$1-1 \to 11 \leftrightarrow -1-1 \leftarrow -11 \qquad \text{(Attractive neutral cycle)}.$$

(3) $\epsilon < -1, 1 < \delta$:

$$11 \to -11 \to -1-1 \to 1-1 \to 11 \qquad \text{(Attractive 4-cycle)}.$$

(4) $-1 < \epsilon, \delta < 1$:

$$11 \leftrightarrow -1-1,\ 1-1 \leftrightarrow -11 \qquad \text{(Neutral cycles)}.$$

(5) $\epsilon = -1$ or 1, or $\delta = -1$ or 1: Structurally unstable.

Example 6.3.5

$$E = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 2 & -1 & -2 & 2 \\ 2 & -2 & -1 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix},$$

$$\begin{array}{lclclcl} 1-111 & \to & 1111\partial, & -11-1-1 & \to & -1-1-1-1\partial \\ & \nearrow & & & \nearrow & \\ 11-11 & & & -1-11-1 & & \end{array}$$

(Non-attractive loops),

$$1-11-1 \leftrightarrow -1-111,\ -11-11 \leftrightarrow 11-1-1 \qquad \text{(Non-attractive 2-cycles)},$$

$$\begin{array}{lclclcl} 111-1 & \to & 1-1-11 & \leftrightarrow & -111-1 & \leftarrow & -1-1-11 \\ & \nearrow & & & & \nwarrow & \\ -1111 & & & & & & 1-1-1-1 \end{array}$$

(Neutral 2-cycle).

SCY$(F) = \{(1111), (-1-1-1-1), (1-11-1, -1-111), (-11-11, 11-1-1)\}$ is not attractive.

Example 6.3.6

$$E = \begin{bmatrix} -1 & 2 & 2 & 2 \\ 4 & -1 & 2 & 2 \\ 2 & 2 & -1 & 2 \\ 2 & 2 & 2 & -1 \end{bmatrix},$$

$$
\begin{array}{ccccc}
1-1-1-1 & \to & -11-1-1 & & \\
 & & & \searrow & \\
 & & -1-11-1 & \to & -1-1-1-1\partial \\
 & & & \nearrow & \\
 & & -1-1-11 & &
\end{array} \qquad \text{(Attractive loop)},
$$

$$
\begin{array}{ccccc}
-1111 & \to & 1-111 & & \\
 & & & \searrow & \\
 & & 11-11 & \to & 1111\partial \\
 & & & \nearrow & \\
 & & 111-1 & &
\end{array} \qquad \text{(Attractive loop)},
$$

$$
11-1-1 \leftrightarrow -1-111, 1-11-1 \leftrightarrow -11-11, 1-1-11 \leftrightarrow -111-1 \quad \text{(Neutral cycles)}.
$$

$(-1-1-1-1)$ and (1111) are neither stable in Amari (1972) nor a k-attractor in Cottrell (1988).

Example 6.3.7

$$
E = \begin{bmatrix} -1 & -2 & 2 & 2 \\ 2 & -1 & -2 & 2 \\ 2 & 2 & -1 & 2 \\ 2 & -2 & 2 & -1 \end{bmatrix},
$$

$$
\begin{array}{ccccccc}
-1111 & \to & 1-11-1 & & 1-111 & & \\
 & & & \searrow & & \searrow & \\
 & & -1-11-1 & \to & 1-1-11 & \to & 1111\partial \\
 & & & \nearrow & & & \\
111-1 & \to & -1-111 & & & &
\end{array}
$$

(Non-attractive loop),

$$
\begin{array}{ccccccc}
1-1-1-1 & \to & -11-11 & & -11-1-1 & & \\
 & & & \searrow & & \searrow & \\
 & & 11-11 & \to & -111-1 & \to & -1-1-1-1\partial \\
 & & & \nearrow & & & \\
-1-1-11 & \to & 11-1-1 & & & &
\end{array}
$$

(Non-attractive loop),

$\mathrm{SCY}(F) = \{(1111), (-1-1-1-1)\}$ is a disconnected minimal attractor.

6.4 PDNN-definable transformations

In this section we characterize the PDNN-definable transformations for spontaneous firing rate 1/2 defined by:

Definition 6.4.1 If a transformation F of $\{-1,1\}^n$ can be defined by $Fx = \mathrm{Sgn}(Ex)$ for an $n \times n$ real matrix E such that $E_{ii} = -1$, then F is called DNN-definable.

