CHAPTER 9 ATTRACTORS IN NON-AUTONOMOUS NEURAL NETWORKS

Abstract. Non-autonomous primitive dynamical neural networks (PDNNs) that incorporate a spontaneous firing rate of 1/2 per unit time are constructed. The concept of attractors is redefined so that they should be invariant under shifts in arrival of the periodic input sequence. If asymptotic properties for an attractor depend on both initial states and input, then the attractor is called bi-dependent. Then we show how non-autonomous PDNNs described in Chapter 6 are modified by input in corresponding non-autonomous PDNNs, so that a non-attractive limit cycle becomes a bi-dependent attractive cyclic orbit, a non-unique attractor becomes unique, and an attractor consisting of more than one cycles becomes an attractive cyclic orbit.

9.1 Introduction

In the last two chapters, we introduced McCulloch-Pitts networks of self-oscillatory neurons with spontaneous firing rates 1/2 and 1/3 respectively, and showed the existence of attractors. In the first order neural networks described in Chapter 6, $\mathbf{N} = \{1, 2, ..., n\}$ represents neurons, and the state space $\{-1, 1\}^{\mathbf{N}}$ of the PDNN (primitive dynamical neural network) is a finite metric space with the integer-valued Hamming distance d_H. The PDNN of spontaneous firing rate 1/2 is a finite state dynamical system (FSDS) on the state space $\{-1, 1\}^{\mathbf{N}}$ generated by the threshold transformation F of $\{-1, 1\}^{\mathbf{N}}$ defined by

$$Fx = \mathrm{Sgn}(Ex), \quad x(t+1) = F(x(t)).$$

This model is autonomous, that is, stable periodic firing patterns that are represented by attractors are completely determined by the efficacy matrices of synaptic connections and the initial states of the neurons at time $t = 0$. However, the dynamics of any biological system depends on information that changes at every time and that is input from the outside of the system, from neurons of other nervous systems and/or from external stimulus. Further, in autonomous networks, if a minimal attractor consists of more than one cycle, then there are some fluidity of shifting from one pattern to another caused by noise, even with a change in firing rate in some cases. This problem may be solved only in a non-autonomous model with input from outside the network.

Therefore, we want here to construct a PDNN in which $x(t+1) \in X = \{-1, 1\}^{\mathbf{N}}$ depends on $x(t)$ and the external input $r(t)$. Let us assume that $r(t) \in Y = \{-1, 1\}^{\mathbf{N}\prime}$, where $\mathbf{N}\prime = \{n+1, n+2, ..., 2n\}$ is the set of external input elements, and that $r(t)$ is a periodic sequence defined as $r(t) = V_{t\%k}$ by some k-sequence $V = (V_0, ..., V_{k-1})$ in Y.

The elemens of $\mathbf{N}\prime$ represent receptor neurons or interneurons. Let the synaptic connections between $\mathbf{N}$ and $\mathbf{N}\prime$ be represented by an efficacy matrix C. Then we define

$$F(x, y) = \mathrm{Sgn}(Ex + Cy), \tag{9.1.1}$$

$$x(t+1) = F(x(t), r(t)). \tag{9.1.2}$$

In this case, for each given input V, a dynamical system φ_V on $X = \{-1,1\}^n$ generated by F and V is defined as $\varphi_V : X \times \mathbf{Z}_+ \to X, \varphi_V(x,t) = x(t)$, and $x(0) = x$. As in Chapter 6, if $x(t) = -x(t-1)$ for every t, then the orbit $x(0), x(1), ...$ is called *neutral*.

In the following, both $\{-1,1\}^{\mathbf{N}}$ and $\{-1,1\}^{\mathbf{N}\prime}$ are identified with $\{-1,1\}^n$. The input k-sequence V may be expressed as $V = (i_1 w^1, i_2 w^2, .., i_j w^j)$ such that $i_1 + ... + i_j = k$ and $w^h \in \{-1,1\}^n$. For example, for $n = 2$, $V = ((1,1),(1,1),(-1,1),(-1,-1),(-1,-1)) = (2(1,1), 1(-1,1), 2(-1,-1))$.

We are concerned with properties that are invariant under shifts in arrival of the input k-sequence. Therefore, we apply the definition of attractors given in the next section, which should also be referred to for the following discussion.

There are two extreme cases in the dynamical system φ_V. The first case is that $\min_{x\in\{-1,1\}^{\mathbf{N}}}|E_i x|$ is sufficiently greater than $\max_j|C_{ij}|$ for every i so that $x(t+1) = \mathrm{Sgn}(Ex(t))$. In this case we have an autonomous PDNN described in Chapter 6. The second case is that $\min_j|C_i V_j|$ is sufficiently greater than $\max_j|E_{ij}|$ for every i so that $x(t+1) = \mathrm{Sgn}(Cr(t))$. In this case, the sequence $(\mathrm{Sgn}(CV_0), ..., \mathrm{Sgn}(CV_{k-1}))$ is clearly an attractor. For example, if V is a k-sequence of period p, and if $C = cI$, where I is the identity matrix, such that c is sufficiently greater than $|E_i|$ for every i, then $x(t+1) = r(t)$, so that V is a unique attractor. However, we are not interested in such extreme cases and concerned with only *bi-dependent* attractors.

Section 9.2 formally defines various concepts concerning autonomous and non-autonomous PDNNs and particularly attractors. Section 9.3 describes Boolean representations of non-autonomous PDNNs. The existence of bi-dependent attractive loops is shown in Section 9.4. The existence of bi-dependent, attractive, cyclic orbits in skew-circular PDNNs is shown in Section 9.6. In the final Section 9.7, we show how a non-unique attractor in a cicular autonomous PDNN becomes unique but bi-dependent in a corresponding non-autonomous PDNN.

9.2 Definition of attractors

A sequence V of X is called *periodic* if there exists some k such that $V_i = V_{i\%k}$ for every $i = 0, 1, ...$ The minimal such k is called the *period*, and V is denoted by the k-sequence $(V_0, V_1, .., V_{k-1})$; in particular, if $V_i \neq V_j$ for every $i \neq j$, $0 \leq i, j \leq k-1$ then V is called *cyclic* (See Chapter 6.1).

