Standard Dilations of Tuples and Representations of Certain C^*-algebras

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Thesis submitted to Indian Statistical Institute
in partial fulfillment of the requirements
for the award of degree of

Doctor of Philosophy

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BANGALORE
30^{th} MAY 2004

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Acknowledgements

I am grateful and deeply indepted to my guru Professor B. V. Rajarama Bhat for his teaching and guidance. I have learnt a lot about operator algebras from conversation and discussions with him. His friendly attitude and encouragement was also of great help during my thesis work.

Besides Prof. Bhat I have joint publications with Dr. J. Zacharias and Dr. T. Bhattacharyya. I am thankful to them for letting me use these publications as part of my thesis and for the knowledge I gained while doing these research works. I acknowledge the support I received from my parents I would like to express my gratitude to all my teachers, faculty of Stat-Math Unit, Indian Statistical Institute Bangalore Center, especially Professor G. Misra and to Dr. Arupkumar Pal (ISI, Delhi Center) for training I received from them and for their generous help in my academic pursuit. I am grateful to Indian Statistical Institute for providing me research fellowship and excellent environment for research.

I am thankful to Professor M. Schurmann and his collegues at Greifswald, Germany for imparting knowledge on some aspects of operator algebras and bialgebras, and to NBHM for funding me to visit Ernst-Moritz-Arndt-Universität, Greifswald, Germany to attend a short term school on quantum probability and related topics which was helpful to me in my research. I wish to thank my friends at the Institute who have also made my stay here enjoyable during my thesis work.

Chapter 1

Introduction

In 1953 Sz.-Nagy [SF] showed that every contraction on a Hilbert space dilates to a unitary and moreover the dilation is unique under a natural minimality assumption. This was a fundamental result and using this, one has simpler proofs of von Neumann's inequality and other strong results. In fact von Neumann's inequality is a necessary and sufficient condition for a Banach space to become a Hilbert space ([Pi], page: 26). Ando [An] obtained minimal dilation to commuting unitaries for pairs of commuting contractions. But Varopoulos, Crabb-Davie and Parrott (see [Pi]) gave examples to establish that simultaneous dilation to n commuting unitaries is not possible for n commuting contractions with $n \geq 3$. Eventually a successful theory of minimal dilation to isometries with orthogonal ranges called *minimal isometric dilation* or *standard noncommuting dilation* was developed by Bunce, Frazho and Popescu ([Bu][Fr1-2][Po1-5]). This dilation is for a class of operator tuples defined as follows.

DEFINITION 1.0.1 A *contractive n-tuple,* or a *row contraction* is a n-tuple $\underline{T} = (T_1, \ldots, T_n)$ of bounded operators on a Hilbert space $\mathcal{H}$ such that $T_1T_1^* + \cdots + T_nT_n^* \leq I$.

In this thesis we are mainly concerned with three types of dilations of tuples, one of which is the minimal isometric dilation as referred above, the other two introduced by us namely standard q-commuting dilation (standard commuting dilation is a particular case of this) and minimal Cuntz-Krieger dilation. The standard commuting dilation was introduced by Drury, Arveson and Popescu (refer [Du][Ar4][Po5]) and was used crucially in the study of multivariate analogue of von Neumann's inequality, Poisson transforms, operator spaces, invariants of Hilbert modules and others. Athavale [At1-2] and Agler [Ag] used this type of dilations while considering reproducing kernel Hilbert spaces.

Popescu [Po1-5] has shown that many of the single operator dilation theory of Sz-Nagy and Foias holds for minimal isometric dilation of n-tuples also (in fact for infinite sequence of tuples also). Davidson, Kribs, Pitts and Shpigel ([DKS][DP1-2]) have given a fine decomposition of the WOT-closed algebras generated by this dilations and used them to get invariants for similarity. Popescu and Kribs also used this dilation to develop noncommutative analogue of Arveson's curvature invariants and Euler characteristics of Hilbert modules.

In Chapter 2 of this thesis which is based on joint work with Bhat and Bhattacharyya [BBD] we obtain a fundamental relationship between the minimal isometric dilation and the standard commuting dilation. We introduce the concept of 'maximal piece' of a tuple of operators satisfying certain relations with respect to a set of polynomials. This concept becomes useful to study all types of dilations that we are interested in this thesis. For each type we make appropriate choice of the set of polynomials. In this Chapter we select the set of polynomials so that it induces commuting property in the maximal piece and we call this as 'maximal commuting piece'. An alternate procedure of constructing minimal isometric dilation using positive definite kernel is also given followed by a summary of generalization of von Neumann's inequality. In Section 2.2 we show that the minimal isometric dilation of the standard commuting dilation of a commuting tuple coincides with the minimal isometric dilation of the tuple. Here as well as while answering similar questions related to other dilations we use the concrete presentation of minimal isometric dilation given by Popescu [Po1]. In section 2.3 we give a complete classification of the representations of Cuntz algebras which arise out of the dilation of commuting contractive tuples. We are able to do this by showing that these representations are related to the GNS representations of Cuntz states. This one is a major achievements of the theory of identifying the standard commuting dilation as maximal commuting piece. Then we quote the result from [DKS] giving the complete description of WOT-closed algebras generated by minimal isometric dilation. Recently, there has been a lot of effort to study such representations in connection with wavelet theory, see for instance the papers [BJ1-2] of Bratteli and Jorgensen. The notion of spherical unitaries of Athavale and Arveson is very useful in this.

In [De], we studied the dilations of 'q-commuting tuples' and these are defined as follows:

DEFINITION 1.0.2 A n-tuple $\underline{T} = (T_1, \ldots, T_n)$ is said to be *q-commuting* if $T_jT_i = q_{ij}T_iT_j$ for all $1 \leq i, j \leq n$, where q_{ij} are non-zero complex numbers. (To avoid trivialities we assume that $q_{ij} = q_{ji}^{-1}$).

When the matrix $q = (q_{ij})_{n\times n}$ is chosen to be the one which has all entries 1, this dilation becomes the standard commuting dilation. Chapter 3 in this thesis is based on Dey [De]. For a q-commuting tuple on a finite dimensional Hilbert space $\mathcal{H}$ say of dimension m, q_{ij} are either 0 or m^{th}-roots of unity. But there are no such restrictions on infinite dimensional Hilbert spaces. There are some interesting results about dilations of q-commuting tuples in a work of Bhat and Bhattacharyya [BB]. The notion of q-commuting tuples and related dilations become important because of its occurrence in Quantum Theory ([Co][Ma][Pr]). Also such tuples are simplest noncommuting tuples and so are helpful when one enquires about the form of certain notions for noncommuting tuples, which already exist for commuting tuples. We are interested in checking whether the standard q-commuting dilation sits inside the minimal isometric dilation, and this would be a generalization of an important result we have in previous Chapter. For answering this question affirmatively we need a new q-Fock space (one particular case of which is the symmetric Fock space). These are also subspaces of full Fock space like the symmetric Fock space and each comes through some representation of permutation groups. In fact these representations become unitarily equivalent for different choices of matrix q. These q-Fock spaces are different from those of Bożejko, Speicher and Jorgensen. We give another description for this through a particular representation of permutation group. We give a formula for the projection of full Fock space onto this space. On this Fock space we consider a special tuple of q-commuting operators and show that it is unitarily equivalent to the tuple of shift operators of [BB]. The universal properties of standard q-commuting dilation using methods similar to those used in [Po4] for minimal isometric dilation have been discussed in Section 3.3. In Section 3.4, we calculate the distribution of $S_i + S_i^*$ with respect to the vacuum expectation for standard tuple $\underline{S}$ associated with $\Gamma_q(\mathbb{C}^n)$ and study some properties of the related operator spaces.

We look at the minimal Cuntz-Krieger dilation in Chapter 4 which is based on joint work with Bhat and Zacharias [BDZ]. Let $A = (a_{ij})_{n\times n}$ be a $0-1$-matrix, i.e., $a_{ij} \in \{0, 1\}$ and each row and column of A has atleast one entry 1. Then Cuntz Krieger algebra $\mathcal{O}_A$ is defined as follows:

DEFINITION 1.0.3 *Cuntz-Krieger algebra $\mathcal{O}_A$* is the unital C^*-algebra generated by n partial isometries $s_1, \cdots, s_n$ with orthogonal ranges satisfying

$$s_i s_j = a_{ij} s_i s_j, \qquad s_i^* s_i = \sum_{j=1}^n a_{ij} s_j s_j^*,$$
$$I = \sum_{i=1}^n s_i s_i^*.$$

The above equations are called *Cuntz-Krieger relations.* Cuntz-Krieger dilations are related to Cuntz-Krieger algebras and they arise when the following type of operator tuples are dilated:

DEFINITION 1.0.4 A n-tuple $\underline{T}$ is said to satisfy *A-relations* if $T_i T_j = a_{ij} T_i T_j$ for $1 \leq i, j \leq n$ where $A = (a_{ij})_{n\times n}$.

Cuntz-Krieger algebras are some simple C^*-algebras which are not stably isomorphic to Cuntz algebras and this were studied by J. Cuntz and W. Krieger [CK] in connection with topological Markov chains. Minimal Cuntz-Krieger dilation is introduced as study of this dilation sheds light on the structure of Cuntz-Krieger algebras (and Cuntz-Krieger-Toeplitz algebras) and its decomposition. This dilation is constructed using Popescu's Poisson transform. An alternate construction of minimal Cuntz-Krieger dilation using positive definite kernels is also given. Here we address the question as to how the minimal Cuntz-Krieger dilation and minimal isometric dilation are related (Section 4.3). This helps us in classifying Cuntz-Krieger algebras arising out of dilation of contractive tuples. For this we need to consider a subspace of the full Fock space which we call A-Fock space, obtained by enforcing A-relations on the m-particle spaces. Section 4.4 starts with some results on universal properties of minimal Cuntz-Krieger dilation which generalize the results related to minimal isometric dilation of [Po4]. Further the related WOT-closed algebra generated by the operators constituting the minimal Cuntz-Krieger dilation tuple has been studied using similar methods as that of Davidson, Kribs, Pitts and Sphigel ([DKS][DP1-2]).

In Chapter 5 we discuss some examples of these dilations. Of special mention is that the Fermionic Fock space has been realised as a space associated with the maximal piece for certain set of polynomials.

We will consider complex and separable Hilbert spaces. For a subspace $\mathcal{H}$ of a Hilbert space, $P_{\mathcal{H}}$ will denote the orthogonal projection onto $\mathcal{H}$. For any Hilbert space $\mathcal{K}$, we have the full Fock space over $\mathcal{K}$ denoted by $\Gamma(\mathcal{K})$ and the Boson (or symmetric) Fock space over $\mathcal{K}$ denoted by $\Gamma_s(\mathcal{K})$ defined as,

$$\Gamma(\mathcal{K}) = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K}^{\otimes^2} \oplus \cdots \oplus \mathcal{K}^{\otimes^m} \oplus \cdots,$$

$$\Gamma_s(\mathcal{K}) = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K}^{\circledS^2} \oplus \cdots \oplus \mathcal{K}^{\circledS^m} \oplus \cdots,$$

where $\mathcal{K}^{\circledS^m}$ denotes the m-fold symmetric tensor product. We will consider the Boson Fock space as a subspace of the full Fock space in the natural way. We denote the vacuum vector $1 \oplus 0 \oplus \cdots$ by ω. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. The left creation operator V_i and the right creation operators Y_i in $\Gamma(\mathbb{C}^n)$ are defined by

$$V_i x = e_i \otimes x, \qquad Y_i x = x \otimes e_i$$

where $1 \leq i \leq n$ and $x \in \Gamma(\mathbb{C}^n)$ (Of course, here $e_i \otimes \omega$ and $\omega \otimes e_i$ is interpreted as e_i). V_i's are clearly isometries with orthogonal ranges and so are Y_i's. We denote the tuples $(V_1, \cdots, V_n)$ and $(Y_1, \cdots, Y_n)$ by $\underline{V}$ and $\underline{Y}$ respectively. Also $\sum V_i V_i^* = \sum Y_i Y_i^* = I - E_0 \leq I$, where E_0 is the projection on to the vacuum space.

For operator tuples $(T_1, \ldots, T_n)$, we need to consider the products of the form $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$, where each $\alpha_k \in \{1, 2, \ldots, n\}$. We would have the following notation for such products. Let Λ denote the set $\{1, 2, \ldots, n\}$ and Λ^m denote the m-fold cartesian product of Λ for $m \geq 1$. Given $\alpha = (\alpha_1, \ldots, \alpha_m)$ in Λ^m, $\underline{T}^\alpha$ will mean the operator $T_{\alpha_1} T_{\alpha_2} \cdots T_{\alpha_m}$. Let $\tilde{\Lambda}$ denote $\cup_{m=0}^{\infty} \Lambda^m$, where Λ^0 is just the set $\{0\}$ by convention and by $\underline{T}^0$ we would mean the identity operator of the Hilbert space where T_i's are acting. Let $\mathcal{S}_m$ denote the group of permutation on m symbols $\{1, 2, \cdots, m\}$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. For $\alpha \in \tilde{\Lambda}$, e^α will denote the vector $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m}$ in the full Fock space $\Gamma(\mathbb{C}^n)$ and e^0 will denote the vacuum vector ω.

Preliminaries

We begin with a brief description of dilation theory of single contraction and von Neumann inequality. Let V be an isometry on a Hilbert space $\mathcal{K}$. Then $\mathcal{K}$ can be decomposed uniquely into an orthogonal sum of reducing subspaces $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, for V such that the compression of V to $\mathcal{K}_0$ and $\mathcal{K}_1$ are unitary and unilateral shift respectively. Here $\mathcal{K}_0 = \cap_{n=0}^{\infty} V^n \mathcal{K}$ and $\mathcal{K}_1 = \oplus_{n=0}^{\infty} V^n (\mathcal{K} \ominus V\mathcal{K})$. This decomposition is called *Wold decomposition* for a single isometry. Sz.-Nagy showed that for every contraction T, i.e., $\|T\| \leq 1$ on a Hilbert space $\mathcal{H}$, there exists a minimal isometric dilation V which is unique upto unitary equivalence. By this one means that V is a isometry on some Hilbert space $\mathcal{K}$ which keeps $\mathcal{H}^\perp$ invariant,

$$P_{\mathcal{H}} V|_{\mathcal{H}} = T \quad \text{and} \quad \mathcal{K} = \overline{\text{span}}\{V^m(h) : h \in \mathcal{H}, m \in \mathbb{N} \cup \{0\}\}.$$

To obtain such a dilation we define $D_T := (I - T^*T)^{\frac{1}{2}}$, $\mathcal{D}_T$ as the range of D_T and $V : \mathcal{H} \oplus (l^2 \otimes \mathcal{D}_T) \to \mathcal{H} \oplus (l^2 \otimes \mathcal{D}_T)$ by $V(h_0, h_1, \cdots) = (Th_0, D_T h_0, h_1, \cdots)$ where $h_0 \in H$ and $h_i \in \mathcal{D}_T, i > 0$. From this one can further obtain the minimal unitary dilation U on $\tilde{\mathcal{K}}$ by using the well-known result that an isometry can be extended to a unitary on a space containing the space $\mathcal{H}$ and then arranging the minimality by taking compression to appropriate space. From this one can prove that for any polynomial p, we get

$$\|p(T)\| \leq \sup_{|z| \leq 1} |p(z)|.$$

This is called *von Neumann's inequality.* Indeed $p(T) = P_{\mathcal{H}} p(U)|_{\mathcal{H}}$ and hence

$$\|p(T)\| \leq \|p(U)\| = \sup\{|p(z)| : z \in \text{ spectrum } U\} \leq \sup_{|z|=1} |p(z)|.$$

Ando[An] has shown the following dilation result for a 2-tuple $\underline{T} = (T_1, T_2)$:

THEOREM 1.0.5 *(Ando's Theorem) Let $\underline{T} = (T_1, T_2)$ be commuting 2-tuple on $\mathcal{H}$, such that $\|T_1\| \leq 1, \|T_2\| \leq 1$. Then there exists 2-tuple $\underline{U} = (U_1, U_2)$ consisting of commuting unitaries such that*

$$P_{\mathcal{H}} U_1^n U_2^k|_{\mathcal{H}} = T_1^n T_2^k \qquad \forall n, k \geq 0.$$

From this we get the generalization of von Neumann inequality in 2-variables which is the following: For a polynomial p in 2 commuting variables

$$\|p(T_1, T_2)\| \leq \sup_{|z_1|,|z_2| \leq 1} \{|p(z_1, z_2)|\}.$$

Following this Varopoulos and Crabb-Davie [Pi] gave examples to show that dilation to commuting unitaries is not possible and that even the natural generalization of von Neumann inequality for tuple of commuting contractions does not hold. Parrott [Pa] came out with an example in which the dilation to commuting unitary is not possible but for $n = 3$ case but von Neumann inequality is satisfied. If we define $c_n = \sup\{p(T_1, \cdots, T_n) : |T_i| \leq 1, 1 \leq i \leq n\}$ then c_3 is infact greater than 1. Question of determining c_n still remains open. Fang [Fa] has given a generalization of the inequality using Bohr's radius. Another canonical generalization of von Neumann inequality would have been the following: For p being a polynomial in n commuting variables ($n \geq 2$) and $\underline{T}$ being a contractive n-tuple, is it always true that

$$\|p(T_1, \cdots, T_n)\| \leq \sup\{|p(z_1, \cdots, z_n)| : \sum_i |z_i|^2 \leq 1\}? \tag{1.1}$$

Arveson showed (refer [Ar3]) not only that this generalization does not hold but even showed that for some tuples $\underline{T}$ we can make choices of polynomial p such that the right hand side of equation (1.1) less than 1, but the left hand side is unbounded. We would briefly talk about the generalized versions of von Neumann inequality given by Bożejko, Popescu and Arveson in Section 2.2.

Chapter 2

Standard Dilations of Commuting Tuples

2.1 Maximal Commuting Piece and Dilation

DEFINITION 2.1.1 Let $\mathcal{H}, \mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ is a closed subspace of $\mathcal{L}$. Suppose $\underline{T}, \underline{R}$ are n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$ respectively. Then $\underline{R}$ is called a *dilation* of $\underline{T}$ if

$$R_i^* u = T_i^* u$$

for all $u \in \mathcal{H}, 1 \leq i \leq n$. In such a case $\underline{T}$ is called a *piece* of $\underline{R}$. If further $\underline{T}$ is a commuting tuple (i.e., $T_iT_j = T_jT_i$, for all i, j), then it is called a commuting piece of $\underline{R}$. A dilation $\underline{R}$ of $\underline{T}$ is said to be a *minimal dilation* if $\overline{\text{span}}\{\underline{R}^\alpha h : \alpha \in \tilde{\Lambda}, h \in \mathcal{H}\} = \mathcal{L}$.

In this Definition we note that if $\underline{R}$ is a dilation of $\underline{T}$, then $\mathcal{H}$ is a co-invariant subspace of $\underline{R}$, that is, it is left invariant by all R_i^*. It is standard (see [Ha]) to call $(R_1^*, \ldots, R_n^*)$ as an *extension* of $(T_1^*, \ldots, T_n^*)$ and $(T_1^*, \ldots, T_n^*)$ as a *part* of $(R_1^*, \ldots, R_n^*)$. In such a situation it is easy to see that for any $\alpha, \beta \in \tilde{\Lambda}$, $\underline{T}^\alpha(\underline{T}^\beta)^*$ is the compression of $\underline{R}^\alpha(\underline{R}^\beta)^*$ to $\mathcal{H}$, that is,

$$\underline{T}^\alpha(\underline{T}^\beta)^* = P_{\mathcal{H}}\underline{R}^\alpha(\underline{R}^\beta)^*|_{\mathcal{H}}. \tag{2.1}$$

We may extend this relation to any polynomials p, q in n-noncommuting variables to have

$$p(\underline{T})(q(\underline{T}))^* = P_{\mathcal{H}}p(\underline{R})(q(\underline{R}))^*|_{\mathcal{H}}.$$

Usually it is property (2.1) is all that one demands of a dilation. But we have imposed a condition of co-invariance in Definition 2.1.1, as it is very convenient to have it this way for our purposes.

We begin with a n-tuple of bounded operators $\underline{R}$ on a Hilbert space $\mathcal{L}$ and introduce a general notion in terms of finite set of polynomials a particular case of which we would need in this Chapter and some other case would be useful in later Chapters.

Let $\mathcal{R}$ be the WOT-closed algebra generated by R_i's, and let $\{p_\xi\}_{\xi \in \mathcal{I}}$ be some polynomials in n-noncommuting variables with finite indexing set $\mathcal{I}$. Consider

$$\begin{aligned} \mathcal{C}(\underline{R}) &= \{\mathcal{M} : \mathcal{M} \text{ is a co-invariant subspace for each } R_i, \\ &\quad (p_\xi(\underline{R}))^* h = 0, \forall h \in \mathcal{M}, 1 \leq i \leq n, \xi \in \mathcal{I}\}. \end{aligned}$$

So $\mathcal{C}(\underline{R})$ consists of all co-invariant subspaces of a n-tuple of operators $\underline{R}$ such that the compressions form a tuple $\underline{R}^p = (R_1^p, \ldots, R_n^p)$ satisfying $p_\xi(\underline{R}^p) = 0$ for all $\xi \in \mathcal{I}$. It is a complete lattice, in the sense that arbitrary intersections and span closures of arbitrary unions of such spaces are again in this collection. Therefore it has a maximal element which we denote by $\mathcal{L}^p(\underline{R})$ (or by $\mathcal{L}^p$ when the tuple under consideration is clear).

DEFINITION 2.1.2 The *maximal piece* of $\underline{R}$ with respect to $\{p_\xi\}_{\xi\in\mathcal{I}}$ is defined as the piece obtained by compressing $\underline{R}$ to the maximal element $\mathcal{L}^p(\underline{R})$ of $\mathcal{C}(\underline{R})$ denoted by $\underline{R}^p = (R_1^p, \ldots, R_n^p)$. The maximal piece is said to be *trivial* if the space $\mathcal{L}^p(\underline{R})$ is the zero space.

It is quite easy to get tuples with trivial maximal piece, as tuples with no non-trivial co-invariant subspaces have this property. Of course, our main interest lies in tuples with non-trivial maximal pieces. The following Lemma gives more concrete description of the maximal piece.