Theorem 6.4.2 A self-dual threshold transformation F of $\{-1,1\}$ is PDNN-definable if and only if $\mathrm{Var}(i^- F) \leq \mathrm{Var}(F)$ for every i.

Proof. Let F be a self-dual threshold transformation of $\{-1,1\}^n$. Then by Proposition 4.3.1 of Chapter 4, there exists an $n \times n$ real matrix W such that $Fx = \mathrm{Sgn}(Wx)$, and there is no point x such that $(Wx)_i = 0$ for some i. If $W_{ii} = 0$ for some i, then replacing W_{ii} with some ϵ_{ii} such that $|\epsilon_{ii}|$ is sufficiently small does not change $\mathrm{Sgn}(Wx)$. Therefore, we obtain a matrix $W\prime$ such that $W\prime_{ii} \neq 0$ for every i and $\mathrm{Sgn}(W\prime x) = \mathrm{Sgn}(Wx)$. Next, divide the ith row of $W\prime$ by $|W\prime_{ii}|$ to obtain $W\prime\prime$. Suppose $W\prime\prime_{ii} = 1$ for some i, and $\mathrm{Sgn}((W\prime\prime x)_i) = \mathrm{Sgn}((W\prime\prime i^- x)_i)$ for every x, then replacing $W\prime\prime_{ii} = 1$ with -1 does not change $\mathrm{Sgn}((W\prime\prime x)_i)$ for every x.

Suppose F is not PDNN-definable. Then we obtain a matrix W such that $Fx = \mathrm{Sgn}(Wx)$, $W_{ii} = 1$ for some i, and $(Wq)_i > 0$ and $(W(i^- q))_i < 0$ for some q. Therefore, $(W(-i^- q))_i > 0$. Thus $d_H((q, -i^- q) = n-1$, and both q and $-i^- q$ are on the same side of the hyper plane of $\mathbf{R}^n$ defined by $(Wx)_i = 0$. Further, since $W_{ii} = 1$, we have $q_i = 1$ and so $(-i^- q)_i = 1$. Therefore, for any $r \in \{-1, 1\}^n$ such that $r_1 = 1$, at least one of r and $-(i^- r)$ must be on the same side of the hyper plane as q. Therefore the set $\{x \mid x_i = 1, and (Fx)_i = 1, x \in (-1, 1)^n\}$ contains at least $2^{n-2} + 1$. Therefore, $\mathrm{Var}(i^- F) > \mathrm{Var}(F)$.

Suppose F is PDNN-definable. Then, by the above result, there exists a real matrix E such that $Fx = \mathrm{Sgn}(Ex)$ for every x and $E_{ii} = -1$ for every i. Let E_i be the ith row-vector of E and $E\prime_i$ be the row-vector obtained by replacing -1 with 1 for the ith coordinate of E_i. Then

$$\begin{array}{rl} & |\{x \mid x_i = 1, (E_i, x) < 0, x \in \{-1, 1\}^n\}| \\ \geq & |\{x \mid x_i = 1, (E\prime_i, x) < 0, x \in \{-1, 1\}^n\}| \\ = & |\{x \mid x_i = 1, (-E_i, x) < 0, x \in \{-1, 1\}^n\}|, \end{array}$$

where $(\cdot, \cdot)$ denotes the inner product. Therefore $\mathrm{Var}(i^- F) \leq \mathrm{Var}(F)$ for every I. $\square$

Corollary 6.4.3 Let F be a self-dual threshold transformation of $\{-1, 1\}^n$. Then F is orthogonally equivalent to a PDNN-definable threshold transformation of $\{-1, 1\}^n$.

Proof. Let F be a self-dual threshold transformation of $\{-1, 1\}^n$. Let $J = \{i \mid i \in \mathbf{N}, \mathrm{Var}(i^- F) > \mathrm{Var}(F)\}$. If $G = J^- F$, then $\mathrm{Var}(i^- G) \leq \mathrm{Var}(G)$ for every i. Therefore, G is PDNN-definable. $\square$

Corollary 6.4.4 If F is a self-dual threshold transformation such that $(Fx)_i = x_i$ for every x for some i, then F is not PDNN-definable.