Let X and Y be finite metric spaces. Let F be a mapping: $X \times Y \to X$ and $V = (V_0, V_1, .., V_{k-1})$ be a k-sequence of Y. Then F defines a mapping $\varphi_V : X \times \mathbf{Z}_+ \to X$ by

$$\begin{aligned} \varphi_V(x,0) &= x, \\ \varphi_V(x,t) &= F(\varphi_V(x,t-1), V_{(t-1)\%k}) \text{ for } t \geq 1, \end{aligned}$$

Then φ_V is called a *finite-state dynamical system* (FSDS) on X *generated by* F *and input* V. If x is a point of X and $W = \sigma^j V$ for some j, then the sequence

$$\mathrm{Orb}_{F,V}x = (\varphi_V(x,0), \varphi_V(x,1), \varphi_V(x,2), ...)$$

is called an orbit starting at the point x. If S is a subset of X, then $\mathrm{Orb}_{F,V}S$ denotes the set of all orbits starting at some $x \in S$. Further, let $\mathrm{PO}(F,V)$ denote the set of all periodic orbits in the FSDS φ_V. A periodic orbit $W = (W_0, W_1, .., W_{m-1}) \in \mathrm{PO}(F,V)$ is called a *limit orbit* of x and denoted by $\Omega_{F,V}x$, if there exists some $j \in \mathbf{Z}_+$ such that $j \equiv 0 \mod k$ and $\varphi_V(x,t) = W_{(t-j)\%m}$ for every $t \geq j$. For a subset S of X, $\Omega_{F,V}S$ is the union $\bigcup_{x\in S}\Omega_{F,V}x$. Let σ denote the round shift defined by $\sigma W=(W_1,...,W_{m-1},W_0)$ for any m-sequence $W=(W_0,W_1,..,W_{m-1})$ of any length m. Further, if $\Phi=\{U,..,W\}$, then let $\sigma\Phi=\{\sigma U,...,\sigma W\}$.

Definition 9.2.1 A subset Φ of $\mathrm{PO}(F,V)$ is called *attractive* or an *attractor* in the FSDS generated by F and V, if there exists some neighborhood $U_\epsilon(\mathrm{Im}\Phi)$ such that

(1) $F(U_\epsilon(\mathrm{Im}\Phi)\times\mathbf{V})\subseteq U_\epsilon\mathrm{Im}\Phi$.
(2) $\Omega_{F,\sigma^jV}(U_\epsilon(\mathrm{Im}\Phi))=\sigma^j\Phi$ for any j.

In particular, if Φ consists of one periodic orbit, the periodic orbit is called an attractor. The *basin* for an attractor Φ is the set of all points x such that $\Omega_{F,\sigma^jV}x\in\sigma^j\Phi$ for any j. If the basin for an attractor is not the entire space X, then the attractor is called *dependent on initial states*. If an attractor in the FSDS generated by F and V is not an attractor in the FSDS generated by F and some input $V\prime$, then the attractor is called $dependent on input$. If an attractor is dependent on both initial states and input, then the attractor is called $bi-dependent$.

9.3 Boolean and extended representations

Let l and o be the elements of $\mathbf{Q}^n$ such that $l_i=1$ and $o_i=0$ or every i. The mapping $F:\{-1,1\}^n\times\{-1,1\}^n\to\{-1,1\}^n$ defined by (9.1.1) can be represented by an equivalent mapping $G:\mathbf{Q}^n\times\mathbf{Q}^n\to\mathbf{Q}^n$ defined by

$$G=\mathrm{Bool}\circ F\circ\mathrm{Sgn}.$$

Specifically,

$$G(x,y)=\mathrm{Bool}(2(Ex+Cy)-(E+C)l), \tag{9.3.1}$$

since $\mathrm{Sgn}x=2x-l$ if $x\in\mathbf{Q}^n$. Further, (9.1.2) is replaced by

$$x(t+1)=G(x(t)),s(t)), \tag{9.3.2}$$

where $s=\mathrm{Bool}r$.

Let p_i and q_i be respectively the projection functions: $\mathbf{Q}^n\times\mathbf{Q}^n\to\mathbf{Q}$ defined by

$$p_i(x,y)=x_i,\quad q_i(x,y)=y_i.$$

Let $G:\mathbf{Q}^n\times\mathbf{Q}^n\to\mathbf{Q}^n$ and $G=(G_1,...,G_n)$, where $G_i=p_i\circ G$. G can be further represented by

$$G=[g_1,...,g_n],\ \text{where}\ g_i=p_i\cdot\neg G_i. \tag{9.3.3}$$

The extended representation $G^\#$ of G is defined as in the previous chapters by

$$(G^\#x)_i=\begin{cases} d_{SH}(P_{\mathbf{N}\setminus\{i\}}x,g_i|1) & \text{if } x_i=1,\\ d_{SH}(P_{\mathbf{N}\setminus\{i\}}x,(\bar{\neg}g_i)|0) & \text{if } x_i=0\end{cases}$$

for $i=1,..,n$. It is clear that

$$x_i\neq(Gx)_i \text{ if and only if } (G^\#x)_i\leq 0.$$

If T is an isometry of $\mathbf{Q}^n$, then T also defines an isometry of $\mathbf{Q}^n\times\mathbf{Q}^n$ by $T(x,y)=(Tx,Ty)$. With this definition, let Tg be defined for an isometry of $\mathbf{Q}^n$ and a function $g:\mathbf{Q}^n\times\mathbf{Q}^n\to\mathbf{Q}$ by the Pólya action as $Tg=g\circ T^{-1}$. Let ρ be the circular permutation $(1,2,..,n)$ of $\mathbf{N}$. A mapping $G:\mathbf{Q}^n\times\mathbf{Q}^n\to\mathbf{Q}^n$ is called *circular* if $G\circ\rho=\rho\circ G$. G is called *skew-circular* if $G\circ\rho n^-=\rho n^-\circ G$. Then G is circular iff $g_i=\rho^{I-1}g_1$ for every i in (9.3.3); G is denoted by $\langle g_1\rangle$ in this case. G

is skew-circular if and only if $g_i = (\rho n^-)^{i-1} g_1$ for every i; G is denoted by $\langle\langle g_1 \rangle\rangle$ in this case.