LEMMA 2.1.3 *Let $\underline{R}$ be a n-tuple of bounded operators on a Hilbert space $\mathcal{L}$. Let $\mathcal{K}_\xi = \overline{span}\{\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta h : h \in \mathcal{L}$ and $\alpha, \beta \in \tilde{\Lambda}\}$ for all $\xi \in \mathcal{I}$, and $\mathcal{K} = \overline{span} \cup_{\xi\in\mathcal{I}} \mathcal{K}_\xi$. Then $\mathcal{L}^p(\underline{R}) = \mathcal{K}^\perp$. In other words, $\mathcal{L}^p(\underline{R}) = \{h \in \mathcal{L} : (\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta)^* h = 0, \forall \xi \in \mathcal{I}$ and $\alpha, \beta \in \tilde{\Lambda}\}$.*

PROOF: Firstly $\mathcal{K}^\perp$ is a co-invariant subspace of $\underline{R}$ is obvious as each R_i leaves $\mathcal{K}$ invariant. Now for $i, j \in \{1, 2, \ldots, n\}$, and $h_1 \in \mathcal{K}^\perp, h_2 \in \mathcal{L}$,

$$\langle (p_\xi(\underline{R}))^* h_1, h_2\rangle = \langle h_1, p_\xi(\underline{R}) h_2\rangle = 0.$$

So we get $(p_\xi(\underline{R}))^* h_1 = 0$ and hence $\mathcal{K}^\perp \in \mathcal{C}(\underline{R})$. Now if $\mathcal{M}$ is an element of $\mathcal{C}(\underline{R})$, take $i, j \in \{1, \ldots, n\}, \alpha \in \tilde{\Lambda}, h_1 \in \mathcal{M}, h \in \mathcal{L}$. We have

$$\langle h_1, \underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta h\rangle = \langle (\underline{R}^\alpha p_\xi(\underline{R})\underline{R}^\beta)^* h_1, h\rangle = 0$$

as $(\underline{R}^\alpha)^* h_1 \in \mathcal{M}$. Hence $\mathcal{M}$ is a subspace of $\mathcal{K}^\perp$. Now the last statement is easy to see. □

Even if we take $\mathcal{M}_\xi := \overline{\text{span}}\{\underline{R}^\alpha p_\xi(\underline{R}) h : h \in \mathcal{L}$ and $\alpha \in \tilde{\Lambda}\}$ for all $\xi \in \mathcal{I}$, and $\mathcal{M} = \overline{\text{span}}\{\cup_{\xi\in\mathcal{I}} \mathcal{M}_\xi\}$, still we get $\mathcal{L}^p(\underline{R}) = \mathcal{M}^\perp$.

COROLLARY 2.1.4 *Suppose $\underline{R}$, $\underline{T}$ are n-tuples of operators on two Hilbert spaces $\mathcal{L}, \mathcal{M}$. Then the maximal piece of $(R_1 \oplus T_1, \ldots, R_n \oplus T_n)$ acting on $\mathcal{L} \oplus \mathcal{M}$ is $(R_1^p \oplus T_1^p, \ldots, R_n^p \oplus T_n^p)$ acting on $\mathcal{L}^p \oplus \mathcal{M}^p$. The maximal piece of $(R_1 \otimes I, \ldots, R_n \otimes I)$ acting on $\mathcal{L} \otimes \mathcal{M}$ is $(R_1^p \otimes I, \ldots, R_n^p \otimes I)$ acting on $\mathcal{L}^p \otimes \mathcal{M}$.*

PROOF: Clear from Corollary 2.1.7. □

Now let us see how the maximal piece behaves with respect to the operation of taking dilations. Before considering specific dilations we have the following general statement.

PROPOSITION 2.1.5 *Suppose $\underline{T}, \underline{R}$ are n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$, with $\mathcal{H} \subseteq \mathcal{L}$, such that $\underline{R}$ is a dilation of $\underline{T}$. Then $\mathcal{H}^p(\underline{T}) = \mathcal{L}^p(\underline{R}) \bigcap \mathcal{H}$ and $\underline{R}^p$ is a dilation of $\underline{T}^p$.*

PROOF: We have $R_i^* h = T_i^* h$, for $h \in \mathcal{H}$. Therefore, $(p_\xi(\underline{R}))^*(\underline{R}^\alpha)^* h = (p_\xi(\underline{T}))^*(\underline{T}^\alpha)^* h$ for $h \in \mathcal{H}$, $1 \leq i, j \leq n$, and $\alpha \in \tilde{\Lambda}$. Now the first part of the result is clear from Corollary 2.1.7. Further for $h \in \mathcal{L}^p(\underline{R})$, $R_i^* h = (R_i^p)^* h$ and so for $h \in \mathcal{H}^p(\underline{T}) = \mathcal{L}^p(\underline{R}) \bigcap \mathcal{H}$, $(R_i^p)^* h = R_i^* h = T_i^* h = (T_i^p)^* h$. This proves the claim. □

Suppose $\mathcal{J}$ stands for the two sided closed ideal

$$\overline{\text{span}} \cup_{\xi \in \mathcal{I}} \{\underline{R}^{\alpha} p_{\xi}(\underline{R}) \underline{R}^{\beta} : \alpha, \beta \in \tilde{\Lambda}\}$$

of $\mathcal{R}$. Then $\mathcal{R}/\mathcal{J}$ is generated by $\tilde{R}_i, 1 \leq i \leq n$ where $\tilde{R}_i = R_i + \mathcal{J}$. From the above Proposition it follows that $\underline{\tilde{R}} = (\tilde{R}_1, \cdots, \tilde{R}_n)$ satisfies $p_{\xi}(\underline{\tilde{R}}) = 0$. Further if the WOT-closed algebra generated by $R_1^p, \cdots, R_n^p$ is denoted by $\mathcal{R}^p$ then there is a algebra homomorphism from $\mathcal{R}/\mathcal{J}$ to $\mathcal{R}^p$ which maps $\tilde{R}_i$ to R_i^p for $1 \leq i \leq n$. This homomorphism is given by the map $\Psi : \mathcal{R}/\mathcal{J} \to \mathcal{R}^p$ such that for any polynomial q in n-noncommuting variables $\Psi(q(\underline{R}) + \mathcal{J}) = q(\underline{R}^p)$. To check well definedness we start with some polynomial q such that $q(\underline{R}) \in \mathcal{J}$. So $q(\underline{R}) \in \overline{\text{span}} \cup_{\xi \in \mathcal{I}} \{\underline{R}^{\alpha} p_{\xi}(\underline{R}) \underline{R}^{\beta} : \alpha, \beta \in \tilde{\Lambda}\}$. And so for $k, h \in \mathcal{K}^p$ as R_i^* leaves $\mathcal{K}^p$ invariant we have

$$\begin{aligned} \langle q(\underline{R}^p) k, h \rangle &= \langle P_{\mathcal{K}^p} q(\underline{R}) P_{\mathcal{K}^p} k, h \rangle \\ &= \langle q(\underline{R}) k, h \rangle = \langle k, (q(\underline{R}))^* h \rangle = 0. \end{aligned}$$

For the last equality we use Corollary 2.1.7. Hence $\Psi(q(\underline{R}) + \mathcal{J}) = 0$, i.e., Ψ is well defined. It is trivial to check homomorphism property of Ψ.

Definition 2.1.6 Suppose $\underline{R} = (R_1, \cdots, R_n)$ is a n-tuple of operators on a Hilbert space $\mathcal{L}$. Then the *maximal commuting piece* of $\underline{R}$ is defined as the maximal piece obtained for a indexed set of polynomial $p_{ij}(z) = z_i z_j - z_j z_i, (i,j) \in \{1, \cdots, n\} \times \{1, \cdots, n\} = \mathcal{J}$. In this case we denote $\underline{R}^p$ by $\underline{R}^c$ and $\mathcal{L}^p(\underline{R})$ by $\mathcal{L}^c(\underline{R})$. The space $\mathcal{L}^c(\underline{R})$ is called the *maximal commuting subspace.*

Corollary 2.1.7 *Let $\underline{R}$ be a n-tuple of bounded operators on a Hilbert space $\mathcal{L}$. Let $\mathcal{K}_{ij} = \overline{span}\{\underline{R}^{\alpha}(R_i R_j - R_j R_i) h : h \in \mathcal{L}, \alpha \in \tilde{\Lambda}\}$ for all $1 \leq i, j \leq n$, and $\mathcal{K} = \overline{span} \cup_{i,j=1}^n \mathcal{K}_{ij}$. Then $\mathcal{L}^c(\underline{R}) = \mathcal{K}^{\perp}$. In other words, $\mathcal{L}^c(\underline{R}) = \{h \in \mathcal{L} : (R_i^* R_j^* - R_j^* R_i^*)(\underline{R}^{\alpha})^* h = 0, \forall 1 \leq i, j \leq n, \alpha \in \tilde{\Lambda}\}$.*

Proof: Follows from Lemma 2.1.3. □

Proposition 2.1.8 *Let $\underline{V} = (V_1, \ldots, V_n)$ and $\underline{S} = (S_1, \ldots S_n)$ be standard contractive tuples on full Fock space $\Gamma(\mathbb{C}^n)$ and Boson Fock space $\Gamma_s(\mathbb{C}^n)$ respectively. Then the maximal commuting piece of $\underline{V}$ is $\underline{S}$.*

Proof: As we have already noted in Chapter 1, $\underline{S}$ is a commuting piece of $\underline{V}$. To show maximality we make use of Corollary 2.1.7. Suppose $x \in \Gamma(\mathbb{C}^n)$ and $\langle x, \underline{V}^{\alpha}(V_i V_j - V_j V_i) y \rangle = 0$ for all $\alpha \in \tilde{\Lambda}, 1 \leq i, j \leq n$ and $y \in \Gamma(\mathbb{C}^n)$. We wish to show that $x \in \Gamma_s(\mathbb{C}^n)$. Suppose x_m is the m-particle component of x, that is, $x = \oplus_{m \geq 0} x_m$ with $x_m \in (\mathbb{C}^n)^{\otimes^m}$ for $m \geq 0$. For $m \geq 2$ and any permutation σ of $\{1, 2, \ldots, m\}$ we need to show that the unitary $U_{\sigma} : (\mathbb{C}^n)^{\otimes^m} \to (\mathbb{C}^n)^{\otimes^m}$, defined by

$$U_{\sigma}(u_1 \otimes \cdots \otimes u_m) = u_{\sigma^{-1}(1)} \otimes \cdots \otimes u_{\sigma^{-1}(m)},$$

leaves x_m fixed. Since the group of permutations of $\{1, 2, \ldots, m\}$ is generated by permutations $\{(1,2), \ldots, (m-1, m)\}$ it is enough to verify $U_{\sigma}(x_m) = x_m$ for permutations σ of the form $(i, i+1)$. So fix m and i with $m \geq 2$ and $1 \leq i \leq (m-1)$. We have

$$\langle \oplus_p x_p, \underline{V}^{\alpha}(V_k V_l - V_l V_k) y \rangle = 0,$$

for every $y \in \Gamma(\mathbb{C}^n)$, $1 \leq k, l \leq n$. As α is arbitrary, this means that

$$\langle x_m, z \otimes (e_k \otimes e_l - e_l \otimes e_k) \otimes w \rangle = 0$$

for any $z \in (\mathbb{C}^n)^{\otimes(i-1)}$, $w \in (\mathbb{C}^n)^{\otimes(m-i-1)}$. This clearly implies $U_\sigma(x_m) = x_m$, for $\sigma = (i, i+1)$. □

In [Po1] Popescu showed the existence of minimal isometric dilation for an infinite sequence of operator which forms a row contraction by generalizing Schäffer construction [Sc] of one variable case. But we would now give this Popescu's result and proof for contractive n-tuple.

THEOREM 2.1.9 *For every contractive n-tuple of operators on Hilbert space $\mathcal{H}$ there exists a minimal isometric dilation which is unique up to unitary equivalence.*

PROOF: Let us define $\mathcal{K} = \mathcal{H} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathcal{D})$ where $\mathcal{D}$ is the closure of the range of operator

$$D : \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ copies}} \to \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ copies}}$$

and D is the positive square root of

$$D^2 = [\delta_{ij} I - T_i^* T_j]_{n \times n}.$$

Whenever it is convenient for us we identify $\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ copies}}$ with $\mathbb{C}^n \otimes \mathcal{H}$ so that

$$(h_1, \ldots, h_n) = \sum_{i=1}^{n} e_i \otimes h_i.$$

Then

$$D(h_1, \ldots, h_n) = D(\sum_{i=1}^{n} e_i \otimes h_i) = \sum_{i=1}^{n} e_i \otimes (h_i - \sum_{j=1}^{n} T_i^* T_j h_j).$$

Let for $1 \leq i \leq n, h \in \mathcal{H}$ and $d_\alpha \in \mathcal{D}$

$$\tilde{V}_i(h \oplus \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha) := T_i h \oplus D(e_i \otimes h) \oplus e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes d_\alpha \tag{2.2}$$

for $h \in \mathcal{H}$, $d_\alpha \in \mathcal{D}$ for $\alpha \in \tilde{\Lambda}$, and $1 \leq i \leq n$ ($\mathbb{C}^n \omega \otimes \mathcal{D}$ has been identified with $\mathcal{D}$). These are isometries as

$$\begin{aligned}
& \langle \tilde{V}_i(h \oplus \sum_\alpha e^\alpha \otimes d_\alpha), \tilde{V}_i(h' \oplus \sum_\beta e^\beta \otimes d'_\beta) \rangle \\
= \; & \langle T_i^* T_i h, h' \rangle + \langle D^2(e_i \otimes h), e_i \otimes h' \rangle + \langle e_i \otimes \sum_\alpha e^\alpha \otimes d_\alpha, e_i \otimes \sum_\beta e^\beta \otimes d'_\beta \rangle \\
= \; & \langle (T_i^* T_i + I - T_i^* T_i) h, h' \rangle + \langle \sum_\alpha e^\alpha \otimes d_\alpha, \sum_\beta e^\beta \otimes d'_\beta \rangle \\
= \; & \langle h \oplus \sum_\alpha e^\alpha \otimes d_\alpha, h \oplus \sum_\beta e^\beta \otimes d'_\beta \rangle.
\end{aligned}$$

Also, for $i \neq j$

$$\begin{aligned} & \langle \tilde{V}_i(h \oplus \sum_\alpha e^\alpha \otimes d_\alpha), \tilde{V}_j(h' \oplus \sum_\beta e^\beta \otimes d'_\beta)\rangle \\ = & \langle T_j^* T_i h, h'\rangle + \langle D^2(e_i \otimes h), e_j \otimes h'\rangle \\ = & \langle T_j^* T_i h, h'\rangle + \langle -T_j^* T_i h, h'\rangle = 0. \end{aligned}$$

So $\tilde{V}_i$'s are isometries with orthogonal ranges. Also $\tilde{V}_i^*$ leaves $\mathcal{H}$ invariant as $\tilde{V}_i^* h = T_i^* h$. For when $h, h' \in \mathcal{H}$ and $d_\alpha \in \mathcal{D}$

$$\begin{aligned} \langle V_i^* h, h' \oplus (\sum_\alpha e^\alpha \otimes d_\alpha)\rangle &= \langle h, \tilde{V}_i(h' \oplus \sum_\alpha e^\alpha \otimes d_\alpha)\rangle \\ &= \langle h, T_i h'\rangle = \langle T_i^* h, h'\rangle = \langle T_i^* h, h' \oplus (\sum_\alpha e^\alpha \otimes d_\alpha)\rangle. \end{aligned}$$

Next we would check if this dilation is minimal. Clearly $\sum_i \tilde{V}_i \mathcal{H} = \mathcal{H} \oplus (\omega \otimes \mathcal{D}) = \mathcal{H} \oplus \mathcal{D}$. Further

$$\sum_{|\alpha| \leq m} \underline{\tilde{V}}^\alpha \mathcal{H} = \mathcal{H} \oplus (\omega \otimes \mathcal{D}) \oplus \sum_i (V_i \otimes I)(\omega \otimes \mathcal{D}) \oplus \cdots \oplus \sum_{|\alpha| = m} (\underline{V}^\alpha \otimes I)(\omega \otimes \mathcal{D})$$

Thus $\overline{\text{span}}\{\underline{\tilde{V}}^\alpha h : h \in \mathcal{H}, \alpha \in \tilde{\Lambda}\} = \mathcal{H} \oplus \Gamma(\mathbb{C}^n) \otimes \mathcal{D} = \mathcal{K}$. The uniqueness follows using similar arguements as that of Theorem 4.1 of [SF]. □

Now we will give another method of constructing the minimal isometric dilation of a contractive n-tuple. For a contractive tuple $\underline{T} = (T_1, \cdots, T_n)$ on Hilbert space $\mathcal{H}$ define a set

$$\mathcal{M}_0 = \{(\alpha, u) : \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\}.$$

For some $\alpha, \beta \in \tilde{\Lambda}$ if $\alpha = 0$, or $|\alpha| \leq |\beta|$ and $\alpha_i = \beta_i$ for $1 \leq i \leq |\alpha|$ then we write $\alpha \subseteq \beta$. Further if $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$ we define

$$\gamma = \begin{cases} 0 & \text{if } |\alpha| = |\beta| \\ (\beta_{|\alpha|+1}, \cdots, \beta_{|\beta|}) & \text{if } |\alpha| < |\beta| \\ (\alpha_{|\beta|+1}, \cdots, \alpha_{|\alpha|}) & \text{if } |\beta| < |\alpha|. \end{cases}$$

Let $u, v \in \mathcal{H}$ be arbitrary. Consider a map $\tilde{K} : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{C}$ defined as follows:

$$\tilde{K}((\alpha, u), (\beta, v)) = \begin{cases} \langle u, \underline{T}^\gamma v\rangle & \text{if } \alpha \subseteq \beta \\ \langle u, (\underline{T}^\gamma)^* v\rangle & \text{if } \alpha \supseteq \beta \\ 0 & \text{otherwise.} \end{cases}$$

We would show that $\tilde{K}$ is positive definite kernel. For this we consider the matrix $N^{(m)} = (N^{(m)}_{\alpha,\beta})$ where matrix $N^{(m)}$ is written as block matrix in terms of $N^{(m)}_{\alpha,\beta}$, and rows and columns of the block matrix are indexed by $\alpha, \beta \in \tilde{\Lambda}$ and $|\alpha|, |\beta| \leq m$. (For all the matrices denoted by notations of the type $A^{(m)}$ below are in the form of block matrices indexed by $\alpha, \beta \in \tilde{\Lambda}$ and $|\alpha|, |\beta| \leq m$). Here

$$N^{(m)}_{\alpha,\beta} := \begin{cases} \underline{T}^\gamma & \text{if } \alpha \subseteq \beta \\ (\underline{T}^\gamma)^* & \text{if } \alpha \supseteq \beta \\ 0 & \text{otherwise.} \end{cases}$$

We would show that $K^{(m)}$ is positive which would clearly imply that $\tilde{K}$ is positive definite kernel. Here we use induction to show this. First we define the matrices $L^{(m)}$ and $F^{(m)}$ as

$$L^{(m)}_{\alpha,\beta} = \begin{cases} T_\beta & \text{if } |\beta| = 1, |\alpha| = 0 \\ I & \text{if } \alpha = \beta, 0 < |\alpha| \\ 0 & \text{otherwise} \end{cases} \quad \text{and } F^{(m)}_{\alpha,\beta} = \begin{cases} I - \sum T_i T_i^* & \text{if } |\alpha|, |\beta| = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $N^{(1)} = L^{(1)}(L^{(1)})^* + F^{(1)}$. Also for $m > 1$

$$N^{(m)} = L^{(m)} M^{(m)} (L^{(m)})^* + F^{(m)} \text{ where}$$

$$M^{(m)} = \begin{pmatrix} 0 & 0 \\ 0 & N^{(m-1)} \otimes I_{\mathbb{C}^n} \end{pmatrix}.$$

Infact for $|\alpha|, |\beta| \leq m$

$$M^{(m)}_{\alpha,\beta} := \begin{cases} 0 & \text{if } \alpha \text{ or } \beta = 0 \\ \underline{T}^\gamma & \text{if } \alpha \subseteq \beta, 0 < |\alpha| \\ (\underline{T}^\gamma)^* & \text{otherwise} \end{cases}$$

and $N^{(m-1)}$ is positive by hypothesis. So we get $N^{(m)}$ to be positive.

Hence there is a Hilbert space $\mathcal{K}$ and a injective map $\lambda : \mathcal{M}_0 \to \mathcal{K}$ such that $\overline{\text{span}}\{\lambda(\alpha, u) : 1 \leq i \leq n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \mathcal{K}$ and

$$\langle \lambda(\alpha, u), \lambda(\beta, v) \rangle = \tilde{K}((\alpha, u), (\beta, v)).$$

We claim that the tuple $\underline{\tilde{V}} = (\tilde{V}_1, \cdots, \tilde{V}_n)$ consisting of maps $\tilde{V}_i : \mathcal{K} \to \mathcal{K}$ defined as

$$\tilde{V}_i \lambda((\alpha_1, \cdots, \alpha_m), u) = \lambda((i, \alpha_1, \cdots, \alpha_m), u),$$

is the minimal isometric dilation. That these are isometries with orthogonal ranges is clear from the following equations and the definition of kernel $\tilde{K}$:

$$\begin{aligned} & \langle \tilde{V}_i \lambda((\alpha_1, \cdots, \alpha_m), u), \tilde{V}_j \lambda((\beta_1, \cdots, \beta_k), v) \rangle \\ = & \langle \lambda((i, \alpha_1, \cdots, \alpha_m), u), \lambda((j, \beta_1, \cdots, \beta_k), v) \rangle \\ = & \tilde{K}(((i, \alpha_1, \cdots, \alpha_m), u), (j, \beta_1, \cdots, \beta_k), v))) \\ = & \delta_{ij} \tilde{K}(((\alpha_1, \cdots, \alpha_m), u), (\beta_1, \cdots, \beta_k), v))) \\ = & \delta_{ij} \langle \lambda((\alpha_1, \cdots, \alpha_m), u), \lambda((\beta_1, \cdots, \beta_k), v) \rangle. \end{aligned}$$

Minimality holds as

$$\begin{aligned} & \overline{\text{span}}\{\underline{\tilde{V}}^\beta \lambda(\alpha, u) : \alpha \text{ and } \beta \in \tilde{\Lambda}, u \in \mathcal{H}\} \\ = & \overline{\text{span}}\{\lambda(\alpha, u) : 1 \leq i \leq n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \mathcal{K}. \end{aligned}$$

Finally notice that $\tilde{V}_i^*$'s leaves $\mathcal{H}$ invariant as

$$\begin{aligned} \langle \tilde{V}_i^* \lambda(0, u), \lambda((\beta_1, \cdots, \beta_m), v) \rangle &= \langle \lambda(0, u), \tilde{V}_i \lambda((\beta_1, \cdots, \beta_m), v) \rangle \\ &= \langle \lambda(0, u), \lambda((i, \beta_1, \cdots, \beta_m), v) \rangle \\ &= \tilde{K}((0, u), ((i, \beta_1, \cdots, \beta_m), v)) \\ &= \tilde{K}((0, T_i^* u), ((\beta_1, \cdots, \beta_m), v)) \\ &= \langle \lambda(0, T_i^* u), \lambda((\beta_1, \cdots, \beta_m), v) \rangle. \end{aligned}$$

DEFINITION 2.1.10 Let $\underline{T} = (T_1, \ldots, T_n)$ be a contractive tuple on a Hilbert space $\mathcal{H}$. The operator $\Delta_{\underline{T}} = [I - (T_1T_1^* + \cdots + T_nT_n^*)]^{\frac{1}{2}}$ is called the *defect operator* of $\underline{T}$ and the subspace $\overline{\Delta_{\underline{T}}(\mathcal{H})}$ is called the *defect space* of $\underline{T}$. The tuple $\underline{T}$ is said to be *pure* if $\sum_{\alpha\in\Lambda^m} \underline{T}^\alpha(\underline{T}^\alpha)^*$ converges to zero in strong operator topology as m tends to infinity.

Suppose $\sum T_iT_i^* = I$, then it is easy to see that $\sum_{\alpha\in\Lambda^m} \underline{T}^\alpha(\underline{T}^\alpha)^* = I$ for all m and there is no-way this sequence can converge to zero. So in the pure case the defect operator and the defect spaces are non-trivial.