Corollary 6.4.5 Let T be an orthogonal transformation of $\{-1, 1\}^n$. Then T is PDNN-definable if and only if for every i there exists some x depending on i such that $(Tx)_i \neq x_i$.

Proof. Let D be the matrix representing T, i.e., $Tx = Dx$ for every x. If $(Tx)_i = x_i$ for every x for some i, that is, $D_{ii} = 1$, then $\mathrm{Var}(i^- T) > \mathrm{Var}(T)$, so that T is not PDNN-definable. Let $D_{ii} = 0$ or -1 for every i. If $D_{ii} = 0$ for some i, then replacing D_{ii} with some $\epsilon_{ii} < 0$ such that $|\epsilon_{ii}|$ is sufficiently small does not change $\mathrm{Sgn}(Dx)$. Therefore we obtain a matrix $D\prime$ such that $D\prime_{ii} = -1$ for every i and $\mathrm{Sgn}(D\prime x) = Dx$, so that T is PDNN-definable. $\square$

Proposition 6.4.6 If F is PDNN-definable, and G is orthogonally similar to F, then G is also PDNN-definable.

Proof. Let F be PDNN-definable and G be a threshold transformation similar to F. By definition, $Fx = \mathrm{Sgn}(Ex)$, $E_{ii} = -1$ for every i, and there exists an orthogonal transformation T of $\{-1,1\}^n$ such that $G = T^{-1}FT$. Let D be the matrix representing T. Then $Gx = T^{-1}(FTx) = D^{-1}\mathrm{Sgn}(EDx) = \mathrm{Sgn}(D^{-1}EDx)$. It is clear that $(D^{-1}ED)_{ii} = -1$ for every i. Therefore, G is PDNN-definable. □

If a transformation H of $\{-1,1\}^n$ is the direct product of transformations F of $\{-1,1\}^m$ and G of $\{-1,1\}^{n-m}$, i.e.,

$$H\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Fx \\ Gy \end{bmatrix}$$

then clearly H is PDNN-definable if and only if both F and G are PDNN-definable. In this case, we call the PDNN generated by H the *direct product* of the PDNN generated by F and the PDNN generated by G.

We now switch from transformations of $\{-1,1\}^n$ to corresponding transformations of $\mathbf{Q}^n$. First, the following proposition immediately follows from Theorem 6.4.2.

Proposition 6.4.7 A self-dual transformation $[f_1, ..., f_n]$ of $\mathbf{Q}^n$ is PDNN-definable, if and only if $|f_i| \geq 2^{n-2}$ for every i. $\langle f \rangle$ is PDNN-definable iff $|f| \geq 2^{n-2}$. $\langle\langle f \rangle\rangle$ is PDNN-definable iff $|f| \geq 2^{n-2}$.

Corollary 6.4.8 If F is a minimal threshold transformation, then $\bar{\neg}F$ is PDNN-definable.

6.5 Attractive loops

The following proposition indicates basic properties of attractive loops of the present PDNNs. In particular, Proposition 6.5.1 (a) implies that there is no connected attractor consisting of more than two loops.

Proposition 6.5.1 Let F be a PDNN-definable self-dual transformation of $\mathbf{Q}^n$. (a) If (q) and (r) are different loops of F, then $d_H(q,r) \geq 2$. (b) If $Fx = q$ for every x such that $d_H(x,q) = 1$, then (q) is an attractive loop.

Proof. (a) Without loss of generality let $q = l$ and $r = 1^- l$, where $l = 1 \cdots 1$, and (q) and (r) be loops of a PDNN-definable transformation F of $\mathbf{Q}^n$. If $F = [f_1, f_2, ..., f_n]$, then $1^- F = [p_1 \cdot \neg f_1, f_2, ..., f_n]$ by Proposition 2.4.1 of Chapter 2. Therefore $q \in x_1 \cdot \neg f_1$ and $\bar{\neg}r = 1^- o \in p_1 \cdot \neg f_1$, since (q) and (r) are loops of F. Therefore, as the proof of Theorem 6.4.2, $p_1 \cdot \neg f_1$ contains at least $2^{n-2} + 1$ points. Therefore $\mathrm{Var}(i^- F) > \mathrm{Var}(F)$, which contradicts the fact that F is PDNN-definable. (b). Let F be a PDNN-definable transformation of $\{-1,1\}^n$ defined by $Fx = \mathrm{Sgn}(Ex)$ for every $x \in \{-1,1\}^n$. Without loss of generality let $q = l$, and let $F(i^- q) = q$ for every i. Then $E(i^- q) > o$ for $i = 1, 2, .., n$. Therefore, adding these n inequalities, we obtain $(n-2)E(q) > o$, so that $Fq = q$. □

A PDNN is called symmetric if the matrix E is symmetric in Definition 6.3.2. From Goles-Chacc's theorem (Theorem 6.2.1) it follows.