As in previous chapters, if a group $\mathbf{G}$ acts on a set X, then an equivalence relation $\sim_{\mathbf{G}}$ on X can be defined by $x \sim_{\mathbf{G}} y$ if there is an element $\tau \in \mathbf{G}$ such that $\tau x = y$. Each equivalence class with respect to the equivalence relation $\sim_{\mathbf{G}}$, that is, $\mathrm{Orb}_{\mathbf{G}} x$ is denoted by $[x]$. Also, $\mathrm{Orb}_{\mathbf{G}} x$ for a subset $S \subseteq X$ is denoted by $[S]$. Further, the equivalence relation $\sim_{\mathbf{G}}$ is extended to the set of all non-empty subsets of X, that is, $A \sim_{\mathbf{G}} B$ if $\mathrm{Orb}_{\mathbf{G}} A = \mathrm{Orb}_{\mathbf{G}} B$.

As in Chapter 6, it is often more convenient to consider $-F$ in place of F in (9.1.1), or equivalently, $\bar{\neg} G$ in place of G in (9.3.1). Note that $F(x,y) = -F(-x,-y)$, that is, $G(x,y) = \bar{\neg} G(\bar{\neg} x, \bar{\neg} y)$. Therefore, as shown in the following diagram, an orbit x of φ_V generated by G and V can be obtained from a corresponding orbit x' of $\psi_{V'}$ generated by $\bar{\neg} G$ and V', where $V'_i = \bar{\neg} V_i$ if i is even, $V'_i = V_I$ if i is odd, and $x'(0) = \bar{\neg} x(0)$. Specifically, $x(t) = \bar{\neg} x'(t)$ if t is even, and $x(t) = x'(t)$ if t is odd.

$$
\begin{array}{ll}
\varphi_V & \psi_{V'} \\
x(0), & x'(0) = \bar{\neg} x(0), \\
V_0. & V'_0 = \bar{\neg} V_0. \\
x(1) = G(x(0)), v_0), & x'(1) = \bar{\neg} G(x'(0), v'_0) \\
 & = \bar{\neg} G(\bar{\neg} x(0), \bar{\neg} V_0) \\
 & = x(1), \\
V_1. & V'_1 = V_1. \\
x(2) = G(x(1), v_1), & x'(2) = \bar{\neg} G(x'(1), V'_1) \\
 & = \bar{\neg} G(x(1), v_1) \\
 & = \bar{\neg} x(2), \\
V_2. & V'_2 = \bar{\neg} V_2. \\
x(3) = G(x(2), v_2), & x'(3) = \bar{\neg} G(x'(2), v'_2) \\
 & = \bar{\neg} G(\bar{\neg} x(2), \bar{\neg} V_2) \\
 & = x(3), \\
V_3. & V'_3 = V_3. \\
\ldots\ldots\ldots & \ldots\ldots\ldots
\end{array}
$$

9.4. Attractive loops

Let $d(x)$ be the density of $x \in \mathbf{Q}^m$, i.e. $d(x) = |\{i \mid x_i = 1\}|$.

Example 9.4.1 Let $E_{ij} = \epsilon > 0$ for every $i \neq j$ and $C = \epsilon L$ in (9.1.1), where $L_{ij} = 1$ for every i, j.

Lemma 9.4.2 In the PDNN of Example 9.4.1, the transformation G defined by (9.3.1) is

$$
G = \begin{cases}
< p_1 >, & \text{if } 0 < \epsilon < 1/(2n-1); \\
< p_1 \cdot S_1(\neg p_2, .., \neg p_n, \neg q_1, .., \neg q_n) >, & \text{if } 1/(2n-1) < \epsilon < 1/(2n-3); \\
< p_1 \cdot S_2(\neg p_2, .., \neg p_n, \neg q_1, .., \neg q_n) >, & \text{if } 1/(2n-3) < \epsilon < 1/(2n-5); \\
\cdots & \cdots
\end{cases}
$$

Proof. $G_i(x,y) = \mathrm{Bool}((2(d(x \cdot y)) - 2n - 1)\epsilon - 1)$, if $x_i = 1$. Therefore, $(x,y) \in g_i$ if and only if $x_i = 1$ and $(2d(x \cdot y) - 2n - 1)\epsilon - 1 < 0$. Then, $(x,y) \in g_i$ iff $(x_i = 1, d(x \cdot y) \leq 2n,$ and $\epsilon < 1/(2n-1))$, or $(x_i = 1, d(x \cdot y) \leq 2n-1,$ and $\epsilon < 1/(2n-3))$

or ($x_i = 1, d(x \cdot y) \leq 2n-2,$ and $\epsilon < 1/(2n-5)$), or so forth. Therefore, the results of this lemma is obtained. □

Let $0 < 1/(2n-3) < \epsilon < 1/(2n-5)$ in Example 9.4.1. Then by Lemma 9.4.2, $< p_1 \cdot S_2(\neg p_2, .., \neg p_n, \neg q_1, .., \neg q_n) >$. Therefore, (l) is a unique attractive loop for the input 1-sequence $V = (l)$, since $G(x, l) = x$ if and only if $x = l$, and $G(U_1(l), l) = l$. Similarly, (o) is a unique attractive loop for the input 1-sequence $V = (o)$. On the other hand, (l) is a loop for $V = (i^- l)$, but if $x \in U_1(l)$, then $G(x, i^- l) = \bar{\neg} x$ and $G(\bar{\neg} x, i^- l) = x$. Therefore, (l) is not an attractor for $V = (i^- l)$. However, the asymptotic properties of this PDNN are not completely determined by its input. For example, if $x \in U_2(l)$, then $G(x, l) = \bar{\neg} x$ and $G(\bar{\neg} x, l) = x$, that is, x is on a neutral orbit and does not converge to the attractor (l). Therefore, this attractor is bi-dependent.