First we restrict our attention to pure tuples. The reason for this is that it is very easy to write down standard dilations for pure tuples. So let $\mathcal{H}$ be a complex, separable Hilbert space and let $\underline{T}$ be a pure contractive tuple on $\mathcal{H}$. Take $\tilde{\mathcal{H}} = \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$, and define an operator $K : \mathcal{H} \to \tilde{\mathcal{H}}$ by

$$Kh = \sum_\alpha e^\alpha \otimes \Delta_{\underline{T}}(\underline{T}^\alpha)^* h, \tag{2.3}$$

where the sum is taken over all $\alpha \in \tilde{\Lambda}$ ([Po5],[AP1]). For $h \in \overline{\Delta_{\underline{T}}(\mathcal{H})}$

$$\begin{aligned}\langle K^*(\sum_\alpha e^\alpha \otimes h_\alpha), h\rangle &= \langle \sum_\alpha e^\alpha \otimes h_\alpha, \sum_\beta e^\beta \otimes \Delta_{\underline{T}}(\underline{T}^\beta)^* h\rangle \\ &= \sum_\alpha \langle h_\alpha, \Delta_{\underline{T}}(\underline{T}^\alpha)^* h\rangle = \langle \sum_\alpha \underline{T}^\alpha \Delta_{\underline{T}} h_\alpha, h\rangle.\end{aligned}$$

So $K^*(\sum_\alpha e^\alpha \otimes h_\alpha) = \sum_\alpha \underline{T}^\alpha \Delta_{\underline{T}} h_\alpha$ and

$$\begin{aligned}K^*Kh &= K^*(\sum_\alpha e^\alpha \otimes \Delta_{\underline{T}}(\underline{T}^\alpha)^* h) = \sum_\alpha \underline{T}^\alpha (\Delta_{\underline{T}})^2 (\underline{T}^\alpha)^* h \\ &= h - \lim_{|\alpha|\to\infty} \sum_\alpha \underline{T}^\alpha(\underline{T})^* h = h.\end{aligned}$$

Thus K is an isometry. Moreover

$$\begin{aligned}K^*(\underline{V}^\alpha(\underline{V}^\beta)^* \otimes I)Kh &= K^*(\underline{V}^\alpha(\underline{V}^\beta)^* \otimes I)\sum_\gamma e^\gamma \otimes \Delta_{\underline{T}}(\underline{T}^\gamma)^* h \\ &= K^*(\sum_\epsilon e^\alpha \otimes e^\epsilon \otimes \Delta_{\underline{T}}(\underline{T}^\epsilon)^*(\underline{T}^\beta)^* h) \\ &= \underline{T}^\alpha(\sum_\epsilon T^\epsilon (\Delta_{\underline{T}})^2 (\underline{T}^\epsilon)^*)(\underline{T}^\beta)^* h = \underline{T}^\alpha(\underline{T}^\beta)^* h.\end{aligned}$$

Now $\mathcal{H}$ is considered as a subspace of $\tilde{\mathcal{H}}$ by identifying vectors $h \in \mathcal{H}$ with $Ah \in \tilde{\mathcal{H}}$. Then by noting that each $V_i^* \otimes I$ leaves the range of K invariant and $\underline{T}^\alpha = K^*(\underline{V}^\alpha \otimes I)K$ for all $\alpha \in \tilde{\Lambda}$ it is seen that the tuple $\underline{\tilde{V}} = (V_1 \otimes I, \ldots, V_n \otimes I)$ of operators on $\tilde{\mathcal{H}}$ is a realization of the minimal isometric dilation of $\underline{T}$. Now if $\underline{T}$ is a commuting tuple, it is easy to see that the range of K is contained in $\tilde{\mathcal{H}}_s = \Gamma_s(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$. In other words now $\mathcal{H}$ can be considered as a subspace of $\tilde{\mathcal{H}}_s$. Moreover, $\underline{\tilde{S}} = (S_1 \otimes I, \ldots, S_n \otimes I)$, as a tuple of operators in $\tilde{\mathcal{H}}_s$ is a realization of the standard commuting dilation of $(T_1, \ldots T_n)$. More abstractly, if $\underline{T}$ is commuting and pure, the standard commuting dilation of it is got by embedding $\mathcal{H}$ isometrically in $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{K}$, for some Hilbert space $\mathcal{K}$, such that $(S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}})$ is a dilation of $\underline{T}$ and $\overline{\text{span}}\{(\underline{S}^\alpha \otimes I_{\mathcal{K}})h : h \in \mathcal{H}, \alpha \in \tilde{\Lambda}\} = \Gamma_s(\mathbb{C}^n) \otimes \mathcal{K}$. Up to unitary equivalence there is unique such dilation and $\dim(\mathcal{K}) = \text{rank }(\Delta_{\underline{T}})$.

THEOREM 2.1.11 *Let $\underline{T}$ be a pure contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece $\underline{\tilde{V}}^c$ of the minimal isometric dilation $\underline{\tilde{V}}$ of $\underline{T}$ is a realization of the standard commuting dilation of $\underline{T}^c$ if and only if $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^c(\underline{T}))}$. In such a case rank $(\Delta_{\underline{T}}) =$ rank $(\Delta_{\underline{T}^c}) =$ rank $(\Delta_{\underline{\tilde{V}}}) =$ rank $(\Delta_{\underline{\tilde{V}}^c})$.*

PROOF: We denote $\mathcal{H}^c(\underline{T}), \overline{\Delta_{\underline{T}}(\mathcal{H})}$ and $\overline{\Delta_{\underline{T}}(\mathcal{H}^c(\underline{T}))}$ by $\mathcal{H}^c, \mathcal{M}$, and $\mathcal{M}^c$ respectively. It is obvious that $\underline{T}^c$ is also a pure contractive tuple. We already know from Proposition 2.1.5 that $\underline{\tilde{V}}^c = (\underline{S} \otimes I_{\mathcal{M}})$ on $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}$ is a dilation of $\underline{T}^c$. It is the standard dilation if and only if $\mathcal{L} := \overline{\text{span}}\{(\underline{S}^\alpha \otimes I_{\mathcal{M}})Kh : h \in \mathcal{H}^c, \alpha \in \tilde{\Lambda}\}$ is equal to $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}$, where $K : \mathcal{H} \to \tilde{\mathcal{H}}$ is the isometry defined by (2.3).

From the definition of K, using the commutativity of the operators T_i, it is clear that for $h \in \mathcal{H}^c$, $Kh \in \Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}^c$. Hence $\mathcal{L} \subseteq \Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}^c$. Further, as $(\underline{S} \otimes I_{\mathcal{M}})$ is a dilation, $(S_i^* \otimes I_{\mathcal{M}})$ leaves $K(\mathcal{H}^c)$ invariant. Therefore, $((I - \sum S_i S_i^*) \otimes I_{\mathcal{M}})Kh \in \mathcal{L}$ for $h \in \mathcal{H}^c$. But $(I - \sum S_i S_i^*)$ being the projection onto the vacuum space, $((I - \sum S_i S_i^*) \otimes I_{\mathcal{M}})Kh = \omega \otimes \Delta_{\underline{T}} h$. As $\{\underline{S}^\alpha \omega, \alpha \in \tilde{\Lambda}\}$ spans whole of $\Gamma_s(\mathbb{C}^n)$ we get that $\Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}^c \subseteq \mathcal{L}$. Hence $\mathcal{L} = \Gamma_s(\mathbb{C}^n) \otimes \mathcal{M}^c$ and this way we have proved the first claim.

Now suppose $\underline{\tilde{V}}^c$ is a realization of the standard commuting dilation of $\underline{T}^c$. This in particular means that rank $(\Delta_{\underline{T}^c}) =$ rank $(\Delta_{\underline{\tilde{V}}^c})$. Also as $\underline{\tilde{V}}$ is the minimal isometric dilation of $\underline{T}$, rank $(\Delta_{\underline{T}}) =$ rank $(\Delta_{\underline{\tilde{V}}})$. Further as $\underline{\tilde{V}}^c = (\underline{S} \otimes I_{\mathcal{M}})$, rank $(\Delta_{\underline{\tilde{V}}^c}) = \dim(\mathcal{M}) =$ rank $(\Delta_{\underline{T}})$. □

We may ask whether the equality of ranks in this Theorem is good enough to make a converse statement. To answer this we make use of the following simple Lemma.

LEMMA 2.1.12 *Suppose*

$$M = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}$$

is a bounded positive operator on some Hilbert space. Then rank $(A) =$ rank $\begin{pmatrix} A \\ B \end{pmatrix}$

PROOF: Without loss of generality we can assume that M is a contraction. Then it is a folklore Theorem that there exists a contraction D such that $B = C^{\frac{1}{2}} D A^{\frac{1}{2}}$. Now

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} \\ C^{\frac{1}{2}} D \end{pmatrix} \left(A^{\frac{1}{2}} \right),$$

and hence rank $\begin{pmatrix} A \\ B \end{pmatrix} \leq$ rank $A^{\frac{1}{2}}$. But A being positive, rank $A =$ rank $A^{\frac{1}{2}}$. Therefore rank $\begin{pmatrix} A \\ B \end{pmatrix} \leq$ rank $(A) \leq$ rank $\begin{pmatrix} A \\ B \end{pmatrix}$. □

REMARK 2.1.13 *Let $\underline{T}$ be a pure contractive tuple on a Hilbert space $\mathcal{H}$ with minimal isometric dilation $\underline{\tilde{V}}$. If rank $\Delta_{\underline{T}}$ and rank $\Delta_{\underline{T}^c}$ are finite and equal then $\underline{\tilde{V}}^c$ is a realization of the standard commuting dilation of $\underline{T}^c$.*

PROOF: In view of Theorem 2.1.11 we need to show that $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^c(\underline{T}))}$. Since $\overline{\Delta_{\underline{T}}(\mathcal{H})} \supseteq \overline{\Delta_{\underline{T}}(\mathcal{H}^c(\underline{T}))}$, and these spaces are now finite dimensional, it suffices to show that their dimensions are equal or rank $(\Delta_{\underline{T}}) =$ rank $(\Delta_{\underline{T}} P_{\mathcal{H}^c})$. Clearly rank $(\Delta_{\underline{T}}) \geq$

rank $(\Delta_{\underline{T}} P_{\mathcal{H}^c})$. Also by assumption, rank $(\Delta_{\underline{T}})$ = rank $(\Delta_{\underline{T}^c})$. By positivity rank $(\Delta_{\underline{T}^c})$ = rank $(\Delta^2_{\underline{T}^c})$. And then by previous Lemma rank $(\Delta^2_{\underline{T}^c})$ = rank $(P_{\mathcal{H}^c}(\Delta^2_{\underline{T}})P_{\mathcal{H}^c})$ = rank $(\Delta^2_{\underline{T}} P_{\mathcal{H}^c})$ $\leq$ rank $(\Delta_{\underline{T}} P_{\mathcal{H}^c})$. □

If both the ranks are infinite then we can not ensure that $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^c(\underline{T}))}$ is seen by the following example.

EXAMPLE 2.1.14 Let $\underline{R} = (R_1, R_2)$ be a commuting pure contractive 2-tuple on an infinite dimensional Hilbert space $\mathcal{H}_0$ (We can even take R_1, R_2 as scalars) such that $\overline{\Delta_{\underline{R}}(\mathcal{H}_0)}$ is infinite dimensional. Take $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}^2$, and let T_1, T_2 be operators on $\mathcal{H}$ defined by

$$T_1 = \begin{pmatrix} R_1 & & \\ & 0 & t_1 \\ & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} R_2 & & \\ & 0 & 0 \\ & t_2 & 0 \end{pmatrix},$$

where t_1, t_2 are any two scalars, $0 < t_1, t_2 < 1$. Then $\underline{T} = (T_1, T_2)$ is a pure contractive tuple. Making use of Corollary 2.1.4, $\mathcal{H}^c(\underline{T}) = \mathcal{H}_0$ (thought of as a subspace of $\mathcal{H}$ in the natural way) and the maximal commuting piece of $\underline{T}$ is (R_1, R_2), and therefore rank $(\Delta_{\underline{T}^c})$ = rank $(\Delta_{\underline{T}}) = \infty$. But $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{R}}(\mathcal{H}_0)} \oplus \mathbb{C}^2$.

We do not know how to extend Theorem 2.1.11 to contractive tuples which are not necessarily pure.

We would discuss in brief the generalized versions of von Neumann's inequality given by Bożejko, Popescu and Arveson using the dilation theory. More references on this inequality can be found in Chapter 1 of [Pi]. In [Bo] Bożejko gave the following extension of this inequality:

THEOREM 2.1.15 *Let $\underline{T}$ be a n-tuple of contractions on some Hilbert space and p be a polynomial in n noncommuting variables. Then*

$$\|p(\underline{T})\| \leq \sup\{\|p(\underline{U})\|\}$$

where the supremum runs over all possible n-tuples $\underline{U} = (U_1, \cdots, U_n)$ of unitary operators on this Hilbert space.

Using minimal isometric dilation Popescu derived the following result:

THEOREM 2.1.16 *For any contractive n-tuple $\underline{T}$ and any polynomial p in n noncommuting variables*

$$\|p(\underline{T})\| \leq \|p(\underline{V})\|.$$

Drury [Dr] and later Arveson [Ar4] investigated this inequality in light of standard commuting dilation. Arveson gave the following example to show that one of the natural generalization of von Neumann's inequality given by equation (1.1) fails in a big way.

EXAMPLE 2.1.17 Assume $n \geq 2$ and $a_0, a_1, a_2, \cdots$ to be sequence of complex numbers such that

$$\sum_i |a_i| = 1, \ \sum_i |a_i|^2 i^{(n-1)/2} = \infty.$$

Let f_N be polynomials in commuting n-varables defined as

$$f_N(z_1,\cdots,z_n)=\sum_{i=1}^{N}\frac{a_i}{s^i}(z_1\cdots z_n)^i \text{ where } s=\sqrt{\frac{1}{n^n}}.$$

Then it was shown in [Ar4] that $\sup_{\sum_i |z_i|^2\leq 1}\|f_N(z_1,\cdots,z_n)\|\leq 1$ and f_N converges uniformly over the closed unit ball but

$$\lim_{N\to\infty}\|f_N(\underline{S})\|=\infty.$$

This leads to the realization that no effective model theory for $n\geq 2$ could be based on subnormal operators. Now we state the Arveson's version [Ar4] of von Neumann's inequality.

THEOREM 2.1.18 *Let $\underline{T}$ be a commuting contractive n-tuple and p be a polynomial in n commuting variables. Then*

$$\|p(\underline{T})\|\leq\|p(\underline{S})\|.$$

2.2 Commuting Tuples

In this Section we wish to consider commutative contractive tuples. Let us begin with describing the way one obtains two standard dilations for such tuples.

Recall standard tuples $\underline{V}$ and $\underline{S}$ on Fock spaces $\Gamma(\mathbb{C}^n)$, and $\Gamma_s(\mathbb{C}^n)$ respectively, introduced in Chapter 1. Let $C^*(\underline{V})$, and $C^*(\underline{S})$ be unital C^* algebras generated by them. For any $\alpha,\beta\in\tilde{\Lambda}$, $\underline{V}^\alpha(I-\sum V_iV_i^*)(\underline{V}^\beta)^*$ is the rank one operator $x\mapsto\langle e^\beta,x\rangle e^\alpha$, formed by basis vectors e^α,e^β. So $C^*(\underline{V})$ contains all compact operators. In a similar way we see that $C^*(\underline{S})$ also contains all compact operators of $\Gamma_s(\mathbb{C}^n)$. As $V_i^*V_j=\delta_{ij}I$, it is easy to see that $C^*(\underline{V})=\overline{\text{span}}\ \{\underline{V}^\alpha(\underline{V}^\beta)^*:\alpha,\beta\in\tilde{\Lambda}\}$. By explicit computation commutators $[S_i^*,S_j]$ are compact for all i,j ([Ar4], Proposition 5.3, or [BB]). Therefore we can also obtain $C^*(\underline{S})=\overline{\text{span}}\ \{\underline{S}^\alpha(\underline{S}^\beta)^*:\alpha,\beta\in\tilde{\Lambda}\}$.

Suppose $\underline{T}$ is a contractive tuple on a Hilbert space $\mathcal{H}$. We obtain a certain completely positive map (Popescu's Poisson transform) from $C^*(\underline{V})$ to $\mathcal{B}(\mathcal{H})$, as follows. For $0<r<1$ the tuple $r\underline{T}=(rT_1,\ldots,rT_n)$ is clearly a pure contraction. So by (2.3) we have an isometry $K_r:\mathcal{H}\to\Gamma(\mathbb{C}^n)\otimes\overline{\Delta_r(\mathcal{H})}$ defined by

$$K_rh=\sum_\alpha e^\alpha\otimes\Delta_r((r\underline{T})^\alpha)^*h,\quad h\in\mathcal{H},$$

where $\Delta_r=(I-r^2\sum T_iT_i^*)^{\frac{1}{2}}$. So for every $0<r<1$ we have a completely positive map $\psi_r:C^*(\underline{V})\to\mathcal{B}(\mathcal{H})$ defined by

$$\psi_r(X)=K_r^*(X\otimes I)K_r,\quad X\in C^*(\underline{V}).$$

By taking limit as r increases to 1 (See [Po5] or [AP1] for details), we obtain a unital completely positive map ψ from $C^*(\underline{V})$ to $\mathcal{B}(\mathcal{H})$ satisfying

$$\psi(\underline{V}^\alpha(\underline{V}^\beta)^*)=\underline{T}^\alpha(\underline{T}^\beta)^*\ \text{ for }\alpha,\beta\in\tilde{\Lambda}.$$

As $C^*(\underline{V}) = \overline{\text{span}}\ \{\underline{V}^\alpha(\underline{V}^\beta)^* : \alpha,\beta \in \tilde{\Lambda}\}$, ψ is the unique such completely positive map. Now consider the minimal Stinespring dilation of ψ. So we have a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$, and a unital $*$-homomorphism $\pi : C^*(\underline{V}) \to \mathcal{B}(\tilde{\mathcal{H}})$, such that

$$\psi(X) = P_{\mathcal{H}}\pi(X)|_{\mathcal{H}} \ \ \forall X \in C^*(\underline{V}),$$

and $\overline{\text{span}}\,\{\pi(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}\} = \tilde{\mathcal{H}}$. Taking $\underline{\tilde{V}} = (\tilde{V}_1,\ldots,\tilde{V}_n) = (\pi(V_1),\ldots,\pi(V_n))$, one verifies that each $(\tilde{V}_i)^*$ leaves $\mathcal{H}$ invariant and $\underline{\tilde{V}}$ is the unique minimal isometric dilation of $\underline{T}$.

In a similar fashion if $\underline{T}$ is commuting by considering $C^*(\underline{S})$ instead of $C^*(\underline{V})$, and restricting K_r in the range to $\Gamma_s(\mathbb{C}^n)$, and taking limits as before (See [Ar4], [Po5], [AP1]) we obtain the unique unital completely positive map $\phi : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H})$, satisfying

$$\phi(\underline{S}^\alpha(\underline{S}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^* \qquad \alpha,\beta \in \tilde{\Lambda}.$$

Consider the minimal Stinespring dilation of ϕ. Here we have a Hilbert space $\mathcal{H}_1$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\pi_1 : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H}_1)$, such that

$$\phi(X) = P_{\mathcal{H}}\pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S}),$$

and $\overline{\text{span}}\,\{\pi_1(X)h : X \in C^*(\underline{S}), h \in \mathcal{H}\} = \mathcal{H}_1$. Taking $\underline{\tilde{S}} = (\tilde{S}_1,\ldots,\tilde{S}_n) = (\pi_1(S_1),\ldots,\pi_1(S_n))$, $\underline{\tilde{S}}$ is the standard commuting dilation of $\underline{T}$ by definition (It is not difficult to verify that it is a minimal dilation in the sense of our Definition 2.1.1). As minimal Stinespring dilation is unique up to unitary equivalence, standard commuting dilation is also unique up to unitary equivalence.

THEOREM 2.2.1 *Suppose $\underline{T}$ is a commuting contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece of the minimal isometric dilation of $\underline{T}$ is a realization of the standard commuting dilation of $\underline{T}$.*

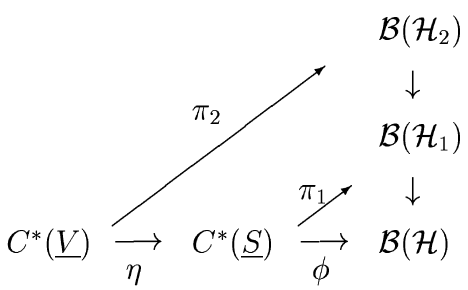
Our approach to prove this theorem is as follows. First we consider the standard commuting dilation of $\underline{T}$ on a Hilbert space $\mathcal{H}_1$ as described above. Now the standard tuple $\underline{S}$ is also a contractive tuple. So we have a unique unital completely positive map $\eta : C^*(\underline{V}) \to C^*(\underline{S})$, satisfying

$$\eta(\underline{V}^\alpha(\underline{V}^\beta)^*) = \underline{S}^\alpha(\underline{S}^\beta)^* \quad \alpha,\beta \in \tilde{\Lambda}.$$

Now clearly $\psi = \phi \circ \eta$. Consider the minimal Stinespring dilation of the composed map $\pi_1 \circ \eta : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_1)$. Here we obtain a Hilbert space $\mathcal{H}_2$ containing $\mathcal{H}_1$ and a unital $*$-homomorphism $\pi_2 : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_2)$, such that

$$\pi_1 \circ \eta(X) = P_{\mathcal{H}_1}\pi_2(X)|_{\mathcal{H}_1}, \quad \forall X \in C^*(\underline{V}),$$

and $\overline{\text{span}}\ \{\pi_2(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_1\} = \mathcal{H}_2$. Now we have a commuting diagram as follows



where all the down arrows are compression maps, horizontal arrows are unital completely positive maps and diagonal arrows are unital $*$-homomorphisms.

Taking $\underline{\hat{V}} = (\hat{V}_1, \ldots, \hat{V}_n) = (\pi_2(V_1), \ldots, \pi_2(V_n))$, we need to show (i) $\underline{\hat{V}}$ is the minimal isometric dilation of $\underline{T}$ and (ii) $\underline{\tilde{S}} = (\pi_1(S_1), \ldots, \pi_1(S_n))$ is the maximal commuting piece of $\underline{\hat{V}}$. Due to uniqueness up to unitary equivalence of minimal Stinespring dilation, we have (i) if we can show that π_2 is a minimal dilation of $\psi = \phi \circ \eta$. For proving this we actually make use of (ii). At first we prove (ii) in a very special case.

DEFINITION 2.2.2 A n-tuple $\underline{T} = (T_1, \ldots, T_n)$ of operators on a Hilbert space $\mathcal{H}$ is called a *spherical unitary* if it is commuting, each T_i is normal, and $T_1T_1^* + \cdots + T_nT_n^* = I$.

Actually, if $\mathcal{H}$ is a finite dimensional Hilbert space and $\underline{T}$ is a commuting tuple on $\mathcal{H}$ satisfying $\sum T_iT_i^* = I$, then it is automatically a spherical unitary, that is, each T_i is normal. This is the case because here standard commuting dilation of $\underline{T}$ is a tuple of normal operators and hence each T_i^* is subnormal (or see [At1] for this result) and all finite dimensional subnormal operators are normal (see [Ha]).