Theorem 6.5.2 SCY(F) of any symmetric PDNN generated by F is either empty or consists of loops or/and 2-cycles.

The above theorem means that in symmetric PDNNs, any neuron ultimately either performs a spontaneous firing or continues to fire at the maximum rate 1 or does not fire at all.

Now, in order to find an attractor in symmetric PDNNs, consider the simple PDNN generated by $F = \mathrm{Sgn}(Ex)$, where

$$E = \begin{bmatrix} -1 & \epsilon & \cdot & \cdot & \cdot & \cdot & \epsilon \\ \epsilon & -1 & \epsilon & \cdot & \cdot & \cdot & \epsilon \\ \epsilon & \epsilon & -1 & \epsilon & \cdot & \cdot & \epsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \epsilon & \cdot & \cdot & \cdot & \cdot & \epsilon & -1 \end{bmatrix}, \tag{6.5.1}$$

Let the transformation of $\mathbf{Q}^n$ corresponding to F be $G = [g_1, ..., g_n]$. Let $d(x)$ be the density of x, i.e. $d(x) = |\{i \mid x_i = 1\}|$. Then $(Fx)_i = \mathrm{Sgn}((2d(x) - n - 1)\epsilon - 1)$, if $x_i = 1$. Therefore, $x \in g_i$ if and only if $x_i = 1$ and $(2d(x) - n - 1)\epsilon - 1 < 0$.

First let $\epsilon > 0$. Then, $x \in g_i$ iff ($x_i = 1$, $d(x) \leq n$, and $\epsilon < 1/(n-1)$), or ($x_i = 1$, $d(x) \leq n - 1$, $\epsilon < 1/(n-3)$) or ($x_i = 1$, $d(x) \leq n - 2$, and $\epsilon < 1/(n-5)$), or so forth. Therefore, if $0 < \epsilon < 1/(n-1)$ then $G = \langle p_1 \rangle$ and this PDNN is generated by $\bar{\neg}$. If $1/(n-1) < \epsilon < 1/(n-3)$, then

$$G = \langle p_1 \cdot S_1\{\neg p_2, .., \neg p_n\}\rangle, \tag{6.5.2}$$

and (l) and (o) are loops. If $1/(n-3) < \epsilon < 1/(n-5)$ then

$$G = \langle p_1 \cdot S_2\{\neg p_2, .., \neg p_n\}\rangle, \tag{6.5.3}$$

and (l) and (o) are attractors, $GU_1 l = l$ and $GU_1 o = o$, and $U_1 l$ and $U_1 o$ are respectively the basins for the attractors. If $1/(n-5) < \epsilon < 1/(n-7)$ then

$$G = \langle p_1 \cdot S_3\{\neg p_2, .., \neg p_n\}\rangle, \tag{6.5.4}$$

and $U_2 l$ and $U_2 o$ are the basins for the attractors, and so forth. This sequence ends, when $1/(n - (n-1)) < \epsilon$ for even n, where

$$G = \langle p_1 \cdot S_{n/2}\{\neg p_2, .., \neg p_n\}\rangle, \tag{6.5.5}$$

and $U_{n/2-1} l$ and $U_{n/2-1} o$ are the basins for the attractors (l) and (o). The sequence ends, when $1/(n - (n-2)) = 1/2 < \epsilon$ for odd n, where

$$G = \langle p_1 \cdot S_{(n-1)/2}\{\neg p_2, .., \neg p_n\}\rangle, \tag{6.5.6}$$

and $U_{(n-1)/2-1} l$ and $U_{(n-1)/2-1} o$ are the basins for the attractors (l) and (o).

Further, points outside the basins for the attractors are on neutral cycles. In particular, if $d(x) = n/2$ or $d(x) = (n+1)/2$, then $Gx = \bar{\neg} x$, so that any point with density $n/2$ or $(n+1)/2$ is always on a neutral cycle.