Next, let $0 < 1/(2n-5) < \epsilon < 1/(2n-7)$. If $V = (l)$, then l is a unique attractive loop for V, since $G(U_2 l, l) = l$. Similarly, (o) is a unique attractive loop for $V = (o)$, and l is an attractor for $V = (i^- l)$. Further, l is also an attractor for $V = (i^- l)$, since $G(U_1 l, i^- l) = l$. But if $x \in U_1 l$, then $G(x, \{i,j\}^- l) = -x$ and $G(-x, \{i,j\}^- l) = x$ for $i \neq j$. Therefore, l is not an attractor for $V = (\{i,j\}^- l)$ such that $i \neq j$. In summary, we have obtained:

Theorem 9.4.3 There exists a PDNN that has a bi-dependent attractive loop.

9.5. An attractiveness condition

Let $H : \mathbf{Q}^n \times \mathbf{Q}^n \to \mathbf{Q}^n$ and $V = (V_0, V_1, .., V_{k-1})$ be a cyclic sequence in $\mathbf{Q}^n$. Then H can define a transformation H_V of $\mathbf{Q}^n \times \mathbf{V}$ by $H_V(x, V_i) = (H(x, V_i), V_{(i+1)\%k})$. Let T be an isometry of $\mathbf{Q}^n$ such that $TV_i = V_{(i+1)\%k}$ for every i. Then $H_V(x, V_i) = (H(x, V_i), TV_i)$.

Further, let H be commutative with T, i.e. $H \circ T = T \circ H$. Then H_V is commutative with T. Let $[S]$ for a subset S of $\mathbf{Q}^n \times \mathbf{Q}^n$ denote $\mathrm{Orb}_{\langle T \rangle} S$. Then, as described in Chapter 2.6, H_V defines a transformation $H_V^{\sim}$ of $[\mathbf{Q}^n \times \mathbf{V}]$ by $H_V^{\sim}[x, y] = [H_V(x, y)]$. Under these conditions, the following proposition holds.

Proposition 9.5.1 Let $W = (W_0, W_1, .., W_{k-1})$ be a cyclic sequence in $\mathbf{Q}^n$ such that $TW_i = W_{(i+1)\%k}$ with the length k being the same as that of V. Let $H_V^{\sim}[U_\epsilon \mathbf{W} \times \mathbf{V}] \subseteq [U_\epsilon \mathbf{W} \times \mathbf{V}]$ for some $\epsilon > 0$, and assume that $\mathrm{CY}(H_V^{\sim} | [U_\epsilon \mathbf{W} \times \mathbf{V}])$ consists of some loops $[Y_1, V_0]\partial, .., [Y_m, V_0]\partial$ for $Y_i \in \mathbf{W}$. Then $\Phi = \mathrm{Orb}_T\{Y_1, .., Y_m\}$ is an attractor of H.

Proof. (i) For the proof of $H(U_\epsilon \mathrm{Im}\Phi \times \mathbf{V}) \subseteq U_\epsilon \mathrm{Im}\Phi$, let $(x, V_i) \in U_\epsilon \mathrm{Im}\Phi \times \mathbf{V} = U_\epsilon \mathbf{W} \times \mathbf{V}$. Then $[x, V_i] \in [U_\epsilon \mathbf{W} \times \mathbf{V}]$, so that $H_V^{\sim}[x, V_i] \in [U_\epsilon \mathbf{W} \times \mathbf{V}]$. Therefore, $H(x, V_i) \in [U_\epsilon \mathbf{W}]$. Since $[U_\epsilon S] = U_\epsilon [S]$ for any S, and $\mathbf{W}$ is $\langle T \rangle$-invariant,we have $H(x, V_i) \in U_\epsilon \mathbf{W} = U_\epsilon \mathrm{Im}\Phi$.

(ii) For the proof of $\Omega_{H, \sigma^j V}(U_\epsilon \mathrm{Im}\Phi) = \sigma^j \Phi$ for any j, let $x \in U_\epsilon \mathrm{Im}\Phi = U_\epsilon \mathbf{W}$. Then, in $\mathrm{GRAPH}(H_V^{\sim} | [U_\epsilon \mathbf{W} \times \mathbf{V}])$, $[x, V_j] \to [(H(x, V_j), V_{(j+1)\%k})] \to ... \to [W_u, V_0]\partial$ for some u such that $W_u = Y_i$ for some i. Therefore, $(x, V_j) \to (H(x, V_j), V_{(j+1)\%k}) \to ... \to (W_{(u+s)\%k}, V_s \to (W_{(j+u)\%k}, V_j) \to ...$ for some s. Therefore, $\Omega_{H, \sigma^j V} x = \sigma^j \mathrm{Orb}_T Y_i$. Therefore, $\Omega_{H, \sigma^j V}(U_\epsilon \mathrm{Im}\Phi) = \sigma^j \Phi$. □

9.6. Attractors in skew-circular PDNNs

Example 9.6.1

$$E = \begin{bmatrix} -1 & \epsilon & \cdot & \cdot & \cdot & \cdot & \epsilon \\ -\epsilon & -1 & \epsilon\cdot & \cdot & \cdot & \epsilon & \\ -\epsilon & -\epsilon & -1 & \epsilon & \cdot & \cdot & \epsilon \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\epsilon & \cdot & \cdot & \cdot & \cdot & -\epsilon & -1 \end{bmatrix}, \tag{9.6.1}$$

where $n \geq 5$ and $1/(n-1) < \epsilon < 1/(n-3)$, and $C = \delta I$.

In Example 9.6.1, if $\delta = 0$, then the transformation G defined by (9.3.1) is

$$G = \langle\langle p_1 \cdot S_1\{\neg p_2, .., \neg p_n\}\rangle\rangle,$$

which is independent of any input V and generates an autonomous PDNN. More specifically in this case, if G' is the transformation of $\mathbf{Q}^n$ defined by $G'x = G(x, a)$ for any a, then $\bar{\neg}G' = \langle\langle p_1 \cdot p_2 \cdots p_n)\rangle\rangle$, so that GRAPH($\bar{\neg}G'$) consists of loops and the unique $2n$-cycle:

$$W = (l, ..., (\rho n^-)^{i-1}l, ..., (\rho n^-)^{2n-1}l).$$

Therefore, this cycle is not attractive in the autonomous FSDS generated by $\bar{\neg}G'$.