Note that if $\underline{T}$ is a spherical unitary we have $\phi(\underline{S}^{\alpha}(I - \sum S_iS_i^*)(\underline{S}^{\beta})^*) = \underline{T}^{\alpha}(I - \sum T_iT_i^*)(\underline{T}^{\beta})^* = 0$ for any $\alpha, \beta \in \tilde{\Lambda}$. This forces that $\phi(X) = 0$ for any compact operator X in $C^*(\underline{S})$. Now as the commutators $[S_i^*, S_j]$ are all compact we see that ϕ is a unital $*$-homomorphism. So the minimal Stinespring dilation of ϕ is itself. So the following result yields Theorem 2.2.1 for spherical unitaries.

THEOREM 2.2.3 *Let $\underline{T}$ be a spherical unitary on a Hilbert space $\mathcal{H}$. Then the maximal commuting piece of the minimal isometric dilation of $\underline{T}$ is $\underline{T}$.*

As proof of this Theorem involves some lengthy computations we prefer to postpone it. But assuming this, we prove the Theorem 2.3.1.

PROOF OF THEOREM 2.3.1 : As $C^*(\underline{S})$ contains the ideal of all compact operators by standard C^*-algebra theory we have a direct sum decomposition of π_1 as follows. Take $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$ where $\mathcal{H}_{1C} = \overline{\text{span}}\{\pi_1(X)h : h \in \mathcal{H}, X \in C^*(\underline{S}) \text{ and } X \text{ is compact}\}$ and $\mathcal{H}_{1N} = \mathcal{H}_1 \ominus \mathcal{H}_{1C}$, Clearly $\mathcal{H}_{1C}$ is a reducing subspace for π_1. Therefore

$$\pi_1(X) = \begin{pmatrix} \pi_{1C}(X) & \\ & \pi_{1N}(X) \end{pmatrix}$$

that is, $\pi_1 = \pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\mathcal{H}_{1C}}\pi_1(X)P_{\mathcal{H}_{1C}}$, $\pi_{1N}(X) = P_{\mathcal{H}_{1N}}\pi_1(X)P_{\mathcal{H}_{1N}}$. As observed by Arveson [Ar4], $\pi_{1C}(X)$ is just the identity representation with some multiplicity. More precisely, $\mathcal{H}_{1C}$ can be factored as $\mathcal{H}_{1C} = \Gamma_s(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$, such that $\pi_{1C}(X) = X \otimes I$, in particular $\pi_{1C}(S_i) = S_i \otimes I$. Also $\pi_{1N}(X) = 0$ for compact X. Therefore, taking $Z_i = \pi_{1N}(S_i)$, $\underline{Z} = (Z_1, \ldots, Z_n)$ is a spherical unitary.

Now as $\pi_1 \circ \eta = (\pi_{1C} \circ \eta) \oplus (\pi_{1N} \circ \eta)$ and the minimal Stinespring dilation of a direct sum of two completely positive maps is the direct sum of minimal Stinespring dilations. So $\mathcal{H}_2$ decomposes as $\mathcal{H}_2 = \mathcal{H}_{2C} \oplus \mathcal{H}_{2N}$, where $\mathcal{H}_{2C}, \mathcal{H}_{2N}$ are orthogonal reducing subspaces of π_2, such that π_2 also decomposes, say $\pi_2 = \pi_{2C} \oplus \pi_{2N}$, with

$$\pi_{1C} \circ \eta(X) = P_{\mathcal{H}_{1C}}\pi_{2C}(X)|_{\mathcal{H}_{1C}}, \quad \pi_{1N} \circ \eta(X) = P_{\mathcal{H}_{1N}}\pi_{2N}(X)|_{\mathcal{H}_{1N}},$$

for $X \in C^*(\underline{V})$ with $\mathcal{H}_{2C} = \overline{\text{span}}\ \{\pi_{2C}(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_{1C}\}$ and $\mathcal{H}_{2N} = \overline{\text{span}}\ \{\pi_{2N}(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_{1N}\}$. It is also not difficult to see that $\mathcal{H}_{2C} = \overline{\text{span}}\ \{\pi_{2C}(X)h :$

$X \in C^*(\underline{V}), X$ compact, $h \in \mathcal{H}_{1C}\}$ and hence $\mathcal{H}_{2C}$ factors as $\mathcal{H}_{2C} = \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ with $\pi_{2C}(V_i) = V_i \otimes I$. Also $(\pi_{2N}(V_1), \ldots, \pi_{2N}(V_n))$ is a minimal isometric dilation of spherical isometry $(Z_1, \ldots, Z_n)$. Now by Proposition 2.1.8, Theorem 2.2.3 and Corollary 2.1.4, we get that $(\pi_1(S_1), \ldots, \pi_1(S_n))$ acting on $\mathcal{H}_1$ is the maximal commuting piece of $(\pi_2(V_1), \ldots, \pi_2(V_n))$.

All that remains to show is that π_2 is the minimal Stinespring dilation of $\phi \circ \eta$. Suppose this is not the case. Then we get a reducing subspace $\mathcal{H}_{20}$ for π_2 by taking $\mathcal{H}_{20} = \overline{\text{span}}\, \{\pi_2(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}\}$. Take $\mathcal{H}_{21} = \mathcal{H}_2 \ominus \mathcal{H}_{20}$ and correspondingly decompose π_2 as $\pi_2 = \pi_{20} \oplus \pi_{21}$,

$$\pi_2(X) = \begin{pmatrix} \pi_{20}(X) & \\ & \pi_{21}(X) \end{pmatrix}$$

Note that we already have $\mathcal{H} \subseteq \mathcal{H}_{20}$. We claim that $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$. Firstly, as $\mathcal{H}_1$ is the space where the maximal commuting piece of $(\pi_2(V_1), \ldots, \pi_2$ $(V_n)) = (\pi_{20}(V_1) \oplus \pi_{21}(V_1), \ldots, \pi_{20}(V_n) \oplus \pi_{21}(V_n))$ acts, by the first part of Corollary 2.1.4, $\mathcal{H}_1$ decomposes as $\mathcal{H}_1 = \mathcal{H}_{10} \oplus \mathcal{H}_{11}$ for some subspaces $\mathcal{H}_{10} \subseteq \mathcal{H}_{20}$, and $\mathcal{H}_{11} \subseteq \mathcal{H}_{21}$. So for $X \in C^*(\underline{V})$, $P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1}$, has the form (see the diagram)

$$P_{\mathcal{H}_1}\pi_2(X)P_{\mathcal{H}_1} = \begin{pmatrix} \pi_{10} \circ \eta(X) & 0 & & \\ 0 & 0 & & \\ & & \pi_{11} \circ \eta(X) & 0 \\ & & 0 & 0 \end{pmatrix}$$

where π_{10}, π_{11} are compressions of π_1 to $\mathcal{H}_{10}$, $\mathcal{H}_{11}$ respectively. As the mapping η from $C^*(\underline{V})$ to $C^*(\underline{S})$ is clearly surjective, it follows that $\mathcal{H}_{10}, \mathcal{H}_{11}$ are reducing subspaces for π_1. Now as $\mathcal{H}$ is contained in $\mathcal{H}_{20}$, in view of minimality of π_1 as a Stinespring dilation, $\mathcal{H}_1 \subseteq \mathcal{H}_{20}$. But then the minimality of π_2 shows that $\mathcal{H}_2 \subseteq \mathcal{H}_{20}$. Therefore, $\mathcal{H}_2 = \mathcal{H}_{20}$. □

Proof of Theorem 2.2.3 : Here we use the presentation of minimal isometric dilation given by Popescu (2.2). In the present case as $\sum T_i T_i^* = I$, by direct computation D^2 is seen to be a projection. So, D which is the positive square root of D^2, is equal to D^2. Also by Fuglede-Putnam theorem ([Ha], [Pu]), $\{T_1, \ldots, T_n, T_1^*, \ldots, T_n^*\}$ forms a commuting family of operators. Then we get

$$D(h_1, \ldots, h_n) = \sum_{i,j=1}^{n} e_i \otimes T_j(T_j^* h_i - T_i^* h_j) = \sum_{i,j=1}^{n} e_i \otimes T_j(h_{ij}) \qquad (2.4)$$

where $h_{ij} = T_j^* h_i - T_i^* h_j$ for $1 \leq i, j \leq n$. Note that $h_{ii} = 0$ and $h_{ji} = -h_{ij}$.

Now we apply Corollary 2.1.7 to the tuple $\underline{\tilde{V}}$ acting on $\tilde{\mathcal{H}}$. Suppose $y \in \mathcal{H}^{\perp} \bigcap \tilde{\mathcal{H}}^c(\underline{\tilde{V}})$. We wish to show that $y = 0$. We assume $y \neq 0$ and arrive at a contradiction. One can decompose y as $y = 0 \oplus \sum_{\alpha \in \tilde{\Lambda}} e^{\alpha} \otimes y_{\alpha}$, with $y_{\alpha} \in \mathcal{D}$. If for some α, $y_{\alpha} \neq 0$, then $\langle \omega \otimes y_{\alpha}, (\underline{\tilde{V}}^{\alpha})^* y \rangle = \langle e^{\alpha} \otimes y_{\alpha}, y \rangle = \langle y_{\alpha}, y_{\alpha} \rangle \neq 0$. Since each $(\tilde{V}_i)^*$ leaves $\tilde{\mathcal{H}}^c(\underline{\tilde{V}})$ invariant, $(\underline{\tilde{V}}^{\alpha})^* y \in \tilde{\mathcal{H}}^c(\underline{\tilde{V}})$. So without loss of generality we can assume $\|y_0\| = 1$.

Taking $\tilde{y}_m = \sum_{\alpha \in \Lambda^m} e^{\alpha} \otimes y_{\alpha}$, we get $y = 0 \oplus \oplus_{m \geq 0}(\tilde{y}_m)$. As $y_0 \in \mathcal{D}$, $y_0 = D(h_1, \ldots, h_n)$, for some $(h_1, \ldots, h_n)$ (Presently D being a projection its range is closed). Set $\tilde{x}_0 = \tilde{y}_0 = y_0$, and for $m \geq 1$,

$$\tilde{x}_m = \sum_{i_1, \ldots, i_{m-1}, i, j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}).$$

Clearly $\tilde{x}_m \in (\mathbb{C}^n)^{\otimes m} \otimes \mathcal{D}$ for all $m \in \mathbb{N}$. From the Definition (2.2) of $\tilde{V}_i$, commutativity of the operators T_i, and the fact that D is projection, we have

$$\begin{aligned}
& \sum_{1\le i<j\le n} (\tilde{V}_i\tilde{V}_j - \tilde{V}_j\tilde{V}_i)h_{ij} \\
= & \sum_{1\le i<j\le n} (T_iT_jh_{ij} - T_jT_ih_{ij}) + \sum_{1\le i<j\le n} D(e_i \otimes T_jh_{ij} - e_j \otimes T_ih_{ij}) \\
& + \sum_{1\le i<j\le n} (e_i \otimes D(e_j \otimes h_{ij}) - e_j \otimes D(e_i \otimes h_{ij})) \\
= & D\{\sum_{1\le i<j\le n} (e_i \otimes T_jh_{ij} - e_j \otimes T_ih_{ij})\} + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) \\
= & D(\sum_{i,j=1}^{n} e_i \otimes T_jh_{ij}) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) \\
= & D^2(h_1, \ldots, h_n) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) \\
= & \tilde{x}_0 + \tilde{x}_1.
\end{aligned}$$

Therefore $\langle y, \tilde{x}_0 + \tilde{x}_1\rangle = 0$ by Corollary 2.1.7. Now for $m \geq 2$

$$\begin{aligned}
& \sum_{i_1,\ldots,i_{m-1}=1}^{n} \tilde{V}_{i_1}\ldots\tilde{V}_{i_{m-1}}(\sum_{i,j=1}^{n}(\tilde{V}_i\tilde{V}_j - \tilde{V}_j\tilde{V}_i)T^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i}) \\
= & \sum_{i_1,\ldots,i_{m-1}=1}^{n} \tilde{V}_{i_1}\ldots\tilde{V}_{i_{m-1}}[\sum_{i,j=1}^{n} D(e_i \otimes T_jT^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i} \\
& -e_j \otimes T_iT^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i}) + \sum_{i,j=1}^{n}\{e_i \otimes D(e_j \otimes T^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i}) \\
& -e_j \otimes D(e_i \otimes T^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i})\}] \\
= & \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes [D(\sum_{i,j=1}^{n} e_i \otimes T_jT^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i} \\
& -e_j \otimes T_iT^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i}) + \{\sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes T^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i}) \\
& -\sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes T^*_{i_1}\ldots T^*_{i_{m-2}}T^*_i h_{i_{m-1}j})\}]
\end{aligned}$$

(in the term above, i and j have been interchanged in the last summation)

$$\begin{aligned}
= & \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes \\
& [D(\sum_{i=1}^{n} e_i \otimes T^*_{i_1}\ldots T^*_{i_{m-2}}h_{i_{m-1}i} - \sum_{i,j=1}^{n} e_j \otimes T_iT^*_{i_1}\ldots T^*_{i_{m-2}}T^*_j h_{i_{m-1}i})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i,j=1}^{n} e_i \otimes D\{e_j \otimes (T_{i_1}^* \dots T_{i_{m-2}}^* T_j^* h_{i_{m-1}i} - T_{i_1}^* \dots T_{i_{m-2}}^* T_i^* h_{i_{m-1}j})\}] \\
= \quad & \sum_{i_1,\dots,i_{m-1}=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes \sum_{i=1}^{n} D(e_i \otimes T_{i_1}^* \dots T_{i_{m-2}}^* h_{i_{m-1}i}) \\
&+ \sum_{i_1,\dots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes e_i \otimes \\
&D(e_j \otimes T_{i_1}^* \dots T_{i_{m-2}}^* (T_j^* T_i^* h_{i_{m-1}} - T_j^* T_{i_{m-1}}^* h_i - T_i^* T_j^* h_{i_{m-1}} + T_i^* T_{i_{m-1}}^* h_j)) \\
= \quad & \sum_{i_1,\dots,i_{m-2},i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-2}} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \dots T_{i_{m-2}}^* h_{ij}) \\
&+ \sum_{i_1,\dots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \dots T_{i_{m-2}}^* \\
&(-T_j^* T_{i_{m-1}}^* h_i + T_i^* T_{i_{m-1}}^* h_j)) \\
&\text{(in the term above, index } i_{m-1} \text{ has been replaced by } i \\
&\text{and } i \text{ has been replaced by } j \text{ in the first summation)} \\
= \quad & \sum_{i_1,\dots,i_{m-2},i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-2}} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \dots T_{i_{m-2}}^* h_{ij}) \\
&- \sum_{i_1,\dots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \dots T_{i_{m-1}}^* h_{ij}) \\
&= \tilde{x}_{m-1} - \tilde{x}_m.
\end{aligned}
$$

So, $\langle y, \tilde{x}_{m-1} - \tilde{x}_m \rangle = 0$.

Next, we would show that $\|\tilde{x}_{m+1}\| = \|\tilde{x}_0\| = 1$ for all $m \in \mathbb{N}$.

$$
\begin{aligned}
&\|\tilde{x}_{m+1}\|^2 = \langle \sum_{i_1,\dots,i_m,i,j=1}^{n} e_{i_1} \otimes \dots \otimes e_{i_m} \otimes e_i \otimes D(e_j \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij}), \\
&\sum_{i'_1,\dots,i'_{m-1},i',j'=1}^{n} e_{i'_1} \otimes \dots \otimes e_{i'_m} \otimes e_{i'} \otimes D(e_{j'} \otimes T_{i'_1}^* \dots T_{i'_m}^* h_{i'j'}) \rangle \\
= \quad & \sum_{i_1,\dots,i_m,i=1}^{n} \langle \sum_{j=1}^{n} D(e_j \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij}), \sum_{j'=1}^{n} D(e_{j'} \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij'}) \rangle \\
= \quad & \sum_{i_1,\dots,i_m,i=1}^{n} \langle D(\sum_{j=1}^{n} e_j \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij}), \sum_{j'=1}^{n} e_{j'} \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij'} \rangle \\
= \quad & \sum_{i_1,..,i_m,i=1}^{n} \langle \sum_{l,k=1}^{n} e_l \otimes T_k (T_k^* T_{i_1}^* \dots T_{i_m}^* h_{il} - T_l^* T_{i_1}^* \dots T_{i_m}^* h_{ik}), \\
&\sum_{j'=1}^{n} e_{j'} \otimes T_{i_1}^* \dots T_{i_m}^* h_{ij'} \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i_1,..,i_m,i,j=1}^{n} \langle \sum_{k=1}^{n} T_k (T_k^* T_{i_1}^* \dots T_{i_m}^* h_{ij} - T_j^* T_{i_1}^* \dots T_{i_m}^* h_{ik}), T_{i_1}^* \dots T_{i_m}^* h_{ij} \rangle \\
&= \sum_{i,j=1}^{n} \langle \sum_{k=1}^{n} (T_k T_k^* h_{ij} - T_k T_j^* h_{ik}), h_{ij} \rangle \\
&= \sum_{i,j=1}^{n} \langle \sum_{k=1}^{n} (T_k T_k^* T_j^* h_i - T_k T_k^* T_i^* h_j - T_k T_j^* T_k^* h_i + T_k T_j^* T_i^* h_k), T_j^* h_i - T_i^* h_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \sum_{k=1}^{n} (T_k T_j^* T_i^* h_k - T_k T_k^* T_i^* h_j), T_j^* h_i - T_i^* h_j \rangle \\
&= \sum_{i,j=1}^{n} \langle \sum_{k=1}^{n} (T_k T_j^* T_i^* h_k) - T_i^* h_j, T_j^* h_i - T_i^* h_j \rangle \\
&= \sum_{i,j=1}^{n} \langle T_j (\sum_{k=1}^{n} T_k T_j^* T_i^* h_k) - T_j T_i^* h_j, h_i \rangle - \sum_{i,j=1}^{n} \langle T_i (\sum_{k=1}^{n} T_k T_j^* T_i^* h_k) \\
&\quad - T_i T_i^* h_j, h_j \rangle \\
&= \sum_{i=1}^{n} \langle \sum_{j=1}^{n} T_j T_j^* (\sum_{k=1}^{n} T_k T_i^* h_k) - \sum_{j=1}^{n} T_j T_i^* h_j, h_i \rangle - \sum_{j=1}^{n} \langle \sum_{i=1}^{n} T_i T_i^* (\sum_{k=1}^{n} T_k T_j^* h_k) \\
&\quad - \sum_{i=1}^{n} T_i T_i^* h_j, h_j \rangle \\
&= \sum_{i=1}^{n} \langle (\sum_{k=1}^{n} T_k T_i^* h_k) - \sum_{j=1}^{n} T_j T_i^* h_j, h_i \rangle - \sum_{j=1}^{n} \langle \sum_{k=1}^{n} T_k T_j^* h_k - h_j, h_j \rangle \\
&= \sum_{j=1}^{n} \langle h_j - \sum_{k=1}^{n} T_k T_j^* h_k), h_j \rangle = \langle D(h_1, \dots, h_n), (h_1, \dots, h_n) \rangle \\
&= \|\tilde{x}_0\|^2 = 1.
\end{aligned}
$$

As $\langle y, \tilde{x}_0 + \tilde{x}_1 \rangle = 0$ and $\langle y, \tilde{x}_m - \tilde{x}_{m+1} \rangle = 0$ for $m \in \mathbb{N}$, we get $\langle y, \tilde{x}_0 + \tilde{x}_{m+1} \rangle = 0$ for $m \in \mathbb{N}$. This implies $1 = \langle \tilde{y}_0, \tilde{y}_0 \rangle = \langle \tilde{y}_0, \tilde{x}_0 \rangle = -\langle \tilde{y}_{m+1}, \tilde{x}_{m+1} \rangle$. By Cauchy-Schwarz inequality, $1 \leq \|\tilde{y}_{m+1}\| \|\tilde{x}_{m+1}\|$, i.e., $1 \leq \|\tilde{y}_{m+1}\|$ for $m \in \mathbb{N}$. This is a contradiction as $y = 0 \oplus \oplus_{m \geq 0} \tilde{y}_m$ is in the Hilbert space $\tilde{\mathcal{H}}$. □

2.3 Representations of Cuntz Algebras and Related WOT-closed Algebras

For $n \geq 2$, the Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by n-isometries $\underline{s} = \{s_1, \dots, s_n\}$, satisfying Cuntz relations: $s_i^* s_j = \delta_{ij} I, 1 \leq i, j \leq n$, and $\sum s_i s_i^* = I$. It admits many unitarily inequivalent representations. Various classes of representations of $\mathcal{O}_n$ have been constructed in [BJ1-2], [DKS]. Given a tuple of contractions $\underline{T} = (T_1, \dots, T_n)$ on a Hilbert space satisfying $\sum T_i T_i^* = I$, we consider its minimal isometric dilation $\underline{\tilde{V}} = (\tilde{V}_1, \dots, \tilde{V}_n)$. We know that the isometries $\tilde{V}_i$ satisfy Cuntz relations and we obtain a representation $\pi_{\underline{T}}$ of the Cuntz algebra $\mathcal{O}_n$ by setting $\pi_{\underline{T}}(s_i) = \tilde{V}_i$. We wish to classify all representations of $\mathcal{O}_n$ we can obtain by dilating *commuting* contractive tuples $\underline{T}$.

Let $\mathcal{C}_n = C(\partial B_n)$ be the C^*-algebra of all continuous complex valued functions on the sphere $\partial B_n = \{(z_1, \ldots, z_n) : \sum |z_i|^2 = 1\}$. We have a distinguished tuple $\underline{z} = (z_1, \ldots, z_n)$ of elements in $\mathcal{C}_n$ consisting of co-ordinate functions. Given any spherical unitary $\underline{Z} = (Z_1, \ldots, Z_n)$ there is a unique representation of $\mathcal{C}_n$ which maps z_i to Z_i. Now given any commuting n-tuple of operators $\underline{T}$, satisfying $\sum T_i T_i^* = I$, we consider its standard commuting dilation $\underline{\tilde{S}} = (\tilde{S}_1, \ldots, \tilde{S}_n)$. Let $\rho_{\underline{T}}$ be the representation of $\mathcal{C}_n$, obtained by taking $\rho_{\underline{T}}(z_i) = \tilde{S}_i$.

DEFINITION 2.3.1 Let π be representation of $\mathcal{O}_n$ on a Hilbert space $\mathcal{L}$ with $\underline{W} = (W_1, \ldots, W_n) = (\pi(s_1), \ldots, \pi(s_n))$. The representation π is said to be *spherical* if $\overline{\text{span}}\ \{\underline{W}^\alpha h : h \in \mathcal{L}^c(\underline{W}), \alpha \in \tilde{\Lambda}\} = \mathcal{L}$, where $\mathcal{L}^c(\underline{W})$ is the space where the maximal commuting piece $\underline{W}^c$ of $\underline{W}$ acts as in Definition 2.1.6.

Note that this Definition means in particular that if π is spherical then the maximal commuting piece $\underline{W}^c$ is non-trivial. We will see that it is actually a spherical unitary. But this is not a justification for calling such representations as spherical, because this happens for any representation of $\mathcal{O}_n$, as long as $\underline{W}^c$ is non-trivial! The actual justification of this Definition is in Theorem 2.3.3.