Next, let $\epsilon < 0$. Then $Gl = \bar{\neg} l$. If $Gx \neq \bar{\neg} x$, $Gx \neq l$, and $Gx \neq \bar{\neg} l$, then $Gx = x$, so that $(2d(x) - n + 1)\epsilon + 1 < 0$ and $(2d(x) - n - 1)\epsilon - 1 > 0$. From the former inequality follows $2d(x) \geq n$, and from the latter follows $2d(x) \leq n$, so that $d(x) = n/2$. Therefore, if $\omega_G x = \mathbf{V}$ for some $V \in \mathrm{SCY}(G)$, then $d(x) = n/2$, so that $\mathrm{SCY}(G)$ is not attractive. From the above arguments, we have obtained:

Theorem 6.5.3 If $n \geq 4$ and $\epsilon > 1/(n-3)$ in the symmetric PDNN generated by F defined by (6.5.1), then $\mathrm{SCY}(G)$ consists of two separate attractive loops (l)

and (o). If $\epsilon < 1/(n-3)$, then $\mathrm{SCY}(G)$ is either empty or not attractive.

Note that all PDNNs and their attractors have their isometrically similar counterparts. The following corollaries are obtained by construction with direct products.

Corollary 6.5.4 There exists a symmetric PDNN generated by F of dimension $n = rm$ for every $r \geq 4$ and every positive integer m such that $\mathrm{SCY}(F)$ consists of 2^m loops, each being attractive.

Corollary 6.5.5 If $n \geq 5$, then there exists a symmetric PDNN of dimension n that has a significant attractive 2-cycle.

As illustrated in Example 6.3.5, even if a symmetric PDNN generated by F is irreducible, i.e., not a direct product, $\mathrm{SCY}(F)$ may contain a significant 2-cycle. While each neuron fires at either maximum or minimum rate when state on a loop is attained, some neurons fires at maximum or minimum rate and the other neurons perform neutral firing when a non-neutral 2-cycle is attained. Therefore, as long as non-neutral neurons are concerned, there is no difference between a non-neutral 2-cycle and a loop in smaller dimension.

6.6 Attractive cycles

PDNNs which are circular and symmetric at the same time were completely described by Theorem 6.5.3 in the last section. In this section we show the existence of non-loop attractors in circular PDNNs. First, the following proposition is a basis for constructing a PDNN with having an attractor.

Proposition 6.6.1 Let F be a self-dual transformation of $\mathbf{Q}^n$, Φ be an attractor in the FSDS generated by F, and Φ be self-dual, i.e. $\bar{\neg}\mathrm{Im}\Phi = \mathrm{Im}\Phi$. Then $\Psi = \mathrm{CY}(\bar{\neg}F|\mathrm{Im}\Phi)$, where $|\mathrm{Im}\Phi$ denotes the restriction to $\mathrm{Im}\Phi$, is an attractor in the FSDS on $\mathbf{Q}^n$ generated by $\bar{\neg}F$.

Proof. Since Φ is attractive, there exists a neighborhood $U_\epsilon(\mathrm{Im}\Phi)$ such that $F(U_\epsilon(\mathrm{Im}\Phi)) \subseteq U_\epsilon(\mathrm{Im}\Phi)$, and $F^k(U_\epsilon(\mathrm{Im}\Phi)) = \mathrm{Im}\Phi$ for some positive integer k. Therefore, $(\bar{\neg}F)(U_\epsilon(\mathrm{Im}\Phi)) \subseteq \bar{\neg}(U_\epsilon(\mathrm{Im}\Phi)) = U_\epsilon(\mathrm{Im}\Phi)$. Since F is self-dual, $(\bar{\neg}F)^k = \bar{\neg}^k F^k$, Therefore, $(\bar{\neg}F)^k(U_\epsilon(\mathrm{Im}\Phi)) = \bar{\neg}^k F^k(U_\epsilon(\mathrm{Im}\Phi)) = \bar{\neg}^k\mathrm{Im}\Phi = \mathrm{Im}\Phi$. Hence, Ψ is an attractor in the FSDS on $\mathbf{Q}^n$ generated by $\bar{\neg}F$, since $\mathrm{Im}\Psi = \mathrm{Im}\Phi$. $\square$

According to Proposition 6.5.1, if F is a minimal threshold transformation, then $\bar{\neg}F$ is PDNN-definable. Therefore, if we have a minimal threshold transformation having an attractor, then we can obtain a PDNN having an attractor by Proposition 6.6.1.The 4-cycle attractors and in the following Theorem 6.6.2 and two-3-cycle attractors in Theorem 6.6.4 are respectively constructed by modifying Examples 4.4.2 and 4.4.3 of Chapter 4, which are one-to-one transformations. In the following, $\mathrm{Orb}_{\langle\bar{\neg},\rho\rangle}D$ for a subset D of $\mathbf{Q}^n$ is denoted by $[D]$.