Now, we assume $(n-1)\epsilon - 1 < \delta$ and $1 - (n-3)\epsilon < \delta < 1 - (n-5)\epsilon$. Then

$$G = \langle\langle p_1 \cdot (S_2\{\neg p_2, .., \neg p_n\} \vee \neg q_1)\rangle\rangle.$$

Let $H = \bar{\neg}G$. Then

$$H = \langle\langle h_1\rangle\rangle, h = p_1 \cdot q_1 \cdot S_{n-2}\{p_2, .., p_n\}.$$

Therefore, $x_i = 1$ and $H_i(x, y) = 0$, iff $y_i = 1$ and $(x = \{1, .., i-1\}^- l$ or $x = j^-\{1, .., i-1\}^- l)$, where $j \neq i$.

Let $a_i = i\%2$ for every i. Then

$$S_{n-2}\{p_2, .., p_n\}((\rho n^-)^{-(i-1)}a) = 0,$$

so that $(\rho n^-)^{-(i-1)}h(a, y) = 0$ for every i for any y. Therefore, $H(a, y) = a$ for any y; similarly $H(\bar{\neg}a, y) = \bar{\neg}a$ for any y, that is, (a) and $(\bar{\neg}a)$ are loops for any input sequence.

Now consider the points l of W and let l^i denote $1, .., i^- l$ and o^i denote $1, .., i^- o$. We have $l_1 = (o^1)_1 = 1$, $S_{n-2}\{p_2, .., p_n\}l = 1$, and $(o^1)_i = 0$ for any $i \neq 1$. Therefore, $H(l, o^1) = l^1$, so that

$$H((\rho n^-)^{i-1}l, (\rho n^-)^{i-1}o^1) = (\rho n^-)^{i-1}l^1 = (\rho n^-)^i l.$$

Therefore, W is a periodic orbit in the FSDS generated by H and the input

$$V = (o^1, \rho n^- o^1, ..., (\rho n^-)^{2n-1}o^1).$$

H is skew-circular, that is, commutative with $T = \rho n^-$. The values of the extended representation $H^\#(l,y)$ for $y \in \mathbf{V}$ are

$$\begin{array}{rcl}
H^\#(l,o^1) &=& (0,1,2,...,n-1),\\
H^\#(l,o^2) &=& (0,0,2,3,...,n-1),\\
H^\#(l,o^3) &=& (0,0,1,3,4,...,n-1),\\
H^\#(l,o^4) &=& (0,0,1,2,4,5,...,n-1),\\
.. && ...\\
H^\#(l,o^{n-1}) &=& (0,0,1,2,...,n-3,n-1),\\
H^\#(l,l) &=& (0,0,1,2,...,n-3,n-2),\\
H^\#(l,l^1) &=& (1,0,1,2,...,n-2),\\
H^\#(l,l^2) &=& (1,1,1,2,3,...,n-2),\\
H^\#(l,l^3) &=& (1,1,2,2,3,...,n-2),\\
H^\#(l,l^4) &=& (1,1,2,3,3,4,...,n-2),\\
.. && ...\\
H^\#(l,l^{n-1}) &=& (1,1,2,...,n-2,n-2),\\
H^\#(l,l^n) &=& (1,1,2,,...,n-2,n-1),
\end{array}$$

Therefore, in GRAPH($H_{\tilde{V}}$),

$$\begin{array}{ccccccccc}
[l,l] & \to & [l,o^{n-1}] & \to & [l,o^{n-2}] & \to & ... & \to & [l,o^2]\\
 & & & & & & & & \downarrow\\
[2^-l,l^2] & \leftarrow & [l,l^1] & & & & & & [l,o^1]\partial\\
 & & & & & & & & \uparrow\\
[l,l^2] & \to & [l,l^3] & \to & ... & \to & [l,l^{n-1}] & \to & [l,l^n]
\end{array} \qquad (9.6.2)$$

Now, consider $H(j^-l,y)$ for $3 \le j \le n-1$ and $y \in \mathbf{V}$. First for $y = l, o^{n-1}, .., o^1$,

$$\begin{array}{rcl}
(H^\#(j^-l,y))_1 &=& 0,\\
(H^\#(j^-l,y))_2 &\ge& 1,\\
(H^\#(j^-l,y))_i &\ge& 2 \text{ if } 3 \le i \ne j.
\end{array}$$

Therefore, $[j^-l,y] \to [1^-l,\rho n^-y] = [l,y]$ or

$$[j^-l,y] \to [1,j^-l,\rho n^-y] = [(j-1)^-l,y],$$

so that

$$\begin{array}{ccccccl}
[j^-l,y] & \to & ... & \to & [2^-l,y] & \text{or} & \\
[j^-l,y] & \to & ... & \to & [k^-l,y] & \text{for some } k. &
\end{array} \qquad (9.6.3)$$

On the other hand, for $y = l^1, l^2, .., l^n$,

$$\begin{array}{rcl}
(H^\#(j^-l,y))_1 &\ge& 1,\\
(H^\#(j^-l,y))_2 &\ge& 1,\\
(H^\#(j^-l,y))_i &\ge& 2 \text{ if } 3 \le i \ne j.
\end{array}$$

Therefore,

$$[j^-l,y] \to [l,\rho n^-y] \text{ or } [j^-l,y] \to [j^-l,\rho n^-y],$$

so that

$$\begin{array}{cccccl}
[j^-l,y] & \to & ... & \to & [j^-l,o^1] & \text{or}\\
[j^-l,y] & \to & ... & \to & [l,l^k] & \text{for some } k \ge 2.
\end{array} \qquad (9.6.4)$$