THEOREM 2.3.2 *Let $\underline{T} = (T_1, \ldots, T_n)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$, satisfying $\sum T_i T_i^* = I$. Then the representation $\pi_{\underline{T}}$ coming from the minimal isometric dilation of $\underline{T}$ is spherical. Suppose $\underline{R} = (R_1, \ldots, R_n)$ is another commuting tuple, possibly on a different Hilbert space, satisfying $\sum R_i R_i^* = I$. Then the representations $\pi_{\underline{T}}$, $\pi_{\underline{R}}$ of $\mathcal{O}_n$ are unitarily equivalent if and only if the representations $\rho_{\underline{T}}$, $\rho_{\underline{R}}$ of $\mathcal{C}_n$ are unitarily equivalent.*

PROOF: In view of Theorem 2.2.1, the maximal commuting piece of the minimal isometric dilation $\underline{\tilde{V}}$ of $\underline{T}$ is a realization of the standard commuting dilation $\underline{\tilde{S}}$ of $\underline{T}$. The first claim follows easily as the space on which the standard commuting dilation acts includes the original space $\mathcal{H}$. So $\underline{\tilde{V}}$ is the minimal isometric dilation of $\underline{\tilde{S}}$. Similar statement holds for the tuple $\underline{R}$. Now the Theorem follows due to uniqueness up to equivalence of minimal isometric dilation of contractive tuples, and unitary equivalence of maximal commuting pieces of unitarily equivalent tuples. □

So this Theorem reduces the classification problem for representations of $\mathcal{O}_n$ arising out of general commuting tuples to that of representations of $\mathcal{C}_n$. But $\mathcal{C}_n$ being a commutative C^*-algebra, its representations are well-understood and is part of standard C^*-algebra theory. We find the description of this theory as presented in Arveson's classic [Ar3] most suitable for our purposes.

Given any point $w = (w_1, \ldots, w_n) \in \partial B_n$, we have a one dimensional representation ϕ_w of $\mathcal{C}_n$, which maps f to $f(w)$. Of course w is a spherical unitary as operator tuple on $\mathbb{C}$. We can construct the minimal isometric dilation $(W_1^w, \ldots, W_n^w)$ of this tuple as in the proof of Theorem 2.2.3 (Schäffer construction). We see that the dilation space is

$$\mathcal{H}^w = \mathbb{C} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}^n_w) \subseteq \mathbb{C} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}^n),$$

where $\mathbb{C}^n_w$ is the subspace of vectors orthogonal to $(\overline{w_1}, \ldots, \overline{w_n})$ in $\mathbb{C}^n$. Further the operators W_i^w are given by

$$W_i^w(h \oplus \sum_\alpha e^\alpha \otimes d_\alpha) = w_i h \oplus D(e_i \otimes h) \oplus e_i \otimes (\sum_\alpha e^\alpha \otimes d_\alpha).$$

We denote the associated representation of $\mathcal{O}_n$ by ρ_w. This representation is known to be irreducible as it is nothing but the GNS representation of the so-called Cuntz state on $\mathcal{O}_n$ (See [DKS], Example 5.1), given by

$$s_{i_1}\cdots s_{i_m}s_{j_1}^*\cdots s_{j_p}^* \mapsto w_{i_1}\cdots w_{i_m}\overline{w_{j_1}}\cdots\overline{w_{j_p}}.$$

Now an arbitrary multiplicity free representation of $\mathcal{C}_n$ can be described as follows (see [Ar3]). Consider a finite Borel measure μ on ∂B_n. Then we get a representation of $\mathcal{C}_n$ on the Hilbert space $L^2(\partial B_n,\mu)$, which sends $f\in\mathcal{C}_n$ to the operator 'multiplication by f'. This representation can be thought of as direct integral of representations ϕ_w with respect to measure μ. Now it is not hard to see that the associated representation of $\mathcal{O}_n$ is simply the direct integral of representations ρ_w with respect to measure μ and acts on $\oint\mathcal{H}^w\mu(dw)$. Finally an arbitrary representation of $\mathcal{C}_n$ is a countable direct sum of such multiplicity free representations. So we have proved the following result.

THEOREM 2.3.3 *Every spherical representation of $\mathcal{O}_n$ is a direct integral of representations $\rho_w, w\in\partial B_n$ (GNS representations of Cuntz states).*

Here we have not bothered to write down as to when two such representations are equivalent. But in view of Theorem 2.3.2, we can do it exactly as in ([Ar3], page:54-55), by keeping track of multiplicities and equivalence classes of measures.

THEOREM 2.3.4 *Let π be a representation of $\mathcal{O}_n$. Then (i) π decomposes uniquely as $\pi=\pi^0\oplus\pi^1$, where π^0 is spherical and $(\pi^1(s_1),\ldots,\pi^1(s_n))$ has trivial maximal commuting piece (Either π^0 or π^1 could also be absent); (ii) The maximal commuting piece of $(\pi(s_1),\ldots,\pi(s_n))$ is either trivial or it is a spherical unitary. (iii) If π is irreducible then either the maximal commuting piece is trivial or it is one dimensional. In the second case, it is unitarily equivalent to GNS representation of a Cuntz state.*

PROOF: Suppose π is a representation of $\mathcal{O}_n$ on a Hilbert space $\mathcal{L}$ and $\underline{W}=(\pi(s_1),\ldots,\pi(s_n))$. Consider the space $\mathcal{L}^0$ generated by $\mathcal{L}^c(\underline{W})$ as $\mathcal{L}^0=\overline{\text{span}}\,\{\underline{W}^\alpha h : h\in\mathcal{L}^c(\underline{W}),\alpha\in\tilde{\Lambda}\}$. Now each W_i^* leaves $\mathcal{L}^c(\underline{W})$ invariant and clearly $\mathcal{O}_n=C^*\{\underline{s}^\alpha(\underline{s}^\beta)^* : \alpha,\beta\in\tilde{\Lambda}\}$. Then it follows that $\mathcal{L}^0$ is a reducing subspace for π. Taking $\mathcal{L}^1=(\mathcal{L}^0)^\perp$, we decompose π as $\pi^0\oplus\pi^1$ with respect to $\mathcal{L}=\mathcal{L}^0\oplus\mathcal{L}^1$. It is clear that this is a decomposition as required by (i). Uniqueness of this decomposition and (ii) follow easily as maximal commuting piece of direct sum of tuples is direct sum of maximal commuting pieces (Corollary 2.1.4) and then (iii) follows from Theorem 2.3.3. □

Let us see as to what happens if we dilate commuting tuples $\underline{T}$, satisfying just $\sum T_iT_i^*\leq I$. In this case, as is well-known, the minimal isometric dilation decomposes as $((V_1\otimes I)\oplus W_1,\ldots,(V_n\otimes I)\oplus W_n)$ where $(V_1,\ldots,V_n)$ is the standard tuple of full Fock space, and $(W_1,\ldots,W_n)$ are isometries satisfying Cuntz relations. If $\underline{T}$ is not pure the term $(W_1,\ldots,W_n)$ is present and we get a representation of $\mathcal{O}_n$. However, as seen in the proof of Theorem 2.2.1, $(W_1,\ldots,W_n)$ is a minimal isometric dilation of a spherical tuple $(Z_1,\ldots,Z_n)$ (the 'spherical part' of the standard commuting dilation of $\underline{T}$) and hence the representation of $\mathcal{O}_n$ we get is still spherical.

On the other hand it is easy to get examples of non-commuting tuples dilating to representations of $\mathcal{O}_n$ which are not spherical. For instance we can consider the tuple

$\underline{R} = (R_1, R_2)$ on $\mathbb{C}^2$ defined by

$$R_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then as $R_1R_1^* + R_2R_2^* = I$, the minimal isometric dilation of (R_1, R_2) satisfies Cuntz relations. We can see that it has trivial commuting piece through a simple application of Corollary 4.3 of [DKS].

Now we quote the result regarding complete description of the minimal isometric dilation of contractive tuples on finite dimensional Hilbert space and the related WOT-closed algebra obtained by Davidson, Pitts and Shpigel [DPS]. For a contractive n-tuple $\underline{T}$ let us denote the WOT-closed algebras generated by $T_1, \cdots, T_n; V_1, \cdots, V_n$ and $\tilde{V}_1, \cdots, \tilde{V}_n$ by $\mathcal{G}; \mathcal{V}$ and $\mathcal{B}$ respectively. Further denote by $\mathcal{M}_n$ the C^*-algebra generated by all $n \times n$ matrices.

THEOREM 2.3.5 *Assume $\underline{T}$ to be contractive n-tuple on some finite dimensional Hilbert space $\mathcal{H}$ and $\underline{\tilde{V}}$ to be its minimal isometric dilation on Hilbert space $\mathcal{H}_2$. Suppose $\tilde{\mathcal{H}}$ denote the subspace spanned by all minimal B^*-invariant subspaces $\mathcal{W}$ on which $P_{\mathcal{W}} \sum T_iT_i^*|_{\mathcal{W}} = I_{\mathcal{W}}$. Then $\mathcal{H}_{2N} = \mathcal{G}[\tilde{\mathcal{H}}]$ and using an indexing set G one can write $\tilde{\mathcal{H}} = \sum_{g\in G} \mathcal{H}_g^{(m_g)}$ where $\mathcal{H}_g$ are minimal B^*-invariant subspaces of dimension d_g and multiplicity m_g. The compression $\tilde{\mathcal{B}}$ of $\mathcal{B}$ to $\tilde{\mathcal{H}}$ is a C^*-algebra and with respect to the above decomposition we can write $\tilde{\mathcal{H}} = \sum_{g\in G} \mathcal{M}_{d_g} \otimes \mathbb{C}^{m_g}$. Denote by $\mathcal{H}_{2g}$ the minimal dilation space of compression of $\underline{T}$ to $\mathcal{H}_g$ and by P_g the projection onto $\mathcal{H}_g$. Then*

$$\mathcal{H}_2 = \sum_{g\in G} (\mathcal{H}_{2g} \otimes \mathbb{C}^{m_g}) \oplus \mathcal{H}_{2N} = \tilde{\mathcal{H}} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}^l)$$

where $\mathcal{H}_{2g} = \mathcal{H}_g \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}^{l_g}), l_g = d_g(n-1)$,

$$l = (n-1) \sum_{g\in G} l_g m_g + \text{ rank } (I - \sum T_iT_i^*).$$

$$\mathcal{G} \simeq \sum_{g\in G} (B(\mathcal{H}_{2g})P_g) \otimes \mathbb{C}^{m_g} + \mathcal{V} \otimes \mathbb{C}^l$$

Finally we remark that if we are to consider the case $n = \infty$, that is, if we have infinite tuples $\{T_1, T_2, \ldots, \}$, then the standard commuting tuple $\{S_1, S_2, \ldots, \}$, no longer consists of essentially normal operators as the commutators $[S_i^*, S_i]$ have infinite dimensional eigenspaces with non-zero eigenvalues. This is a serious obstacle in extending results of Section 2.2 and 2.3, to infinite tuples.

Chapter 3

Standard Dilations of q-commuting Tuples

For a q-commuting n-tuple $\underline{T}$ on a finite dimensional Hilbert space $\mathcal{H}$ say of dimension m, because of the relation

$$\text{Spectrum}(T_iT_j) \cup \{0\} = \text{Spectrum}(T_jT_i) \cup \{0\} = \text{Spectrum}(q_{ij}T_iT_j) \cup \{0\},$$

we get q_{ij} is either 0 or m^{th}-root of unity. So we work with infinite dimensional Hilbert spaces here.

In Section 3.2 we are able to show that the range of the isometry K defined in equation (2.3) is contained in the q-commuting Fock Space tensored with a Hilbert space when $\underline{T}$ is a pure tuple. Using this we are able to give a condition equivalent to the assertion of the Theorem 3.1.6 to hold for q-commuting pure tuple. The proof of the particular case of Theorem 3.2.6 where $\underline{T}$ is also q-spherical unitary (introduced in Section 3.2) is more difficult than the version for commuting tuple and we had to carefully choose the terms and proceed in a way that 'q_{ij}' of the q-commuting tuples get absorbed or cancel out when we simplify the terms. Also unlike commuting tuple case in Chapter 2 we had to use an inequality related to completely positive map before getting the result through norm estimates.

For a q-commuting tuple $\underline{T} = (T_1, \dots, T_n)$, consider the product $T_{x_1}T_{x_2} \dots T_{x_m}$ where $1 \leq x_i \leq n$. If we replace a consecutive pair say $T_{x_i}T_{x_{i+1}}$ of operators in the above product by $q_{x_{i+1}x_i}T_{x_{i+1}}T_{x_i}$ and do finite number of such operations with different choices of consecutive pairs of these operators appearing in the subsequent product of operators after each such operation, we will get a permutation $\sigma \in \mathcal{S}_m$ such that the final product of operators can be written as $kT_{x_{\sigma^{-1}(1)}}T_{x_{\sigma^{-1}(2)}} \dots T_{x_{\sigma^{-1}(m)}}$ for some $k \in \mathbb{C}$, that is, $T_{x_1}T_{x_2} \dots T_{x_m} = kT_{x_{\sigma^{-1}(1)}}T_{x_{\sigma^{-1}(2)}} \dots T_{x_{\sigma^{-1}(m)}}$. For defining q-commuting tuple in Definition 1.0.2 we needed the known fact that this k depends only on σ and x_i, and not on the different choice of above operations that give rise to the same final product of operators $T_{x_{\sigma^{-1}(1)}}T_{x_{\sigma^{-1}(2)}} \dots T_{x_{\sigma^{-1}(m)}}$. It also follows from the Proposition 3.0.1.

Here after whenever we deal with q-commuting tuples we would have another condition on the tuples that $|q_{ij}| = 1$ for $1 \leq i, j \leq n$. However for the Lemma 3.0.4 and Corollary 3.0.5 we don't need this assumption. Let $\underline{T} = (T_1, \dots, T_n)$ be a q-commuting tuple and consider the product $T_{x_1}T_{x_2} \dots T_{x_m}$ where $1 \leq x_i \leq n$. Let $\sigma \in \mathcal{S}_m$. As transpositions of the type $(k, k+1), 1 \leq k \leq m-1$ generates $\mathcal{S}_m$, let σ^{-1} be $\tau_1 \dots \tau_s$ where for each $1 \leq i \leq s$

there exists k_i such that $1 \leq k_i \leq m-1$ and τ_i is a transposition of the form (k_i, k_i+1). Let $\tilde{\sigma}_i = \tau_{i+1} \ldots \tau_s$ for $1 \leq i \leq s-1$ and $\tilde{\sigma}_s$ be the identity permutation. Let us define $y_i = x_{\tilde{\sigma}_i(k_i)}$ and $z_i = x_{\tilde{\sigma}_i(k_i+1)}$. If we substitute $T_{y_s}T_{z_s}$ by $q_{z_s y_s} T_{z_s} T_{y_s}$ corresponding to τ_s, substitute $T_{y_{s-1}}T_{z_{s-1}}$ by $q_{z_{s-1}y_{s-1}}T_{z_{s-1}}T_{y_{s-1}}$ corresponding to τ_{s-1}, and so on till we substitute the corresponding term for τ_1, we would get $q_1^\sigma(x) \ldots q_s^\sigma(x) T_{x_{\sigma^{-1}(1)}} T_{x_{\sigma^{-1}(2)}} \cdots T_{x_{\sigma^{-1}(m)}}$ where $q_i^\sigma(x) = q_{z_i y_i}$. That is $T_{x_1}T_{x_2}\cdots T_{x_m} = q_1^\sigma(x) \ldots q_s^\sigma(x) T_{x_{\sigma^{-1}(1)}} T_{x_{\sigma^{-1}(2)}} \cdots T_{x_{\sigma^{-1}(m)}}$. Let $q^\sigma(x) = q_1^\sigma(x) \ldots q_s^\sigma(x)$ where $q_i^\sigma(x) = q_{z_i y_i}$.

PROPOSITION 3.0.1 *Let $\underline{T} = (T_1, \ldots, T_n)$ be a q-commuting tuple and consider the product $T_{x_1}T_{x_2}\ldots T_{x_m}$ where $1 \leq x_i \leq n$. Suppose $\sigma \in \mathcal{S}_m$ and $q^\sigma(x)$ be as defined above. Then*

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(k)} x_{\sigma^{-1}(i)}},$$

where product is over $\{(i,k) : 1 \leq i < k \leq m, \sigma^{-1}(i) > \sigma^{-1}(k)\}$. In particular $q^\sigma(x)$ does not depend on the decomposition of σ as product of transpositions.

PROOF: We have

$$q^\sigma(x) = q_1^\sigma(x) \ldots q_s^\sigma(x)$$

where $q_i^\sigma(x) = q_{z_i y_i}$. For a pair i, k such that $1 \leq i < k \leq m$ let $k' = \sigma^{-1}(k)$ and $i' = \sigma^{-1}(i)$. Let $\sigma = \tau_1 \cdots \tau_s$ and $\tilde{\sigma}_i$ be as defined above. If $i' > k'$ then there are odd number of transpositions τ_r for $1 \leq r \leq m$ such that they interchange the positions of i' and k' in the image of $\tilde{\sigma}_r$ when we consider the composition $\tau_r\tilde{\sigma}_r$. And for $1 \leq i < k \leq m$ if $i' < k'$ then there are even number of transpositions τ_r for $1 \leq r \leq m$ such that they interchange the positions of i' and k' in the image of $\tilde{\sigma}_r$ when we consider the composition $\tau_r\tilde{\sigma}_r$. For the first transposition in τ_r that interchanges i' and k', the corresponding factor in $q^\sigma(x)$ say $q_r^\sigma(x)$ is $q_{x_{k'}x_{i'}}$, for the second transposition that interchanges i' and k', the corresponding factor is $q_{x_{i'}x_{k'}}$, for the third transposition that interchanges i' and k', the corresponding factor is $q_{x_{k'}x_{i'}}$, and so on. But $(q_{x_{i'}x_{k'}})^{-1} = q_{x_{k'}x_{i'}}$ and so

$$q^\sigma(x) = \prod q_{x_{\sigma^{-1}(k)} x_{\sigma^{-1}(i)}},$$

where product is over $\{(i,k) : 1 \leq i < k \leq m, \sigma^{-1}(i) > \sigma^{-1}(k)\}$. □

Following similar arguements it is easy to see that if $\sigma \in \mathcal{S}_m$ is such that $(x_1, \cdots, x_n) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)})$, then $q^\sigma(x) = 1$.

DEFINITION 3.0.2 Let $\mathcal{H}, \mathcal{L}$ be two Hilbert spaces such that $\mathcal{H}$ be a closed subspace of $\mathcal{L}$ and let $\underline{T}, \underline{R}$ are n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$ respectively. If $\underline{T}$ is a q-commuting tuple (i.e., $T_jT_i = q_{ij}T_iT_j$, for all i,j), then it is called a *q-commuting piece* of $\underline{R}$.

DEFINITION 3.0.3 Let $\underline{R}$ be a n-tuple of operators on a Hilbert space $\mathcal{M}$. The q-commuting piece $\underline{R}^q = (R_1^q, \ldots, R_n^q)$ obtained by compressing $\underline{R}$ to the maximal element $\mathcal{M}^q(\underline{R})$ of $\mathcal{C}^q(\underline{R})$ is called the *maximal q-commuting piece* of $\underline{R}$ and $\mathcal{M}^q(\underline{R})$ is called the *maximal q-commuting subspace.* The maximal q-commuting piece is said to be *trivial* if $\mathcal{M}^q(\underline{R})$ is the zero space.

The following result gives a description for maximal q-commuting piece and is a consequence of Lemma 2.1.3.

LEMMA 3.0.4 *Let $\underline{R} = (R_1, \ldots, R_n)$ be a n-tuple of bounded operators on a Hilbert space $\mathcal{M}$, $\mathcal{K}_{ij} = \overline{span}\{\underline{R}^\alpha(q_{ij}R_iR_j - R_jR_i)h : h \in \mathcal{M}, \alpha \in \tilde{\Lambda}\}$ for all $1 \leq i, j \leq n$, and $\mathcal{K} = \overline{span} \cup_{i,j=1}^n \mathcal{K}_{ij}$. Then $\mathcal{M}^q(\underline{R}) = \mathcal{K}^\perp$ and $\mathcal{M}^q(\underline{R}) = \{h \in \mathcal{M} : (\overline{q}_{ij}R_j^*R_i^* - R_i^*R_j^*)(\underline{R}^\alpha)^*h = 0, \forall 1 \leq i, j \leq n, \alpha \in \tilde{\Lambda}\}$.*

COROLLARY 3.0.5 1. *Suppose $\underline{R}$, $\underline{T}$ are n-tuples of operators on two Hilbert spaces $\mathcal{L}, \mathcal{M}$. Then the maximal q-commuting piece of $(R_1 \oplus T_1, \ldots, R_n \oplus T_n)$ acting on $\mathcal{L} \oplus \mathcal{M}$ is $(R_1^q \oplus T_1^q, \ldots, R_n^q \oplus T_n^q)$ acting on $\mathcal{L}^q \oplus \mathcal{M}^q$ and the maximal q-commuting piece of $(R_1 \otimes I, \ldots, R_n \otimes I)$ acting on $\mathcal{L} \otimes \mathcal{M}$ is $(R_1^q \otimes I, \ldots, R_n^q \otimes I)$ acting on $\mathcal{L}^q \otimes \mathcal{M}$.*

2. *Let $\underline{T}, \underline{R}$ are n-tuples of bounded operators on $\mathcal{H}$, $\mathcal{L}$, with $\mathcal{H} \subseteq \mathcal{L}$, such that $\underline{R}$ is a dilation of $\underline{T}$. Then $\mathcal{H}^q(\underline{T}) = \mathcal{L}^q(\underline{R}) \bigcap \mathcal{H}$ and $\underline{R}^q$ is a dilation of $\underline{T}^q$.*

PROOF: Follows from Lemma 3.0.4, Corollary 2.1.4 and Proposition 2.1.5. □

3.1 A q-Commuting Fock Space

In this Section we would introduce q-commuting Fock space give two descriptions of it. For any Hilbert space $\mathcal{K}$, we have the full Fock space over $\mathcal{K}$ denoted by $\Gamma(\mathcal{K})$ as,

$$\Gamma(\mathcal{K}) = \mathbb{C} \oplus \mathcal{K} \oplus \mathcal{K}^{\otimes^2} \oplus \cdots \oplus \mathcal{K}^{\otimes^m} \oplus \cdots,$$

We denote the vacuum vector $1 \oplus 0 \oplus \cdots$ by ω. For fixed $n \geq 2$, let $\mathbb{C}^n$ be the n-dimensional complex Euclidian space with usual inner product and $\Gamma(\mathbb{C}^n)$ be the full Fock space over $\mathbb{C}^n$. Let $\{e_1, \ldots, e_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$. For $\alpha \in \tilde{\Lambda}$, e^α will denote the vector $e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m}$ in the full Fock space $\Gamma(\mathbb{C}^n)$ and e^0 will denote the vacuum vector ω. Then the (left) creation operators V_i on $\Gamma(\mathbb{C}^n)$ are defined by

$$V_i x = e_i \otimes x$$

where $1 \leq i \leq n$ and $x \in \Gamma(\mathbb{C}^n)$ (here $e_i \otimes \omega$ is interpreted as e_i). It is obvious that the tuple $\underline{V} = (V_1, \ldots, V_n)$ consists of isometries with orthogonal ranges and $\sum V_i V_i^* = I - I_0$, where I_0 is the projection on to the vacuum space. Let us define *q-commuting Fock space* $\Gamma_q(\mathbb{C}^n)$ as the subspace $(\Gamma(\mathbb{C}^n))^q(\underline{V})$ of the full Fock space. Let $\underline{S} = (S_1, \ldots, S_n)$ be the tuple of operators on $\Gamma_q(\mathbb{C}^n)$ where S_i is the compression of V_i to $\Gamma_q(\mathbb{C}^n)$:

$$S_i = P_{\Gamma_q(\mathbb{C}^n)} V_i|_{\Gamma_q(\mathbb{C}^n)}.$$

Clearly each V_i^* leaves $\Gamma_q(\mathbb{C}^n)$ invariant. Observe that vacuum vector is in $\Gamma_q(\mathbb{C}^n)$. Then it is easy to see that $\underline{S}$ satisfies $\sum S_i S_i^* = I^q - I_0^q$ (where I^q, I_0^q are identity, projection onto vacuum space respectively in $\Gamma_q(\mathbb{C}^n)$). So $\underline{V}$ and $\underline{S}$ are contractive tuples, $S_j S_i = q_{ij} S_i S_j$ for all $1 \leq i, j \leq n$, and $S_i^* x = V_i^* x$, for $x \in \Gamma_q(\mathbb{C}^n)$.