Theorem 6.6.2 There exists a circular PDNN generated by F of dimension $4m$ for any $m \geq 1$ such that $\mathrm{SCY}(F)$ is a connected minimal attractor consisting of one 4-cycle.

Proof. Let $F = \langle f \rangle$ be the circular self-dual transformation of $\mathbf{Q}^{4m}$ defined by

$$f = p_1 \cdot p_2 \cdot \neg p_4 \cdot p_6 \cdot \neg p_8 \cdots p_{4m-2} \cdot \neg p_{4m}.$$

Clearly f is a threshold function, so that F is a threshold transformation. Let $a = 11001100 \cdots 1100$ and $A = \mathrm{Orb}_\rho a$. Then $A = (a, \rho a, \rho^2 a, \rho^3 a) \in \mathrm{CY}(F)$, since $Fa = \rho a$.

First, we show that $F(\mathrm{Car}F) \subseteq \mathbf{A}$. Let $x \in f$. Clearly, $x_{4k+2} = 1$ and $x_{4k+4} = 0$ for every k. If $x_{4k+1} = 1$, then $\rho^{-4k} x \in f$; if $x_{4k+1} = 0$, then $\rho^{-4k} x \notin \bar{\neg} f$, since $x_{4k} = 0$. Therefore, $(Fx)_{4k+1} = 0$. We have $x_{4k+2} = 1$, so that $(\rho^{-(4k+1)} x)_1 = 1$, but $x_1 = 1$, so that $(\rho^{-(4k+1)} x)_{4(m-k-1)+4} = 1$. Therefore, $\rho^{-(4k+1)} x \notin f$, so that $(Fx)_{4k+2} = 1$. If $x_{4k+3} = 1$, then $\rho^{-(4k+2)} x \notin f$, since $(\rho^{-(4k+2)} x)_{4m} = 1$; if $x_{4k+3} = 0$, then $\rho^{-(4k+2)} x \in \bar{\neg} f$. Therefore, $(Fx)_{4k+3} = 1$. We have $x_{4k+4} = 0$, so that $(\rho^{-(4k+3)} x)_1 = 0$, but $x_1 = 1$, so that $(\rho^{-(4k+3)} x)_{4(m-k-1)+2} = 1$. Therefore, $\rho^{-(4k+3)} x \notin \bar{\neg} f$, so that $(Fx)_{4k+2} = 0$. Therefore, $Fx = \rho a$. Therefore, $Ff \subseteq \mathbf{A}$, so that $F(\mathrm{Car}F) \subseteq \mathbf{A}$, since $\mathrm{Car}F = [f]$ and $[\mathbf{A}] = \mathbf{A}$.

Next, $U_1 \mathbf{A} \subseteq \mathrm{Car}F$. In fact, if $x = 1^- a$, then $x \in \rho^4 f$. If $x = 2^- a$, then $x \in \bar{\neg} \rho f$. If $x = 3^- a$, then $x \in f$. If $x = 4^- a$, then $x \in \rho^3 f$. Therefore, because of the circularity and self-duality of F, $U_1 \mathbf{A} \subseteq \mathrm{Car}F$.