Now, consider $H(2^{-}l, y)$ for $y \in \mathbf{V}$. First for $y = l, o^{n-1}, .., o^3$,

$$\begin{aligned}(H^{\#}(2^{-}l, y))_1 &= 0,\\ (H^{\#}(2^{-}l, y))_2 &= n-2,\\ (H^{\#}(2^{-}l, y))_3 &= 0,\\ (H^{\#}(2^{-}l, y))_i &\geq 1 \text{ if } i \geq 4.\end{aligned}$$

Therefore,

$$[2^{-}l, y] \rightarrow [l^3, \rho n^{-} y] = [l, (\rho n^{-})^{-2} y]. \tag{9.6.5}$$

Next, $H^{\#}(2^{-}l, o^2) = (0, n-2, 1, 2, .., n-2)$, so that

$$[2^{-}l, o^2] \rightarrow [l^2, o^3] = [l, o^1]. \tag{9.6.6}$$

Also, $H^{\#}(2^{-}l, o^1) = (0, n-3, 1, 2, .., n-2)$, so that

$$[2^{-}l, o^1] \rightarrow [l^2, o^2] = [l, o]. \tag{9.6.7}$$

On the other hand, for $y = l^3, .., l^6$,

$$\begin{aligned}(H^{\#}(2^{-}l, y))_1 &\geq 1,\\ (H^{\#}(2^{-}l, y))_2 &= n-3,\\ (H^{\#}(2^{-}l, y))_i &\geq 1 \text{ if } i \geq 3.\end{aligned}$$

Therefore,

$$[j^{-}l, y] \rightarrow [2^{-}l, \rho n^{-} y] \rightarrow .. \rightarrow [2^{-}, o^1]. \tag{9.6,7}$$

Further, $H^{\#}(2^{-}l, l^2) = (1, n-3, 0, 1, .., n-3)$, so that

$$[(n-2)^{-}l, l] = [2, 3^{-}l, l^3] \leftarrow [2^{-}l, l^2]. \tag{9.6.8}$$

Also, $H^{\#}(2^{-}l, l^1) = (1, n-2, 0, 1, 2, .., n-3)$, so that

$$[(n-2)^{-}l, o^{n-1}] = [2, 3^{-}l, l^2] \leftarrow [2^{-}l, l^1]. \tag{9.6.9}$$

Therefore, $H^{\sim}_{V}[U_1\mathbf{W} \times \mathbf{V}] \subseteq [U_1\mathbf{W} \times \mathbf{V}]$, and $\mathrm{CY}(H^{\sim}_{V}|[U_1\mathbf{W} \times \mathbf{V}])$ consists of a unique loop $[l, o^1]\partial$. To confirm this, it suffices to check the limit cycles of all vertices incident to the left arrows $\leftarrow$ in (9.6.2), (9.6.8), and (9.6.9). Therefore, by Proposition 9.5.1, W is an attractor. By translating from H into $G = \bar{\neg} H$, we have obtained:

Theorem 9.6.2 In the PDNN generated by E of (9.6.1) and the input cyclic sequence

$$V = (l^1, (\rho n^{-})^{(n+1)} l^1, .., (\rho n^{-})^{(n+1)(i-1)} l^1, ...),$$

the cyclic orbit

$$W = (o, (\rho n^{-})^{(n+1)} o, .., (\rho n^{-})^{(n+1)(i-1)} o, ...)$$

is a bi-dependent attractor. If the initial state is either a or $\bar{\neg} a$, where $a_i = i\%2$, then every neuron performs spontaneous firing for any input.

In the above theorem, if n is even, then both V and W are cyclic $2n$-sequences, since $n+1$ and $2n$ are relatively prime. If n is odd, then both V and W are cyclic n-sequences, since $\gcd(n+1, 2n) = 2$. In this case,

$$W\prime = (o^1, (\rho n^{-})^{(n+1)} o^1, .., (\rho n^{-})^{(n+1)(n-1)} o^1)$$

is an attractor in the PDNN generated by E and input

$$V\prime = (l^2, (\rho n^-)^{(n+1)} l^2, .., (\rho n^-)^{(n+1)(n-1)} l^2).$$

In both cases, note that each attractor consists of one cyclic orbit. Therefore, there is no possibility of shifting from one cyclic orbit to another within an attractor, caused by some noise, unlike in some autonomous neural networks. Computational results expect that $U_1(\mathbf{W})$ is the basin for the attractor and that the limit orbit of any point outside the basin is neutral.

9.7 Attractors in circular PDNNs

As an example of autonomous circular transformations, we showed the existence of a non-unique attractive $2m$-cycle in Chapter 7 (Example 7.1.2). We also showed the existence of a non-unique attractor consisting of two $(2m-1)$-cycles (Example 7.1.4). In this section we show how an autonomous network is modified by input in a corresponding non-autonomous network, so that such a non-unique attractor becomes a unique but bi-dependent attractor. The bi-dependence means that asymptotic properties for this attractor are not completely dictated by the input and are also dependent on initial states. Corresponding to Example 7.1.2 of Chapter 7, the following example shows the existence of a bi-dependent non-autonomous circular threshold transformation having a unique attractor.

Example 9.7.1 Let $n = 2m$, and let $H =< h >$: $\mathbf{Q}^{2m} \times \mathbf{Q}^{2m} \to \mathbf{Q}^{2m}$ be defined by

$$h = p_1 \cdot p_m \cdot \neg p_{2m} \cdot \neg q_1 \cdot q_m \cdot \neg q_{2m}.$$

In Example 9.7.1, $x_i = 1$ and $H_i(x, y) = 0$, iff $x_{i+m-1} = 1$, $x_{i+2m-1} = 0$, $y_i = 0$, $y_{i+m-1} = 1$, and $x_{i+2m-1} = 0$.