Let $U_\sigma^{m,q}$ be defined on $(\mathbb{C}^n)^{\otimes^m}$ by

$$U_\sigma^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) = q^\sigma(x) e_{x_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma^{-1}(m)}} \qquad (3.1)$$

on the standard basis vectors and extended linearly on $(\mathbb{C}^n)^{\otimes^m}$. As $|q_{ij}| = 1$ for $1 \leq i, j \leq n$, U_σ^m is unitary and U_σ^m extends uniquely to a unitary operator on $(\mathbb{C}^n)^{\otimes^m}$. Let

$$(\mathbb{C}^n)^{\textcircled{q}^m} = \{u \in (\mathbb{C}^n)^{\otimes^m} : U_\sigma^{m,q} u = u \ \forall \sigma \in \mathcal{S}_m\}$$

and $(\mathbb{C}^n)^{\textcircled{q}^0} = \mathbb{C}$. From easy combinatorial arguements we observe that

$$\text{dimension of } (\mathbb{C}^n)^{\textcircled{q}^m} = \begin{pmatrix} n+m+1 \\ m \end{pmatrix}.$$

LEMMA 3.1.1 *The map defined from $\mathcal{S}_m$ to $B((\mathbb{C}^n)^{\otimes^m})$ defined by $\sigma \mapsto U_\sigma^{m,q}$ is a unitary representation of the permutation group $\mathcal{S}_m$.*

PROOF: Let $\otimes_{i=1}^m e_{x_i}, \otimes_{i=1}^m e_{y_i} \in (\mathbb{C}^n)^{\otimes^m}, 1 \leq x_i, y_i \leq n$. Suppose there exists $\sigma \in \mathcal{S}_m$ such that $\otimes_{i=1}^m e_{y_i} = \otimes_{i=1}^m e_{x_{\sigma^{-1}(i)}}$. Then $\langle U_\sigma^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle = q^\sigma(x)$ and $\langle \otimes_{i=1}^m e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle = \overline{q^{(\sigma^{-1})}(y)}$. Also

$$q^{(\sigma^{-1})}(y) = \prod q_{y_{\sigma(k)} y_{\sigma(i)}} = \prod q_{x_k x_i}$$

where the products are over $\{(i,k) : 1 \leq i < k \leq m, \sigma(i) > \sigma(k)\}$. If we substitute $k = \sigma^{-1}(i')$ and $i = \sigma^{-1}(k')$ in the last term we get

$$q^{(\sigma^{-1})}(y) = \prod q_{x_{\sigma^{-1}(i')} x_{\sigma^{-1}(k')}} = (\prod q_{x_{\sigma^{-1}(k')} x_{\sigma^{-1}(i')}})^{-1} = (q^\sigma(x))^{-1}$$

where the products are over $\{(i', k') : 1 \leq i' < k' \leq m, \sigma^{-1}(i') > \sigma^{-1}(k')\}$. So

$$q^\sigma(x) = (q^{(\sigma^{-1})}(y))^{-1} = \overline{q^{(\sigma^{-1})}(y)}.$$

The last equality holds as $|q_{ij}| = 1$. This implies $\langle U_\sigma^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle = \langle \otimes_{i=1}^m e_{x_i}, U_{\sigma^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle$. If there does not exists any $\sigma \in \mathcal{S}_m$ such that $\otimes_{i=1}^m e_{y_i} = \otimes_{i=1}^m e_{x_{\sigma^{-1}(i)}}$ then

$$\langle U_{\sigma'}^{m,q}(\otimes_{i=1}^m e_{x_i}), \otimes_{i=1}^m e_{y_i}\rangle = 0 = \langle \otimes_{i=1}^m e_{x_i}, U_{(\sigma')^{-1}}^{m,q}(\otimes_{i=1}^m e_{y_i})\rangle$$

for all $\sigma' \in \mathcal{S}_m$. So $(U_\sigma^{m,q})^* = U_{\sigma^{-1}}^{m,q}$ for $\sigma \in \mathcal{S}_m$, when acting on the basis elements of the $(\mathbb{C}^n)^{\otimes^m}$, and hence is true for all elements $(\mathbb{C}^n)^{\otimes^m}$.

Next let $\sigma \in \mathcal{S}_m$ be equal to $\sigma_1\sigma_2$ for some $\sigma_1, \sigma_2 \in \mathcal{S}_m$. We would show that $U_\sigma^{m,q} = U_{\sigma_1}^{m,q} U_{\sigma_2}^m$. Let $e_x = e_{x_1} \otimes \ldots \otimes e_{x_m}$ where $x_j \in \{1, \ldots, n\}$ for $1 \leq j \leq m$. Let $\sigma_1^{-1} = \tau_1 \ldots \tau_r$ and $\sigma_2^{-1} = \tau_{r+1} \ldots \tau_s$ where for each $1 \leq i \leq s$, there exists k_i such that $1 \leq k_i \leq m-1$ and τ_i is a transposition of the form $(k_i, k_i + 1)$.

$$\begin{aligned} U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) &= U_{\sigma_1}^{m,q}(q^{\sigma_2}(x) e_{x_{\sigma_2^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma_2^{-1}(m)}}) \\ &= q^{\sigma_1}(z) q^{\sigma_2}(x) e_{x_{\sigma_2^{-1}\sigma_1^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma_2^{-1}\sigma_1^{-1}(m)}} \end{aligned}$$

where $e_z = e_{z_1} \otimes \ldots \otimes e_{z_m}$, i.e, $z_i = x_{\sigma_2^{-1}(i)}$. But as $\sigma = \tau_1 \ldots \tau_r \tau_{r+1} \ldots \tau_s$ it is easy to see that $q^\sigma(x) = q^{\sigma_1}(z) q^{\sigma_2}(x)$ using the definition of $q^\sigma(x)$. So we get

$$U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}) = q^\sigma(x) e_{x_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{x_{\sigma^{-1}(m)}} = U_\sigma^{m,q}(e_{x_1} \otimes \ldots \otimes e_{x_m}).$$

And hence $U_{\sigma_1\sigma_2}^{m,q} = U_{\sigma_1}^{m,q} U_{\sigma_2}^{m,q}$. $\square$

In the next Lemma and Theorem we derive a formula for the projection operator onto the q-commuting Fock space.

LEMMA 3.1.2 *Let P_m be the operator on $(\mathbb{C}^n)^{\otimes m}$ defined by*

$$P_m = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} U^{m,q}_\sigma. \tag{3.2}$$

Then P_m is a projection of $(\mathbb{C}^n)^{\otimes m}$ onto $(\mathbb{C}^n)^{@m}$.

PROOF: First we see that

$$P^*_m = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} (U^{m,q}_\sigma)^* = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} U^{m,q}_{\sigma^{-1}} = P_m,$$

Consider a permutation $\sigma' \in \mathcal{S}_m$.

$$P_m U^{m,q}_{\sigma'} = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} U^{m,q}_{\sigma\sigma'} = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} U^{m,q}_\sigma = P_m. \tag{3.3}$$

Similarly $U^{m,q}_{\sigma'} P_m = P_m$. So $P^2_m = P_m$ and hence P_m is a projection. □

THEOREM 3.1.3 $\oplus_{m=0}^\infty (\mathbb{C}^n)^{@m} = \Gamma_q(\mathbb{C}^n)$

PROOF: Let $Q = \oplus_{m=0}^\infty P_m$ be the projection of $\Gamma(\mathbb{C}^n)$ onto $\oplus_{m=0}^\infty (\mathbb{C}^n)^{@m}$ where P_m is a defined in Lemma 3.1.2. We would show that $\oplus_{m=0}^\infty (\mathbb{C}^n)^{@m}$ is left invariant by V_i^*. Let $\otimes_{j=1}^m e_{x_j} \in (\mathbb{C}^n)^{\otimes m}, 1 \leq x_j \leq n$. Then $V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\}$ is zero if none of x_j is equal to i. Otherwise $V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\}$ is some non-zero element belonging to $\oplus_{m=0}^\infty (\mathbb{C}^n)^{@(m-1)}$ because of the following: Suppose $x_j = i$ if and only if $j \in \{r_1, \ldots, r_p\}$, and let $\mathcal{A}_k$ be the set of all $\sigma \in S_m$ such that σ^{-1} sends 1 to $r_k, 1 \leq k \leq p$, then each element of $\mathcal{A}_k$ is a composition $\tau\rho$ where τ is the transposition $(1, r_k)$ and a permutation ρ which keeps r_k fixed and permutes rest of the $m-1$ symbols and viceversa. Let $x = (x_1, \cdots, x_m)$ and $y = (x_{\tau^{-1}(1)}, \cdots, x_{\tau^{-1}(m)})$. As V_i are isometries with orthogonal ranges,

$$V_i^*\{P_m(\otimes_{j=1}^m e_{x_j})\} = V_i^*\{\frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} U^{m,q}_\sigma(\otimes_{j=1}^m e_{x_j})\}$$

$$= \frac{1}{m!} \sum_{k=1}^p V_i^* \{ \sum_{\tau\rho \in \mathcal{A}_k} U^{m,q}_\tau U^{m,q}_\rho (\otimes_{j=1}^m e_{x_j})\}$$

$$= \frac{1}{m!} \sum_{k=1}^p q_{x_1 i} q^\rho(y) V_i^* \{ \sum_{\tau\rho \in \mathcal{A}_k} (e_{x_{\rho^{-1}(r_k)}} \otimes e_{x_{\rho^{-1}(2)}} \otimes \cdots \otimes e_{x_{\rho^{-1}(r_{k-1})}} \otimes$$
$$e_{x_{\rho^{-1}(1)}} \otimes e_{x_{\rho^{-1}(r_{k+1})}} \otimes \cdots \otimes e_{x_{\rho^{-1}(m)}})\}$$

$$= \frac{1}{m!} \sum_{k=1}^p q_{x_1 i} q^\rho(y) V_i^* \{ \sum_{\tau\rho \in \mathcal{A}_k} (e_i \otimes e_{x_{\rho^{-1}(2)}} \otimes \cdots \otimes e_{x_{\rho^{-1}(r_{k-1})}} \otimes e_{x_{\rho^{-1}(1)}}$$
$$\otimes e_{x_{\rho^{-1}(r_{k+1})}} \otimes \cdots \otimes e_{x_{\rho^{-1}(m)}})\}$$

$$= \sum_{k=1}^p \frac{q_{x_1 i}}{m!} \{ q^\rho(y) \sum_{\rho \in \mathcal{S}_{m-1}} (e_{x_{\rho^{-1}(2)}} \otimes \cdots \otimes e_{x_{\rho^{-1}(r_{k-1})}} \otimes e_{x_{\rho^{-1}(1)}}$$
$$\otimes e_{x_{\rho^{-1}(r_{k+1})}} \otimes \cdots \otimes e_{x_{\rho^{-1}(m)}})\}$$

$$= \sum_{k=1}^p a_k(x) P_{m-1}(\otimes_{j=1}^m e_{x_1} \otimes \cdots \otimes \hat{e}_{x_{r_k}} \otimes \cdots \otimes e_{x_m})$$

where $a_k(x)$ are constants and $e_{x_1} \otimes \cdots \otimes \hat{e}_{x_p} \otimes \cdots \otimes e_{x_m}$ denotes the term $e_{x_1} \otimes \cdots \otimes e_{x_{p-1}} \otimes e_{x_{p+1}} \otimes \cdots \otimes e_{x_m}$. This shows that $\oplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^m}$ is left invariant by V_i^*.

Taking $R_i = QV_iQ$ we would show that $\underline{R}$ is q-commuting. For transposition $(1,2)$, let us define $U_{(1,2)}^q$ as $\oplus_{m=o}^{\infty} U_{(1,2)}^{m,q}$ where $U_{(1,2)}^{0,q} = I$ and $U_{(1,2)}^{1,q} = I$. Let $\otimes_{i=1}^k e_{\alpha_i} \in (\mathbb{C}^n)^{\otimes^k}, 1 \leq \alpha_i \leq n$. Using results of Lemma 3.1.2 we get

$$\begin{aligned} R_iR_j\underline{R}^{\alpha}\omega &= QV_iV_j\underline{V}^{\alpha}\omega = QU_{(1,2)}^q V_iV_j(\otimes_{i=1}^k e_{\alpha_i}) \\ &= QU_{(1,2)}^q\{e_i \otimes e_j \otimes (\otimes_{i=1}^k e_{\alpha_i})\} = Qq_{ji}e_j \otimes e_i \otimes (\otimes_{i=1}^k e_{\alpha_i}) \\ &= q_{ji}QV_jV_i\underline{V}^{\alpha}\omega = q_{ji}R_jR_i\underline{R}^{\alpha}\omega. \end{aligned}$$

and clearly $\oplus_{m=0}^{\infty}(\mathbb{C}^n)^{@^m} = \overline{\text{span}}\{\underline{R}^{\alpha}\omega : \alpha \in \tilde{\Lambda}\}$. So $(R_1, \ldots, R_n)$ is a q-commuting piece of $\underline{V}$.

To show maximality we make use of Proposition 3.0.4. Suppose $x \in \Gamma(\mathbb{C}^n)$ and $\langle x, \underline{V}^{\alpha}(q_{ij}V_iV_j - V_jV_i)y\rangle = 0$ for all $\alpha \in \tilde{\Lambda}, 1 \leq i,j \leq n$ and $y \in \Gamma(\mathbb{C}^n)$. We wish to show that $x \in \Gamma_q(\mathbb{C}^n)$. Suppose x_m is the m-particle component of x, i.e., $x = \oplus_{m\geq 0}x_m$ with $x_m \in (\mathbb{C}^n)^{\otimes^m}$ for $m \geq 0$. For $m \geq 2$ and any permutation σ of $\{1,2,\ldots,m\}$ we need to show that the unitary $U_{\sigma}^{m,q} : (\mathbb{C}^n)^{\otimes^m} \to (\mathbb{C}^n)^{\otimes^m}$, defined by equation (3.1) leaves x_m fixed. Since $\mathcal{S}_m$ is generated by the set of transpositions $\{(1,2),\ldots,(m-1,m)\}$ it is enough to verify $U_{\sigma}^{m,q}(x_m) = x_m$ for permutations σ of the form $(i, i+1)$. So fix m and i with $m \geq 2$ and $1 \leq i \leq (m-1)$. We have

$$\langle \oplus_p x_p, \underline{V}^{\alpha}(q_{kl}V_kV_l - V_lV_k)\underline{V}^{\beta}\omega\rangle = 0, \tag{3.4}$$

for every $\beta \in \tilde{\Lambda}, 1 \leq k,l \leq n$. This implies that

$$\langle x_m, e^{\alpha} \otimes (q_{kl}e_k \otimes e_l - e_l \otimes e_k) \otimes e^{\beta}\rangle = 0$$

for any $\alpha \in \Lambda^{i-1}, \beta \in \Lambda^{m-i-1}$. So if

$$x_m = \sum a(s,t,\alpha,\beta)e^{\alpha} \otimes e_s \otimes e_t \otimes e^{\beta}$$

where the sum is over $\alpha \in \Lambda^{i-1}, \beta \in \Lambda^{m-i-1}$ and $1 \leq s,t \leq n$, and $a(s,t,\alpha,\beta)$ are constants, then for fixed α and β it follows from equation (3.4) that $\overline{q}_{kl}a(k,l,\alpha,\beta) = a(l,k,\alpha,\beta)$ or $q_{lk}a(k,l,\alpha,\beta) = a(l,k,\alpha,\beta)$. Therefore for $\sigma = (i, i+1)$

$$\begin{aligned} &U_{\sigma}^{m,q}(a(k,l,\alpha,\beta)e^{\alpha} \otimes e_k \otimes e_l \otimes e^{\beta} + a(l,k,\alpha,\beta)e^{\alpha} \otimes e_l \otimes e_k \otimes e^{\beta}) \\ &= q_{lk}a(k,l,\alpha,\beta)e^{\alpha} \otimes e_l \otimes e_k \otimes e^{\beta} + q_{kl}a(l,k,\alpha,\beta)e^{\alpha} \otimes e_k \otimes e_l \otimes e^{\beta} \\ &\quad = a(l,k,\alpha,\beta)e^{\alpha} \otimes e_l \otimes e_k \otimes e^{\beta} + a(k,l,\alpha,\beta)e^{\alpha} \otimes e_k \otimes e_l \otimes e^{\beta} \end{aligned}$$

This clearly implies $U_{\sigma}^{m,q}(x_m) = x_m$. □

COROLLARY 3.1.4 *For* $u \in (\mathbb{C}^n)^{\otimes^k}, v \in (\mathbb{C}^n)^{\otimes^l}, w \in (\mathbb{C}^n)^{\otimes^m}$

$$P_{k+l+m}\{P_{k+l}(u \otimes v) \otimes w\} = P_{k+l+m}\{u \otimes P_{l+m}(v \otimes w)\}.$$

PROOF: If we identify $\mathcal{S}_{k+l}$ and $\mathcal{S}_{l+m}$ with the subgroups of $\mathcal{S}_{k+l+m}$ such that $\sigma \in \mathcal{S}_{k+l}$ fixes last m elements of $\{1,2,\cdots k+l+m\}$ and $\sigma \in \mathcal{S}_{l+m}$ fixes first k elements of $\{1,2,\cdots,k+l+m\}$, the Corollary follows easily using equation (3.3). □

When $q_{ij} = 1$ for all i, j, we denote $(\mathbb{C}^n)^{\textcircled{q}m}$ by $(\mathbb{C}^n)^{\textcircled{s}m}$ and the q-commuting Fock space $\Gamma_q(\mathbb{C}^n)$ by $\Gamma_s(\mathbb{C}^n)$. This $\Gamma_s(\mathbb{C}^n)$ is called the *symmetric Fock space (or Boson Fock space).* The map $U^{m,q} : \mathcal{S}_m \to B(\mathbb{C}^n)^{\otimes m}$ given by

$$U^{m,q}(\sigma) = U^{m,q}_\sigma$$

gives the representation of $\mathcal{S}_m$ on $B(\mathbb{C}^n)^{\otimes m}$. Let $U^{m,q}_\sigma$ be denoted by $U^{m,s}_\sigma$ where $q_{ij} = 1$ for all i, j. It is easy to see that for all $q = (q_{ij})_{n\times n}$, $|q_{ij}| = 1$, the representations are unitarily equivalent. So there exists unitary $W^{m,q} : (\mathbb{C}^n)^{\otimes m} \to (\mathbb{C}^n)^{\otimes m}$ such that

$$W^{m,q}U^{m,s}_\sigma = U^{m,q}_\sigma W^{m,q}. \tag{3.5}$$

This $W^{m,q}$ is not unique as for $k \in \mathbb{C}$ such that $|k| = 1$, the operator $kW^{m,q}$ is also a unitary satisfying equation (3.5). We will give one such $W^{m,q}$ explicitely.

For $m \in \mathbb{N}, y_i \in \Lambda$ define $W^{m,q}$ over $(\mathbb{C}^n)^{\otimes m}$ as

$$W^{m,q}(e_{y_1} \otimes \ldots \otimes e_{y_m}) = q^{(\tau^{-1})}(x)e_{y_1} \otimes \ldots \otimes e_{y_m}.$$

where $x = (x_1, \cdots, x_m)$ is the tuple got by rearranging $(y_1, \cdots, y_m)$ in nondecreasing order and $\tau \in \mathcal{S}_m$ such that $y_i = x_{\tau(i)}$. From Proposition 3.0.1 its clear that $q^{(\tau^{-1})}(x)$ does not depend upon the choice of τ and

$$\begin{aligned}
& W^{m,q}U^{m,s}_\sigma(e_{y_1} \otimes \ldots \otimes e_{y_m}) = W^{m,q}(e_{y_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{y_{\sigma^{-1}(m)}}) \\
= \;& q^{(\sigma^{-1}\tau)^{-1}}(x)e_{y_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{y_{\sigma^{-1}(m)}} \\
= \;& q^{(\tau^{-1})\sigma}(x)e_{y_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{y_{\sigma^{-1}(m)}} \\
= \;& q^{\sigma}(x_{\tau(1)}, \cdots, x_{\tau(m)})q^{(\tau^{-1})}(x)e_{y_{\sigma^{-1}(1)}} \otimes \ldots \otimes e_{y_{\sigma^{-1}(m)}} \\
= \;& U^{m,q}_\sigma q^{(\tau^{-1})}(x)e_{y_1} \otimes \ldots \otimes e_{y_m} = U^{m,q}_\sigma W^{m,q}(e_{y_1} \otimes \ldots \otimes e_{y_m}).
\end{aligned}$$

So, $W^{m,q}U^{m,s}_\sigma = U^{m,q}_\sigma W^{m,q}$. Denoting the unitary operator $\oplus_{m=0}^{\infty} W^{m,q}$ on $\Gamma(\mathbb{C}^n)$ by W^q where $W^{0,q} = I$, we get

$$W^q P_{\Gamma_S(\mathbb{C}^n)} = P_{\Gamma_q(\mathbb{C}^n)} W^q$$

and for q and q' we get intertwining unitary $W^{q'}(W^q)^*$ such that

$$W^{q'}(W^q)^* P_{\Gamma_q(\mathbb{C}^n)} = P_{\Gamma_{q'}(\mathbb{C}^n)} W^{q'}(W^q)^*.$$

Under Schur product $\mathcal{Q} = \{q = (q_{ij})_{n\times n} : |q_{ij}| = 1\}$ forms a group.