The above arguments have shown that A is an attractor and the only non-loop cycle in the FSDS generated by F. By Proposition 6.5.1, $\bar{\neg} F$ is PDNN-definable. By Proposition 6.6.1, $\mathrm{CY}(\bar{\neg} F | \mathbf{A})$ is also an attractor in the PDNN. Clearly it consists of one 4-cycle. It is also $\mathrm{SCY}(\bar{\neg} F)$ and a connected minimal attractor. $\square$

It is clear from the above proof, a flow graph of F in Theorem 6.6.2 is

$$[f] \to [\neg 1 \cdot 2 \cdot 3 \neg 4 \cdot \neg 5 \cdot 6 \cdot 7 \neg 8 \cdots \neg(4m-3) \cdot (4m-2) \cdot (4m-1) \cdot \neg 4m] \partial.$$

Example 6.6.3 $\mathrm{GRAPH}(\bar{\neg} F)$ in Theorem 6.6.2 for $m = 1$.

$$0000 \leftrightarrow 1111, \qquad 0101 \leftrightarrow 1010,$$

$$\begin{array}{ccccccc}
0010 & & & & & & 1110 \\
 & \searrow & & & & \swarrow & \\
0111 & \to & 1100 & \to & 1001 & \leftarrow & 0100 \\
 & & \uparrow & & \downarrow & & \\
0001 & \to & 0110 & \leftarrow & 0011 & \leftarrow & 1101 \\
 & \nearrow & & & & \nwarrow & \\
1011 & & & & & & 1000
\end{array}$$

Theorem 6.6.4 There exists a circular PDNN generated by F of dimension $3m$ for any $m \geq 2$ such that $\mathrm{SCY}(F)$ is a connected minimal attractor consisting of two 3-cycles.

Proof. Let $F = \langle f \rangle$ be the circular self-dual transformation of $\mathbf{Q}^n$ defined by

$$f = p_1 \cdot p_2 \cdot p_5 \cdot p_8 \cdot p_{11} \cdots p_{3m-1} S_{m-1}\{\neg p_3, \neg p_4, \neg p_6, \neg p_7, .., \neg p_{3m-3}, \neg p_{3m-2}, \neg p_{3m}\}.$$

Clearly f is a threshold function, so that F is a threshold transformation. Let $a = 110110 \cdots 110$ and $A = \mathrm{Orb}_{\rho^{-1}\bar{\neg}} a$. Then $A = (a, \rho^{-1} \bar{\neg} a, .., (\rho^{-1} \bar{\neg})^5 a) \in \mathrm{CY}(F)$, since $Fa = \rho^{-1} \bar{\neg} a$. We will show $F(U_1 \mathbf{A}) \subseteq \mathbf{A}$. In fact, $F(1^- a) = \rho^{-1} \bar{\neg} a$, $F(2^- a) = a$, and $F(3^- a) = \rho^{-1} \bar{\neg} a$. Therefore, because of the circularity and self-duality of F, $F(U_1 \mathbf{A}) \subseteq \mathbf{A}$.

It is shown in the following that $Fx \in \mathbf{A}$ or $F^2 x \in \mathbf{A}$ for every $x \in f$.
(Case 1) $x_3 \cdot x_6 \cdots x_{3(m-1)} \cdot x_{3m} = 1$: Since at least $m - 1$ coordinates of x are 0,

$x_{3k-2} = 0$ for every $2 \leq k \leq m$, so that $x = 1^{-}\rho a$. Therefore, $Fx = \bar{\neg} a \in \mathbf{A}$.
(Case 2) $x_3 \cdot x_6 \cdots x_{3(m-1)} \cdot x_{3m} = 0$: Then $Fx = 01x_3 01x_6 01 \cdots x_{3(m-1)} 01x_{3m}$. Clearly $(F^2x)_{3k-2} = 0$ for every k. Since $x_{3k} = 0$ for at least one k, $(F^2x)_{3k-1} = 1$ for every $1 \leq k \leq m$. If $(Fx)_{3k} = x_{3k} = 1$ for some $1 \leq k \leq m$ then $Fx \notin \rho^{3k-1}f$, so that $(F^2x)_{3k} = 1$. If $(Fx)_{3k} = x_{3k} = 0$ for some $1 \leq k \leq m$, then $\bar{\neg}(Fx) \in \rho^{3k-1}f$, so that $(F^2x)_{3k} = 1$. Therefore, $F^2x = \rho a \in \mathbf{A}$.

The above arguments have shown that A is an attractor and a unique non-loop cycle in the FSDS generated by F. By Proposition 6.5.1 $\bar{\neg}F$ is PDNN-definable. By Proposition 6.6.1, $\mathrm{CY}(\bar{\neg}F|\mathbf{A})$ is also an attractor in the PDNN. Clearly it consists of two 3-cycle. It is also $\mathrm{SCY}(\bar{\neg}F)$ and a connected minimal attractor. $\square$