Let ρ be the cyclic permutation $(1, 2, .., 2m)$. Let o be the zero vector of $\mathbf{Q}^{2m}$ and o^i be defined as in the previous sections. Let W be the $2m$-sequence

$$W = (o^m, \rho o^m, ..., \rho^{2m-1} o^m).$$

Let $v = \rho^{-1} l^{m-1}$ and let the input $2m$-sequence V be

$$V = (v, \rho v, ..., \rho^{2m-1} v).$$

We have

$$H(\rho^i o^m, v) = \begin{cases} \rho\rho^i o^m & \text{if } 0 \le i \le m-3, \\ \rho^i o^{m+1} & \text{if } i = m-2, \\ \rho^i o^m & \text{if } m-1 \le i \le 2m-1. \end{cases} \tag{9.7.1}$$

Therefore, for any i such that $0 \le i \le m-3$, $H(\rho^j \rho^i o^m, \rho^j v) = \rho^{j+1} \rho^i o^m$ for any j. Therefore,

$$\mathrm{Orb}_{H,V}(\rho^i o^m) = (\rho^i o^m, \rho^{i+1} o^m, ..., \rho^{i-1} o^m) = \sigma^i W \in PO(H, V)$$

for $0 \le i \le m-3$. Let

$$\Phi = \{\sigma^i W | 0 \le i \le m-3\}.$$

Our objective is to prove Φ is an attractor in the FSDS generated by H and V.

Let

$$C_i = \{x \mid x \in \mathbf{Q}^{2m}, d_H(\rho^m x, x) = 2i\}.$$

Then clearly $H(x, y) = x$ for any $x \in C_0$ for any $y \in \mathbf{Q}^{2m}$; in particular, (x) is a loop for the input $\sigma^i V$ for any i.

Lemma 9.7.2 $C_0^c = U_{m-1}\mathbf{W}$, where the prefix ^{c} denotes the complement of a subset.

Proof. For $x \in \mathbf{Q}^{2m}$, let j be such that $n_j = |\{i \mid i \leq m, (\rho^j x)_i = 1\}|$ is the maximum. Let $y = \rho^j x$. Let

$$\begin{aligned} \alpha &= |\{i \mid i \leq m, y_i = 1, y_{m+i} = 0\}|, \\ \beta &= |\{i \mid i \leq m, y_i = 0, y_{m+i} = 1\}|, \\ \gamma &= |\{i \mid i \leq m, y_i = 1, y_{m+i} = 1\}|, \\ \delta &= |\{i \mid i \leq m, y_i = 0, y_{m+i} = 0\}|. \end{aligned}$$

Let $x \in C_0^c$. Then $x \in C_k$ for some $k \geq 1$, so that $\beta < \alpha$. Therefore, $d_H(y, o^m) = 2\beta + \gamma + \delta < \alpha + \beta + \gamma + \delta = m$. Therefore, $y \in U_{m-1}o^m$, so that $x \in U_{m-1}\mathbf{W}$. Conversely, let $x \in U_{m-1}\mathbf{W}$. Then $2\beta + \gamma + \delta \leq m - 1$, so that $\beta < \alpha$. Therefore, $\alpha \geq 1$, so that $x \in C_0^c$. □

The following Lemmas are clear.

Lemma 9.7.3 (i) If $x_{i-1} = x_{m+i-1}$, then

$$(H_i(x, v), H_{m+i}(x, v)) = (x_i, x_{m+i})$$

for every i. (ii) If $(H_i(x, v), H_{m+i}(x, v)) = (0, 1)$ then

$$(x_{i-1}, x_{m+i-1}) = (0, 1) \text{ or } (x_{i-1}, x_{m+i-1}) = (0, 1)$$

for every i.

Lemma 9.7.4 (i) If $(x_i, x_{m+i}) = (0, 1)$ for some $1 \leq i \leq m$, then

$$(H_i(x, v), H_{m+i}(x, v)) = (0, 1).$$

(ii) If $(x_1, x_{m+1}) = (0, 1)$, then

$$(H_2(x, v), H_{m+2}(x, v)) = (0, 1).$$

Lemma 9.7.5 (i) If $(x_i, x_{m+i}) = (1, 0)$ for some $1 \leq i \leq m$, and $(x_{i-1}, x_{m+i-1}) \neq (0, 1)$, then

$$(H_i(x, v), H_{m+i}(x, v)) = (1, 0).$$

(ii) If $(x_i, x_{m+i}) = (1, 0)$ for some $1 \leq i \leq m - 2$, and $(x_{i-1}, x_{m+i-1}) = (0, 1)$, then

$$(H_i(x, v), H_{m+i}(x, v)) = (0, 1).$$

Lemma 9.7.6 (i) If $(x_m, x_{2m}) = (1, 0)$, then

$$(H_m(x, v), H_{2m}(x, v)) = (1, 0) \text{ and } (H_1(x, v), H_{m+1}(x, v)) = (0, 1).$$

(ii) If $(x_{m-1}, x_{2m-1}) = (1, 0)$ and $(x_{m-2}, x_{2m-2}) = (0, 1)$, then

$$(H_{m-1}(x, v), H_{2m-1}(x, v)) = (1, 1).$$

(iii) If $(x_{m-2}, x_{2m-2}) = (1, 1)$ and $(x_{m-3}, x_{2m-3}) = (0, 1)$, then

$$(H_{m-2}(x, v), H_{2m-2}(x, v)) = (0, 1).$$

In Example, 9.7.1, H is circular, so that H is commutative with $T = \rho$. Further, $TV_i = V_{(i+1)\%k}$ for every i. Therefore, the transformation H_V of $\mathbf{Q}^n \times \mathbf{V}$ defined by $H_V(x, V_i) = (H(x, V_i), V_{(i+1)\%k})$ is commutative with ρ. Let $[S]$ for a subset S of $\mathbf{Q}^n \times \mathbf{Q}^n$ denote $\mathrm{Orb}_{\langle\rho\rangle} S$. Then, we have $[U_{m-1}\mathbf{W} \times \mathbf{V}] = [U_{m-1}\mathbf{W} \times v]$. Therefore, $[U_{m-1}\mathbf{W} \times \mathbf{V}] = [C_0^c \times v]$ by Lemma 9.7.2. We write $x \to y$ if $y = \rho^{-1}H(x, v)$ i.e. $H_V(x, v) = \rho(y, v)$ in the following. Now let $x \in C_0^c$.