PROPOSITION 3.1.5 *The map from $\mathcal{Q}$ to $B((\mathbb{C}^n)^{\otimes m})$ given by $q \mapsto W^{m,q}$ is a unitary representation of $\mathcal{Q}$.*

PROOF: From the definition of $W^{m,q}$ we get

$$W^{m,q.q'} = W^{m,q}W^{m,q'} \quad \text{and} \quad (W^{m,q})^{-1} = W^{m,q^{-1}}$$

for $q, q' \in \mathcal{Q}$ and $q^{-1} = (q^{-1}_{ij})_{n\times n}$. When q is the identity element of $\mathcal{Q}$, all entries $q_{ij} = 1$ and therefore $W^{m,q}$ is the identity matrix. Hence the Proposition holds. ☐

Let $(\mathbb{C}^n)^{@m}$ be defined as

$$(\mathbb{C}^n)^{@m} = \{u \in (\mathbb{C}^n)^{\otimes m} : U_\sigma^{m,s}(u) = \text{sign}(\sigma)u \quad \forall \sigma \in \mathcal{S}_m\}$$

then the *antisymmetric Fock space* or *Fermion Fock space* $\Gamma_a(\mathbb{C}^n)$ is defined as

$$\Gamma_a(\mathbb{C}^n) = \oplus_{m=0}^{\infty} (\mathbb{C}^n)^{@m}.$$

We observed before that the symmetric Fock space is the q-commuting Fock space where $q_{ij} = 1$. But the antisymmetric Fock space is not equal to any of $\Gamma_q(\mathbb{C}^n)$. But consider the case when $q = (q_{ij})_{n \times n}$ is such that $q_{ij} = -1$ for $1 \le i \ne j \le n$. Then antisymmetric Fock space $\Gamma_a(\mathbb{C}^n)$ is a proper subset of $\Gamma_q(\mathbb{C}^n)$ because of the following: Clearly $(\mathbb{C}^n)^{@m}$ is the set of all $u \in (\mathbb{C}^n)^{@m}$ which are orthogonal to those $P_m e^\beta$ for which there exists $s, t \in \{1, 2, \cdots, m\}, s \ne t$ such that $\beta_s = \beta_t$ (P_m is given by equation (3.2)).

Next we would give another realization of the standard tuple $\underline{S}$. Let $\mathcal{P}$ be the vector space of all polynomials in q-commuting variables $z_1, \ldots, z_n$ that is $z_j z_i = q_{ij} z_i z_j$. Any multi-index $\underline{k}$ is a ordered n-tuple of non-negative integers $(k_1, \ldots, k_n)$. We shall write $k_1 + \ldots + k_n$ as $|\underline{k}|$. The special multi-index which has 0 in all positions except the i^{th} one, where it has 1, is denoted by $\underline{e_i}$. For any non-zero multi-index $\underline{k}$ the monomial $z_1^{k_1} \ldots z_n^{k_n}$ will be denoted by $\underline{z}^{\underline{k}}$ and for the multi-index $\underline{k} = (0, \ldots, 0)$, let $\underline{z}^{\underline{k}}$ be the complex number 1. Let us have the following inner product with it. Declare $\underline{z}^{\underline{k}}$ and $\underline{z}^{\underline{l}}$ orthogonal if $\underline{k}$ is not the same as $\underline{l}$ as ordered multi-indices. Let

$$\|\underline{z}^{\underline{k}}\|^2 = \frac{k_1! \cdots k_n!}{|k|!}.$$

Note that the following inner-product is also refered in [BB] in Definition (1.1) for general case. Now define $\mathcal{H}'$ to be the closure of $\mathcal{P}$ with respect to this inner product. Define a tuple $\underline{S}' = (S_1', \ldots, S_n')$ where each S_i' is defined for $f \in \mathcal{P}$ by

$$S_i' f(z_1, \ldots, z_n) = z_i f(z_1, \ldots, z_n)$$

and S_i is linearly extended to $\mathcal{H}'$. In the case of our standard q-commuting n-tuple $\underline{S}$ of operators on $\Gamma_q(\mathbb{C}^n)$, when $\underline{k} = (k_1, \ldots, k_n)$ let $\underline{S}^{\underline{k}} = S_1^{k_1} \ldots S_n^{k_n}$ and when $\underline{k} = (0, \ldots, 0)$ let $\underline{S}^{\underline{k}} = 1$.

Using (3.2) and the fact that V_i's are isometries with orthogonal ranges for $\underline{k} = (k_1, \ldots, k_n), |\underline{k}| = m$ we get

$$\|\underline{S}^{\underline{k}}\omega\| = \langle P_m \underline{V}^{\underline{k}}\omega, \underline{V}^{\underline{k}}\omega\rangle = \langle \frac{1}{|\underline{k}|!} \sum_{\sigma \in \mathcal{S}_m} U_\sigma^{m,q} \underline{V}^{\underline{k}}\omega, \underline{V}^{\underline{k}}\omega\rangle = \frac{k_1! \cdots k_n!}{|\underline{k}|!}.$$

When we denote $\underline{V}^{\underline{k}}\omega$ as $e_{x_1} \otimes \cdots \otimes e_{x_m}, 1 \le x_i \le n$, then to get the last term of the above equation we used the fact that there are $k_1! \cdots k_n!$ permutations $\sigma \in \mathcal{S}_m$ such that $e_{x_1} \otimes \cdots \otimes e_{x_m} = e_{x_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{x_{\sigma^{-1}(m)}}$. Next we show that the above tuples $\underline{S}'$ and $\underline{S}$ are unitarily equivalent.

PROPOSITION 3.1.6 *Let $\underline{S}' = (S_1', \ldots, S_n')$ be the operator tuples on $\mathcal{H}'$ as introduced above and let $\underline{S} = (S_1, \ldots, S_n)$ be the standard q-commuting tuple of operators on $\Gamma_q(\mathbb{C}^n)$. Then there exists unitary $U : \mathcal{H}' \to \mathcal{H}$ such that $US_i' = S_i U$ for $1 \le i \le n$.*

PROOF : Define $U : \mathcal{P} \to \Gamma_q(\mathbb{C}^n)$ as

$$U(\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}}) = \sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{S}^{\underline{k}}\omega$$

where $b_{\underline{k}}\underline{z}^{\underline{k}} \in \mathcal{P}$, $b_{\underline{k}}$ are constants. As $\|\underline{z}^{\underline{k}}\| = \|\underline{S}^{\underline{k}}\omega\|$ we have

$$\|\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}}\|^2 = \sum_{|\underline{k}|\leq s} |b_{\underline{k}}|^2\|\underline{z}^{\underline{k}}\|^2 = \sum_{|\underline{k}|\leq s} |b_{\underline{k}}|^2\|\underline{S}^{\underline{k}}\omega\|^2 = \|\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{S}^{\underline{k}}\omega\|^2.$$

So we can extend it linearly to $\mathcal{H}'$ and U is a unitary.

$$\begin{aligned} US'_i(\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}}) &= U(z_i\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}}) = q_{1i}^{k_1}\cdots q_{i-1i}^{k_{i-1}}U(\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}+\underline{e}_i}) \\ &= q_{1i}^{k_1}\cdots q_{i-1i}^{k_{i-1}}\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{S}^{\underline{k}+\underline{e}_i}\omega = S_i(\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{S}^{\underline{k}}\omega) \\ &= S_iU(\sum_{|\underline{k}|\leq s} b_{\underline{k}}\underline{z}^{\underline{k}}), \end{aligned}$$

i.e., $US'_i = S_iU$ for $1 \leq i \leq n$. □

For any complex number z, the z-commutator of two operators X, Y is defined as:

$$[X,Y]_z = XY - zYX.$$

The following Lemma holds for $\underline{S}$ as $\underline{S}'$ and $\underline{S}$ are unitarily equivalent and the same properties have been proved for $\underline{S}'$ in [BB].

LEMMA 3.1.7 *1. Each monomial $\underline{S}^{\underline{k}}\omega$ is an eigenvector for $\sum S_i^*S_i - I$, so that it is a diagonal operator on the standard basis. In fact,*

$$\sum_{i=1}^n S_i^*S_i(\underline{S}^{\underline{k}}\omega) = \left(\sum_{i=1}^n \frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2}\right)\underline{S}^{\underline{k}}\omega.$$

Also $\sum S_i^*S_i - I$ is compact.

2. The commutator $[S_i^*, S_i]$ is as follows:

$$[S_i^*, S_i]\underline{S}^{\underline{k}}\omega = \left(\frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2} - \frac{\|\underline{S}^{\underline{k}}\omega\|^2}{\|\underline{S}^{\underline{k}-\underline{e}_i}\omega\|^2}\right)\underline{S}^{\underline{k}}\omega, \text{ when } k_i \neq 0.$$

If $k_i = 0$, then $[S_i^*, S_i]\underline{S}^{\underline{k}}\omega = S_i^*S_i\underline{S}^{\underline{k}}\omega = \frac{\|\underline{S}^{\underline{k}+\underline{e}_i}\omega\|^2}{\|\underline{S}^{\underline{k}}\omega\|^2}\underline{S}^{\underline{k}}\omega$.

3. $[S_i^*, S_j]_{q_{ij}}$ is compact for all $1 \leq i, j \leq n$.

3.2 Dilation of q-Commuting Tuples

Take $\tilde{\mathcal{H}} = \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$, and define an operator $K : \mathcal{H} \to \tilde{\mathcal{H}}$ as in equation (2.3). (this operator was used for q-commuting case by Arias, Popescu [AP1], Bhat and Bhattacharyya [BB]).

LEMMA 3.2.1 *Suppose $\underline{T} = (T_1, \cdots, T_n)$ is a pure q-commuting tuple on a Hilbert Space $\mathcal{H}$ and let K be the operator introduced in equation (2.3). Then there exists a Hilbert space $\mathcal{K}$ such that $(S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}})$ is a dilation of $\underline{T}$ and dim $(\mathcal{K})$ = rank $(\Delta_{\underline{T}})$.*

PROOF: $K(h) = \sum_{\alpha \in \tilde{\Lambda}} e^{\alpha} \otimes \Delta_{\underline{T}}(\underline{T}^{\alpha})^* h$ for $h \in \mathcal{H}$. Let $\mathcal{B}^m$ denote the set of all $\alpha \in \Lambda^m$ such that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m$. So,

$$K(h) = \sum_{m=0}^{\infty} \sum_{\sigma, \alpha} e_{\alpha_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \Delta_{\underline{T}}(T_{\alpha_{\sigma^{-1}(1)}} \cdots T_{\alpha_{\sigma^{-1}(m)}})^* h$$

where the second summation is over $\sigma \in \mathcal{S}_m$ and $\alpha \in \mathcal{B}^m$. Further

$$\begin{aligned} K(h) &= \sum_{m=0}^{\infty} \sum_{\sigma, \alpha} e_{\alpha_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \overline{(q^{\sigma}(\alpha))^{-1}} \Delta_{\underline{T}}(T_{\alpha_1} \cdots T_{\alpha_m})^* h \\ &= \sum_{m=0}^{\infty} \sum_{\sigma, \alpha} q^{\sigma}(\alpha) e_{\alpha_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{\alpha_{\sigma^{-1}(m)}} \otimes \Delta_{\underline{T}}(T_{\alpha_1} \cdots T_{\alpha_m})^* h \\ &= \sum_{m=0}^{\infty} \sum_{\alpha \in \mathcal{B}^m} (m!) P_m e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_m} \otimes \Delta_{\underline{T}}(T_{\alpha_1} \cdots T_{\alpha_m})^* h. \end{aligned}$$

So the range of K is contained in $\tilde{\mathcal{H}}_q := \Gamma_q(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$. □

In other words now $\mathcal{H}$ can be considered as a subspace of $\tilde{\mathcal{H}}_q$. Moreover, $\underline{\tilde{S}} = (S_1 \otimes I, \ldots, S_n \otimes I)$, as a tuple of operators in $\tilde{\mathcal{H}}_q$ is the standard q-commuting dilation of $(T_1, \ldots T_n)$. More abstractly we can get a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ can be isometrically embedded in $\Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$ and $(S_1 \otimes I_{\mathcal{K}}, \ldots, S_n \otimes I_{\mathcal{K}})$ is a dilation of $\underline{T}$ and $\overline{\text{span}}\{(\underline{S}^{\alpha} \otimes I_{\mathcal{K}})h : h \in \mathcal{H}, \alpha \in \tilde{\Lambda}\} = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{K}$. There is a unique such dilation up to unitary equivalence and dim $(\mathcal{K})$ = rank $(\Delta_{\underline{T}})$.

Let $C^*(\underline{V})$, and $C^*(\underline{S})$ be unital C^*-algebras generated by tuples $V_1, \cdots, V_n$ and $S_1, \cdots, S_n, I$ (defined in the Chapter 1) on Fock spaces $\Gamma(\mathbb{C}^n)$ and $\Gamma_q(\mathbb{C}^n)$ respectively. For any $\alpha, \beta \in \tilde{\Lambda}$, $\underline{V}^{\alpha}(I - \sum V_i V_i^*)(\underline{V}^{\beta})^*$ is the rank one operator $x \mapsto \langle e^{\beta}, x \rangle e^{\alpha}$, formed by basis vectors e^{α}, e^{β} and so $C^*(\underline{V})$ contains all compact operators. Similarly we see that $C^*(\underline{S})$ also contains all compact operators of $\Gamma_q(\mathbb{C}^n)$. As $V_i^* V_j = \delta_{ij} I$, it is easy to see that $C^*(\underline{V}) = \overline{\text{span}} \{\underline{V}^{\alpha}(\underline{V}^{\beta})^* : \alpha, \beta \in \tilde{\Lambda}\}$. As q_{ij}-commutators $[S_i^*, S_j]_{q_{ij}}$ are compact for all i, j, we can also get $C^*(\underline{S}) = \overline{\text{span}} \{\underline{S}^{\alpha}(\underline{S}^{\beta})^* : \alpha, \beta \in \tilde{\Lambda}\}$.

Consider a contractive tuple $\underline{T}$ on a Hilbert space $\mathcal{H}$. For $0 < r < 1$ the tuple $r\underline{T} = (rT_1, \ldots, rT_n)$ is clearly pure. If $\underline{T}$ is q-commuting, by considering $C^*(\underline{S})$ instead of $C^*(\underline{V})$, and restricting the range of K_r to $\Gamma_q(\mathbb{C}^n) \otimes \Delta_{\underline{T}}(\mathcal{H})$, and taking limits as r increases to 1 as for the linear map $X \mapsto K_r^*(X \otimes I)K_r$ (similar to that in page: 16) we would get the unique unital completely positive map $\phi : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H})$, (also see [BB]) satisfying

$$\phi(\underline{S}^{\alpha}(\underline{S}^{\beta})^*) = \underline{T}^{\alpha}(\underline{T}^{\beta})^*; \qquad \alpha, \beta \in \tilde{\Lambda}. \tag{3.6}$$

DEFINITION 3.2.2 Let $\underline{T}$ be a q-commuting tuple. Then we have a unique unital completely positive map $\phi : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H})$ satisfying equation (3.6). Consider the minimal Stinespring dilation of ϕ. Here we have a Hilbert space $\mathcal{H}_1$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\pi_1 : C^*(\underline{S}) \to \mathcal{B}(\mathcal{H}_1)$, such that

$$\phi(X) = P_{\mathcal{H}} \pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S}),$$

and $\overline{\text{span}}\ \{\pi_1(X)h : X \in C^*(\underline{S}), h \in \mathcal{H}\} = \mathcal{H}_1$. Let $\tilde{S}_i = \pi_1(S_i)$ and $\underline{\tilde{S}} = (\tilde{S}_1, \ldots, \tilde{S}_n)$. Then $\underline{\tilde{S}}$ is called the *standard q-commuting dilation* of $\underline{T}$.

Standard q-commuting dilation is also unique up to unitary equivalence as minimal Stinespring dilation is unique up to unitary equivalence.

THEOREM 3.2.3 *Let $\underline{T}$ be a pure tuple on a Hilbert space $\mathcal{H}$.*

1. *Then the maximal q-commuting piece $\underline{\tilde{V}}^q$ of the minimal isometric dilation $\underline{\tilde{V}}$ of $\underline{T}$ is a realization of the standard q-commuting dilation of $\underline{T}^q$ if and only if $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$. And if $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$ then rank $(\Delta_{\underline{T}})$ = rank $(\Delta_{\underline{T}^q})$ = rank $(\Delta_{\underline{\tilde{V}}})$ = rank $(\Delta_{\underline{\tilde{V}}^q})$.*

2. *Let the minimal isometric dilation of $\underline{T}$ be $\underline{\tilde{V}}$. If rank $\Delta_{\underline{T}}$ and rank $\Delta_{\underline{T}^q}$ are finite and equal then $\underline{\tilde{V}}^q$ is a realization of the standard q-commuting dilation of $\underline{T}^q$.*

PROOF: The proof is similar to the proofs of that of Theorem 2.1.11 and Remark 2.1.13. □

If the ranks of both $\Delta_{\underline{T}}$ and $\Delta_{\underline{T}^q}$ are infinite then we can not ensure that $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^q(\underline{T}))}$ and hence can not ensure the converse of second part of the Theorem 3.2.3-1, as seen by the following example. For any $n \geq 2$ consider the Hilbert space $\mathcal{H}_0 = \Gamma_q(\mathbb{C}^n) \otimes \mathcal{M}$ where $\mathcal{M}$ is of infinite dimension and let $\underline{R} = (S_1 \otimes I, \cdots, S_n \otimes I)$ be a q-commuting pure n-tuple. Infact one can take any $\underline{R}$ to be any q-commuting pure n-tuple on some Hilbert space $\mathcal{H}_0$ with $\overline{\Delta_{\underline{R}}(\mathcal{H}_0)}$ of infinite dimensional. Suppose $P_k = (p^k_{ij})_{n \times n}$ for $1 \leq k \leq n$ are $n \times n$ matrices with complex entries such that

$$p^k_{ij} = \begin{cases} t_k & \text{if } i = k, j = k+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } p^n_{ij} = \begin{cases} t_n & \text{if } i = n, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

where t_k's are complex numbers satisfying $0 < |t_k| < 1$. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{C}^n$. Take $\underline{T} = (T_1, \cdots, T_n)$ where T_k for $1 \leq k \leq n$ be operators on $\mathcal{H}$ defined by

$$T_k = \begin{pmatrix} R_k & \\ & P_k \end{pmatrix} \text{ for } 1 \leq k \leq n.$$

So $\underline{T}$ is a pure tuple, the maximal q-commuting piece of $\underline{T}$ is $\underline{R}$ and $\mathcal{H}^q(\underline{T}) = \mathcal{H}_0$ (by Corollary 3.0.5). Here rank $(\Delta_{\underline{T}^q})$ = rank $(\Delta_{\underline{T}}) = \infty$ but $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{R}}(\mathcal{H}_0)} \oplus \mathbb{C}^n$.

Consider the case when $\underline{T}$ is a q-commuting tuple on Hilbert space $\mathcal{H}$ satisfying $\sum T_i T_i^* = I$. As $C^*(\underline{S})$ contains the ideal of all compact operators by standard C^*-algebra theory we have a direct sum decomposition of π_1 as follows. Take $\mathcal{H}_1 = \mathcal{H}_{1C} \oplus \mathcal{H}_{1N}$ where $\mathcal{H}_{1C} = \overline{\text{span}}\{\pi_1(X)h : h \in \mathcal{H}, X \in C^*(\underline{S})$ and X is compact$\}$ and $\mathcal{H}_{1N}$ is the orthogonal complement of it. Clearly $\mathcal{H}_{1C}$ is a reducing subspace for π_1. Therefore

$\pi_1 = \pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\mathcal{H}_{1C}}\pi_1(X)P_{\mathcal{H}_{1C}}$, $\pi_{1N}(X) = P_{\mathcal{H}_{1N}}\pi_1(X)P_{\mathcal{H}_{1N}}$. Also $\pi_{1C}(X)$ is just the identity representation with some multiplicity. Infact $\mathcal{H}_{1C}$ can be written as $\mathcal{H}_{1C} = \Gamma_q(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ (see Theorem 4.5 of [BB]) and $\pi_{1N}(X) = 0$ for compact X. But $\Delta_{\underline{T}}(\mathcal{H}) = 0$ and commutators $[S_i^*, S_i]$ are compact. So if we take $W_i = \pi_{1N}(S_i)$, $\underline{W} = (W_1, \ldots, W_n)$ is a tuple of normal operators. It follows that the standard q-commuting dilation of $\underline{T}$ is a tuple of normal operators.

Definition 3.2.4 A q-commuting n-tuple $\underline{T} = (T_1, \ldots, T_n)$ of operators on a Hilbert space $\mathcal{H}$ is called a *q-spherical unitary* if each T_i is normal and $T_1T_1^* + \cdots + T_nT_n^* = I$.

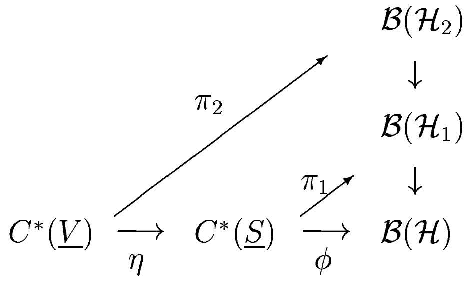
If $\mathcal{H}$ is a finite dimensional Hilbert space and $\underline{T}$ is a q-commuting tuple on $\mathcal{H}$ satisfying $\sum T_iT_i^* = I$, then $\underline{T}$ is a q-spherical unitary because each T_i would be subnormal and all finite dimensional subnormal operators are normal (see [Ha]).

Theorem 3.2.5 *Let $\underline{T}$ is a q-commuting contractive tuple on a Hilbert space $\mathcal{H}$. Then the maximal q-commuting piece of the standard noncommuting dilation of $\underline{T}$ is a realization of the standard q-commuting dilation of $\underline{T}$.*

Proof: Let $\tilde{\underline{S}}$ denote the standard q-commuting dilation of $\underline{T}$ on a Hilbert space $\mathcal{H}_1$ and we follow the notations as in beginning of this section. As $\underline{S}$ is also a contractive tuple, we have a unique unital completely positive map $\eta : C^*(\underline{V}) \to C^*(\underline{S})$, satisfying

$$\eta(\underline{V}^\alpha(\underline{V}^\beta)^*) = \underline{S}^\alpha(\underline{S}^\beta)^*; \qquad \alpha, \beta \in \tilde{\Lambda}.$$

It is easy to see that $\psi = \phi \circ \eta$. Let unital $*$-homomorphism $\pi_2 : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_2)$ for some Hilbert space $\mathcal{H}_2$ containing $\mathcal{H}_1$, be the minimal Stinespring dilation of the map $\pi_1 \circ \eta : C^*(\underline{V}) \to \mathcal{B}(\mathcal{H}_1)$ such that $\pi_1 \circ \eta(X) = P_{\mathcal{H}_1}\pi_2(X)|_{\mathcal{H}_1}, \quad \forall X \in C^*(\underline{V})$, and $\overline{\text{span}}\ \{\pi_2(X)h : X \in C^*(\underline{V}), h \in \mathcal{H}_1\} = \mathcal{H}_2$. We get the following commuting diagram.



where all the down arrows are compression maps, horizontal arrows are unital completely positive maps and diagonal arrows are unital $*$-homomor- phisms. Let $\underline{\hat{V}} = (\hat{V}_1, \ldots, \hat{V}_n)$ where $\hat{V}_i = \pi_2(V_i)$. We would show that $\underline{\hat{V}}$ is the minimal isometric dilation of $\underline{T}$. We have this result if we can show that π_2 is a minimal dilation of $\psi = \phi \circ \eta$ as minimal Stinespring dilation is unique up to unitary equivalence. For this first we show that $\tilde{\underline{S}} = (\pi_1(S_1), \ldots, \pi_1(S_n))$ is the maximal q-commuting piece of $\underline{\hat{V}}$.