Case 1: Assume there is no i such that $1 \leq i \leq m$ and $(x_i, x_{m+i}) = (0, 1)$. Let $1 \leq k \leq m$ be the minimum k such that $(x_k, x_{m+k}) = (1, 0)$. Then, by Lemma 9.7.3 (i), $x \to ... \to x\prime$, where $(x\prime_1, x\prime_{m+1}) = (1, 0)$. If there is some j such that $2 \leq j \leq m$ and $(x\prime_j, x\prime_{m+j}) = (0, 1)$, then go to Case 2. Suppose there is no j such that $2 \leq j \leq m$ and $(x\prime_j, x\prime_{m+j}) = (0, 1)$. Let $x\prime \to x\prime\prime$. First, let $(x\prime_m, x\prime_{2m}) = (1, 0)$. Then, by Lemma 9.7.6 (i), $(x\prime\prime_m, x\prime\prime_{2m}) = (x\prime\prime_{m-1}, x\prime\prime_{2m-1}) = (1, 0)$. Further, there is no j such that $1 \leq j \leq m-2$ and $(x\prime\prime_j, x\prime\prime_{m+j}) = (0, 1)$ by Lemma 9.7.3 (ii). Therefore, $x\prime\prime \to ... \to o^m$. If $(x\prime_m, x\prime_{2m}) \neq (1, 0)$, so that $x\prime_m = x\prime_{2m}$, then $(x\prime\prime_m, x\prime\prime 2m) = (0, 1)$ by Lemma 9.7.3 (i), and we go to Case 2.

Case 2: Assume $(x_k, x_{m+k}) = (0, 1)$ for some $1 \leq k \leq m$. Then $x \to ... \to x\prime$, where $(x\prime_1, x\prime_{m+1}) = (0, 1)$ by Lemma 9.7.4 (i). Let $x\prime \to x\prime\prime$. Then $(x\prime\prime_1, x\prime\prime_{m+1}) = (0, 1)$ by Lemma 9.7.4 (ii), and $(x\prime\prime_m, x\prime\prime_{2m}) = (1, 0)$ by Lemma 9.7.4 (i). Let $x\prime\prime \to y$. Then, $(y_1, y_{m+1}) = (0, 1)$ by Lemma 9.7.4 (ii), $(y_{m-1}, y_{2m-1}) = (1, 0)$ by Lemma 9.7.6 (i), and $(y_m, y_{2m}) = (1, 0)$ by Lemma 9.7.4 (i). Let $y \to y\prime$.

Suppose $(y_{m-2}, y_{2m-2}) = (0, 1)$. Then, $(y\prime_{m-3}, y\prime_{2m-3}) = (0, 1)$ by Lemma 9.7.4 (i), $(y\prime_{m-2}, y\prime_{2m-2}) = (1, 1)$ by Lemma 9.7.6 (ii), $(y\prime_{m-1}, y\prime_{2m-1}) = (1, 0)$ by Lemma 9.7.6 (i), $(y\prime_m, y\prime_{2m}) = (1, 0)$ by Lemma 9.7.4 (i), and $(y\prime_1, y\prime_{m+1}) = (0, 1)$ by Lemma 9.7.4 (ii). Let $y\prime \to y\prime\prime$. Then, $(y\prime\prime_{m-4}, y\prime\prime_{2m-4}) = (0, 1)$ by Lemma 9.7.4 (i), $(y\prime\prime_{m-3}, y\prime\prime_{2m-3}) = (0, 1)$ by Lemma 9.7.6 (iii), $(y\prime\prime_{m-2}, y\prime\prime_{2m-2}) = ((y\prime\prime_{m-1}, y\prime\prime_{2m-1}) = (1, 0)$ by Lemma 9.7.5 (i), $(y\prime\prime_m, y\prime\prime_{2m}) = (1, 0)$ by Lemma 9.7.4 (i), and $(y\prime\prime_1, y\prime\prime_{m+1}) = (0, 1)$ by Lemma 9.7.4 (ii). Therefore, $y\prime\prime ... \to \rho^{m-3}o^m$.

Suppose $(y_{m-2}, y_{2m-2}) \neq (0, 1)$. Then $(y\prime_{m-2}, y\prime_{2m-2}) = (y\prime_{m-1}, y\prime_{2m-1}) = (1, 0)$ by Lemma 9.7.5 (i), $(y\prime_m, y\prime_{2m}) = (1, 0)$ by Lemma 9.7.4 (i), and $(y\prime_1, y\prime_{m+1}) = (0, 1)$ by Lemma 9.7.4 (ii). Therefore, $y\prime \to y\prime\prime$, where $(y\prime\prime_{m-2}, y\prime\prime_{2m-2}) = (y\prime\prime_{m-1}, y\prime\prime_{2m-1}) = (y\prime\prime_m, y\prime\prime_{2m}) = (1, 0)$, and $(y\prime\prime_1, y\prime\prime_{m+1}) = (0, 1)$. Let $y\prime\prime \to z$. Then $(z_{m-3}, z_{2m-3}) = (1, 0)$ or $(0, 1)$ by Lemma 9.7.5 (i), (ii), $(z_{m-2}, z_{2m-2}) = (z_{m-1}, z_{2m-1}) = (z_m, z_{2m}) = (1, 0)$, and $(z_1, z_{m+1}) = (0, 1)$. Therefore, $z \to ... \to \rho^i o^m$ for some $1 \leq i \leq m-3$.

Therefore, Φ is an attractor in the FSDS generated by H of Example 9.7.1 and the input V. By translating from H into $\bar{\neg} H$, we have obtained:

Theorem 9.7.7 In the PDNN generated by $\bar{\neg} H$ for Example 9.7.1 and the input cyclic sequence $\mathrm{Orb}_{\rho\bar{\neg}}\rho^{-1}o^{m-1}$, $\mathrm{Orb}_{\rho^{m+1}}\{\rho^{i(m+1)}l^m | 0 \leq i \leq m-3\}$ is a bi-dependent attractor. Further, any point of C_0 is on a neutral orbit, and C_0^c is the basin for the attractor.