First we consider a particular case when $\underline{T}$ is a q-spherical unitary on a Hilbert space $\mathcal{H}$. In this case we would show that standard q-commuting dilation and the maximal q-commuting piece of the minimal isometric dilation of $\underline{T}$ is itself. We have $\phi(\underline{S}^\alpha(I - \sum S_iS_i^*)(\underline{S}^\beta)^*) = \underline{T}^\alpha(I - \sum T_iT_i^*)(\underline{T}^\beta)^* = 0$ for any $\alpha, \beta \in \tilde{\Lambda}$. This forces that $\phi(X) = 0$ for any compact operator X in $C^*(\underline{S})$. Now as the q_{ij}-commutators $[S_i^*, S_j]_{q_{ij}}$ are all compact we see that ϕ is a unital $*$-homomorphism. So the minimal Stinespring dilation of ϕ is itself and standard q-commuting dilation of $\underline{T}$ is itself. Next we would show that the maximal q-commuting piece of the minimal isometric dilation of $\underline{T}$ is $\underline{T}$. The presentation of the minimal isometric dilation which we would use is taken from [Po1]. The dilation space $\tilde{\mathcal{H}}$ can be decomposed as $\tilde{\mathcal{H}} = \mathcal{H} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathcal{D})$ where $\mathcal{D}$ is the closure of the range of operator D defined in page 14. At some places we would identify $\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ copies}}$ with $\mathbb{C}^n \otimes \mathcal{H}$ so that $(h_1, \ldots, h_n) = \sum_{i=1}^n e_i \otimes h_i$. Let the minimal isometric dilation $\tilde{V}_i$ be as given in equation (2.2). We have

$$T_i T_i^* = T_i^* T_i \text{ and } T_j T_i = q_{ij} T_i T_j \forall 1 \leq i, j \leq n.$$

Also by Fuglede-Putnam Theorem ([Ha] [Pu])

$$T_j^* T_i = \overline{q}_{ij} T_i T_j^* = q_{ji} T_i T_j^* \text{ and } T_j^* T_i^* = q_{ij} T_i^* T_j^* \quad \forall 1 \leq i, j \leq n.$$

As $\sum T_i T_i^* = I$, by direct computation D^2 is seen to be a projection. So, $D = D^2$. Note that $q_{ij}\overline{q}_{ij} = 1$, i.e., $\overline{q}_{ij} = q_{ji}$. Then we get

$$D(h_1, \ldots, h_n) \;=\; \sum_{i,j=1}^{n} e_i \otimes T_j (T_j^* h_i - \overline{q}_{ji} T_i^* h_j) = \sum_{i,j=1}^{n} e_i \otimes T_j (h_{ij})$$

where $h_{ij} = T_j^* h_i - \overline{q}_{ji} T_i^* h_j = T_j^* h_i - q_{ij} T_i^* h_j$ for $1 \leq i, j \leq n$. Note that $h_{ii} = 0$ and $h_{ji} = -\overline{q}_{ij} h_{ij}$.

As clearly $\mathcal{H} \subseteq \tilde{\mathcal{H}}^q(\underline{V})$, lets begin with $y \in \mathcal{H}^{\perp} \bigcap \tilde{\mathcal{H}}^q(\underline{\tilde{V}})$. We wish to show that $y = 0$. Decompose y as $y = 0 \oplus \sum_{\alpha \in \tilde{\Lambda}} e^{\alpha} \otimes y_{\alpha}$, with $y_{\alpha} \in \mathcal{D}$. We assume $y \neq 0$ and arrive at a contradiction. If for some α, $y_{\alpha} \neq 0$, then $\langle \omega \otimes y_{\alpha}, (\underline{\tilde{V}}^{\alpha})^* y \rangle = \langle e^{\alpha} \otimes y_{\alpha}, y \rangle = \langle y_{\alpha}, y_{\alpha} \rangle \neq 0$. Since $(\underline{\tilde{V}}^{\alpha})^* y \in \tilde{\mathcal{H}}^q(\underline{\tilde{V}})$, we can assume $\|y_0\| = 1$ without loss of generality. Taking $\tilde{y}_m = \sum_{\alpha \in \Lambda^m} e^{\alpha} \otimes y_{\alpha}$, we get $y = 0 \oplus \oplus_{m \geq 0} \tilde{y}_m$. D being a projection its range is closed and as $y_0 \in \mathcal{D}$, there exists some $(h_1, \ldots, h_n)$ such that $y_0 = D(h_1, \ldots, h_n)$. Let $\tilde{x_0} = \tilde{y_0} = y_0$ and $\tilde{x_1} = \sum_{i,j=1}^n e_i \otimes D(e_j \otimes h_{ij})$. Further denoting $\prod_{1 \leq r < s \leq m} q_{i_r i_s}$ by p_m, for $m \geq 1$ let

$$\begin{aligned} \tilde{x}_m \;=\; & \sum_{i_1, \ldots, i_{m-1}, i, j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes \\ & D(e_j \otimes p_{m-1} (\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}). \end{aligned}$$

So $\tilde{x}_m \in (\mathbb{C}^n)^{\otimes^m} \otimes \mathcal{D}$ for all $m \in \mathbb{N}$. As $\underline{T}$ is a q-commuting n-tuple and D is a projection, we have

$$\begin{aligned} & \sum_{1 \leq i < j \leq n} (q_{ij} \tilde{V}_i \tilde{V}_j - \tilde{V}_j \tilde{V}_i) q_{ji} h_{ij} = \sum_{1 \leq i < j \leq n} (q_{ij} T_i T_j - T_j T_i) q_{ji} h_{ij} \\ & + \sum_{1 \leq i < j \leq n} D(e_i \otimes T_j h_{ij} - q_{ji} e_j \otimes T_i h_{ij}) + \sum_{1 \leq i < j \leq n} (e_i \otimes D(e_j \otimes h_{ij}) \\ & - q_{ji} e_j \otimes D(e_i \otimes h_{ij})) \end{aligned}$$

$$
\begin{aligned}
&= 0 + D(\sum_{i,j=1}^{n} e_i \otimes T_j h_{ij}) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) \\
&= D^2(h_1, \ldots, h_n) + \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes h_{ij}) = \tilde{x}_0 + \tilde{x}_1.
\end{aligned}
$$

So by Proposition 3.0.4, $\langle y, \tilde{x}_0 + \tilde{x}_1 \rangle = 0$. Next let $m \geq 2$.

$$
\begin{aligned}
& \sum_{i_1,\ldots,i_{m-1}=1}^{n} \tilde{V}_{i_1} \ldots \tilde{V}_{i_{m-1}} \{ \sum_{i,j=1}^{n} (q_{ij} \tilde{V}_i \tilde{V}_j - \tilde{V}_j \tilde{V}_i) p_{m-1} (\prod_{k=1}^{m-2} q_{i_k j}) \\
& (T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j}) \} \\
= & \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \ldots \otimes e_{i_{m-1}} \otimes [\sum_{i,j=1}^{n} D(p_{m-1} (\prod_{k=1}^{m-2} q_{i_k j}) \\
& (q_{ij} e_i \otimes T_j T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j} - e_j \otimes T_i T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j})) \\
& + \sum_{i,j=1}^{n} p_{m-1} (\prod_{k=1}^{m-2} q_{i_k j}) \{ q_{ij} e_i \otimes D(e_j \otimes T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j}) \\
& - e_j \otimes D(e_i \otimes T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j}) \}] \\
= & - \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes \{ \sum_{j=1}^{n} p_{m-1} (\prod_{k=1}^{m-2} q_{i_k j}) \\
& D(e_j \otimes T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j}) \} + \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \ldots \otimes e_{i_{m-1}} \otimes \\
& \{ \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes q_{ij} p_{m-1} (\prod_{k=1}^{m-2} q_{i_k j}) (T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} j})) \\
& - \sum_{i,j=1}^{n} e_i \otimes D(e_j \otimes p_{m-1} (\prod_{k=1}^{m-2} q_{i_k i}) (T_j^* T_{i_1}^* \ldots T_{i_{m-2}}^* h_{i_{m-1} i})) \}
\end{aligned}
$$

(in the term above, i and j have been interchanged in the last summation)

$$
\begin{aligned}
= & - \sum_{i_1,\ldots,i_{m-2},i=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-2}} \otimes e_i \otimes \{ \sum_{j=1}^{n} p_{m-2} q_{i_r i_s} (\prod_{k=1}^{m-2} q_{i_k i} q_{i_k j}) \\
& D(e_j \otimes T_{i_1}^* \ldots T_{i_{m-2}}^* h_{ij}) \} + \sum_{i_1,\ldots,i_{m-1}=1}^{n} e_{i_1} \otimes \ldots \otimes e_{i_{m-1}} \otimes \sum_{i,j=1}^{n} e_i \otimes D(e_j \\
& \otimes \{ p_{m-1} q_{ij} (\prod_{k=1}^{m-2} q_{i_k j}) (T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* T_j^* h_{i_{m-1}} - q_{i_{m-1} j} T_i^* T_{i_1}^* \ldots T_{i_{m-2}}^* T_{i_{m-1}}^* h_j) \\
& - p_{m-1} (\prod_{k=1}^{m-2} q_{i_k i}) (T_j^* T_{i_1}^* \ldots T_{i_{m-2}}^* T_i^* h_{i_{m-1}} - q_{i_{m-1} i} T_j^* T_{i_1}^* \ldots T_{i_{m-2}}^* T_{i_{m-1}}^* h_i)) \}
\end{aligned}
$$

(in the term above, index i_{m-1} has been replaced by i in
the first summation)

$$
\begin{aligned}
= & - \sum_{i_1,\ldots,i_{m-2},i,j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-2}} \otimes e_i \otimes p_{m-2}(\prod_{k=1}^{m-2} q_{i_k i} q_{i_k j}) \\
& D(e_j \otimes T_{i_1}^* \ldots T_{i_{m-2}}^* h_{ij}) \\
& + \sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes \\
& p_{m-1}(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) D(e_j \otimes T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}) = -\tilde{x}_{m-1} + \tilde{x}_m.
\end{aligned}
$$

Hence by proposition 3.0.4, $\langle y, \tilde{x}_{m-1} - \tilde{x}_m \rangle = 0$. Further we compute $\|\tilde{x}_m\|$ for all $m \in \mathbb{N}$.

$$
\begin{aligned}
\|\tilde{x}_m\|^2 &= \langle \sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} e_{i_1} \otimes \cdots \otimes e_{i_{m-1}} \otimes e_i \otimes D(e_j \otimes p_{m-1}(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) \\
& \quad T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}), \sum_{i'_1,\ldots,i'_{m-1},i',j'=1}^{n} e_{i'_1} \otimes \cdots \otimes e_{i'_{m-1}} \otimes e_{i'} \otimes \\
& \quad D(e_{j'} \otimes p_{m-1}(\prod_{k'=1}^{m-1} q_{i'_{k'} i'} q_{i'_{k'} j'}) T_{i'_1}^* \ldots T_{i'_{m-1}}^* h_{i'j'})\rangle \\
&= \sum_{i_1,\ldots,i_{m-1},i=1}^{n} \langle \sum_{j=1}^{n} D(e_j \otimes p_{m-1}(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}), \\
& \quad \sum_{j'=1}^{n} D(e_{j'} \otimes p_{m-1}(\prod_{k'=1}^{m-1} q_{i_{k'} i} q_{i_{k'} j'}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij'})\rangle \\
&= \sum_{i_1,\ldots,i_{m-1},i=1}^{n} \langle D(\sum_{j=1}^{n} e_j \otimes p_{m-1}(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}), \\
& \quad \sum_{j'=1}^{n} e_{j'} \otimes p_{m-1}(\prod_{k'=1}^{m-1} q_{i_{k'} i} q_{i_{k'} j'}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij'}\rangle \\
&= \sum_{i_1,..,i_{m-1},i=1}^{n} \langle p_{m-1}\{\sum_{j,l=1}^{n} (\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j})(e_j \otimes T_l(T_l^* T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij} - q_{jl} \\
& \quad T_j^* T_{i_1}^* \ldots T_{i_{m-1}}^* h_{il}))\}, \sum_{j'=1}^{n} p_{m-1}(\prod_{k'=1}^{m-1} q_{i_{k'} i} q_{i_{k'} j'}) e_{j'} \otimes T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij'}\rangle \\
&= \sum_{i_1,\ldots,i_{m-1},i,j=1}^{n} \langle p_{m-1}(\prod_{k=1}^{m-1} q_{i_k i} q_{i_k j}) \sum_{l=1}^{n} T_l(T_l^* T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij} - \\
& \quad q_{jl} T_j^* T_{i_1}^* \ldots T_{i_{m-1}}^* h_{il}), p_{m-1}(\prod_{k'=1}^{m-1} q_{i_{k'} i} q_{i_{k'} j}) T_{i_1}^* \ldots T_{i_{m-1}}^* h_{ij}\rangle \\
&= \sum_{i,j=1}^{n} \langle h_{ij}, h_{ij}\rangle - \sum_{i_1,\ldots,i_{m-1},i,j,l=1}^{n} \langle T_{i_{m-1}} \ldots T_{i_1} T_j^* T_l T_{i_1}^* \ldots T_{i_{m-1}}^* h_{il}, h_{ij}\rangle.
\end{aligned}
$$

Let $\tau : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be defined by $\tau(X) = \sum_{i=1}^n T_i X T_i^*$ for all $X \in \mathcal{B}(\mathcal{H})$, and let $\tilde{\tau}^m : M_n(\mathcal{B}(\mathcal{H})) \to M_n(\mathcal{B}(\mathcal{H}))$ be defined by $\tilde{\tau}^m(X) = (\tau^m(X_{ij}))_{n\times n}$ for all $X = (X_{ij})_{n\times n} \in M_n(\mathcal{B}(\mathcal{H}))$. As τ is a contractive completely positive map, $\tilde{\tau}^m$ is also a contractive completely positive map.

So we have $\tilde{\tau}^m(D) \leq I$ and

$$\begin{aligned}
\|\tilde{x}_m\|^2 &= \sum_{r=1}^n \langle \tilde{\tau}^{m-1}(D)(h_{r1},\dots,h_{rn}),(h_{r1},\dots,h_{rn})\rangle \\
&\leq \sum_{r=1}^n \langle (h_{r1},\dots,h_{rn}),(h_{r1},\dots,h_{rn})\rangle \\
&= \sum_{r,i=1}^n \langle h_{ri}, h_{ri}\rangle = \sum_{i,r=1}^n \langle T_i^* h_r - \overline{q}_{ir} T_r^* h_i, T_i^* h_r - \overline{q}_{ir} T_r^* h_i\rangle \\
&= \sum_{i,r=1}^n \{\langle T_i^* T_i h_r - T_r^* T_i h_i, h_r\rangle - \langle T_i^* T_r h_r - T_r^* T_r h_i, h_i\rangle\} \\
&= \sum_{r=1}^n \langle h_r - \sum_{i=1}^n T_r^* T_i h_i, h_r\rangle - \sum_{i=1}^n \langle \sum_{r=1}^n T_i^* T_r h_r - h_i, h_i\rangle \\
&= 2\sum_{r=1}^n \langle h_r - \sum_{i=1}^n T_r^* T_i h_i, h_r\rangle \\
&= 2\langle D(h_1,\dots,h_n),(h_1,\dots,h_n)\rangle = 2\|\tilde{x}_0\|^2 = 2.
\end{aligned}$$

As $\langle y, \tilde{x}_0+\tilde{x}_1\rangle = 0$ and $\langle y, \tilde{x}_{m-1}-\tilde{x}_m\rangle = 0$ for $m+1 \in \mathbb{N}$, we get $\langle y, \tilde{x}_0+\tilde{x}_m\rangle = 0$ for $m \in \mathbb{N}$. So $1 = \langle \tilde{y}_0, \tilde{y}_0\rangle = \langle \tilde{y}_0, \tilde{x}_0\rangle = -\langle \tilde{y}_m, \tilde{x}_m\rangle$. By Cauchy-Schwarz inequality, $1 \leq \|\tilde{y}_m\|\|\tilde{x}_m\|$, which implies $\frac{1}{\sqrt{2}} \leq \|\tilde{y}_m\|$ for $m \in \mathbb{N}$. This is a contradiction as $y = 0 \oplus \oplus_{m\geq 0}\tilde{y}_m$ is in the Hilbert space $\tilde{\mathcal{H}}$. This proves the particular case.

Using arguements similar to that of Theorem 2.2.1 of [BBD], the proof of the general case (that is when T_i is not necessarily normal) and the proof of "$\hat{\underline{V}}$ is the minimal isometric dilation of $\underline{T}$", both follows . □

3.3 Universal Properties of Standard q-commuting Dilation

Suppose $\underline{T}$ is q-commuting contractive tuple on $\mathcal{H}$ and $\tilde{\underline{S}}$ is its standard q-commuting dilation. Let $C^*(\tilde{\underline{S}})$ denote the unital C^*-algebra generated by $\tilde{\underline{S}}$. Then the linear map from $C^*(\tilde{\underline{S}})$ to $B(\mathcal{H})$ such that $\tilde{\underline{S}}^\alpha(\tilde{\underline{S}}^\beta)^* \mapsto P_{\mathcal{H}}\tilde{\underline{S}}^\alpha(\tilde{\underline{S}}^\beta)^*|_{\mathcal{H}} = \underline{T}^\alpha(\underline{T}^\beta)^*$ is a unital completely positive map. Now we check some universal properties of standard q-commuting dilations using methods employed by Popescu in Section 2 of [Po4]. Note that if $\hat{\pi}$ is the Stinespring dilation associated with the standard q-commuting dilation $\tilde{\underline{S}}$ then

$$\begin{aligned}
\tilde{S}_i^* \tilde{S}_j &= \hat{\pi}(S_i^* S_j) \\
&\in \overline{\text{span}}\{\hat{\pi}(\underline{S}^\alpha(\underline{S}^\beta)^*) : \alpha,\beta \in \tilde{\Lambda}\} \\
&= \overline{\text{span}}\{\tilde{\underline{S}}^\alpha(\tilde{\underline{S}}^\beta)^* : \alpha,\beta \in \tilde{\Lambda}\}.
\end{aligned}$$

PROPOSITION 3.3.1 *Suppose $\underline{\tilde{S}}$ on Hilbert space $\mathcal{H}_1$ is the standard q-commuting dilation of a q-commuting contractive tuple $\underline{T}$.*

1. *Consider a unital C^*-algebra generated by the entries of the tuple $\underline{d} = (d_1, \cdots, d_n)$, and let that be denoted by $C^*(\underline{d})$. Assume that $\underline{d}$ also satisfy for $1 \leq i, j \leq n$*

$$d_i^* d_j \in \overline{span}\{\underline{d}^\alpha (\underline{d}^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}.$$

Further assume that if for every i, j

$$\tilde{S}_i^* \tilde{S}_j = \sum_{\alpha,\beta} k_{\alpha,\beta,i,j} \underline{\tilde{S}}^\alpha (\underline{\tilde{S}}^\beta)^* \text{ for some } k_{\alpha,\beta,i,j} \in \mathbb{C}$$

$$\text{then } d_i^* d_j = \sum_{\alpha,\beta} k_{\alpha,\beta,i,j} \underline{d}^\alpha (\underline{d}^\beta)^*$$

Let there be a completely positive map $\varrho : C^*(\underline{d}) \to B(\mathcal{H})$ such that $\varrho(\underline{d}^\alpha(\underline{d}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^*$. Then there is a $*$-homomorphism form $C^*(\underline{d})$ to $C^*(\underline{\tilde{S}})$ such that $d_i \mapsto \tilde{S}_i$ for all $1 \leq i \leq n$.

2. *Suppose $\pi : C^*(\underline{T}) \to B(\tilde{\mathcal{K}})$ is a $*$-homomorphism and $\theta : C^*(\underline{\tilde{S}}) \to C^*(\underline{T})$ be the completely positive map obtained by restricting the compression map (to $B(\mathcal{H})$) for $B(\mathcal{H}_1)$ to $C^*(\underline{\tilde{S}})$. Assume the minimal Stinespring dilation of $\pi \circ \theta$ to be $\tilde{\pi}$ such that $\pi \circ \theta(X) = P_{\tilde{\mathcal{K}}} \tilde{\pi}(X)|_{\tilde{\mathcal{K}}}$. Then $(\tilde{\pi}(\tilde{S}_1), \cdots, \tilde{\pi}(\tilde{S}_n))$ is the standard q-commuting dilation of $(\pi(T_1), \cdots, \pi(T_n))$.*

PROOF:

1. Note that

$$C^*(\underline{d}) = \overline{\text{span}}\{\underline{d}^\alpha (\underline{d}^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}.$$

Stinespring decomposition of ϱ gives $\varrho(X) = P_{\mathcal{H}} \pi(X) P_{\mathcal{H}}$ where $\pi : C^*(\underline{d}) \to B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$ such that $\pi(d_i^*)$ leaves $\mathcal{H}$ invariant and

$$\mathcal{K} = \overline{\text{span}}\{\pi(\underline{d}^\alpha)h : h \in \mathcal{H}, \alpha \in \tilde{\Lambda}\}.$$

Define linear map $U : \mathcal{K} \to \mathcal{H}_1$ satisfing

$$U(\pi(\underline{d}^\alpha)h_\alpha) = \underline{\tilde{S}}^\alpha h_\alpha.$$

U is a unitary operator as for $h_\alpha \in \mathcal{H}$

$$\begin{aligned} \|\pi(\sum_{|\alpha| \leq m} \underline{d}^\alpha) h_\alpha\|^2 &= \sum_{|\alpha|,|\beta| \leq m} \langle \varrho((\underline{d}^\beta)^* \underline{d}^\alpha) h_\alpha, h_\beta \rangle \\ &= \sum_{|\alpha|,|\beta| \leq m} \langle (\underline{\tilde{S}}^\beta)^* \underline{\tilde{S}}^\alpha h_\alpha, h_\beta \rangle = \| \sum_{|\alpha| \leq m} \underline{\tilde{S}}^\alpha h_\alpha \|^2 \end{aligned}$$

Also one can see without much effort that $U\pi(d_i) = \tilde{S}_i U$ and hence the map form $C^*(\underline{d})$ to $C^*(\underline{\tilde{S}})$ given by $X \mapsto U\pi(X)U^*$ is a $*$-homomorphism satisfying $d_i \mapsto \tilde{S}_i$ for all $1 \leq i \leq n$.

2. By applying (1) of this Proposition to $\tilde{\pi} \circ \theta$ we can prove the final statement of the Proposition.

□

THEOREM 3.3.2 *Let $\underline{T}$ be q-commuting contractive n-tuple on $\mathcal{H}$ and $\underline{\tilde{S}}$ be its standard q-commuting dilation on the space $\mathcal{H}_1$. Suppose π_1 and π_2 are two $*$-homomorphism of $C^*(\underline{T})$ on Hilbert space $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. Let θ be as defined in the previous Proposition. If X be an operator such that $X\pi_1(Y) = \pi_2(Y)X$ for all $Y \in C^*(\underline{T})$, and $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are the minimal Stinespring dilations of $\pi_1 \circ \theta$ and $\pi_2 \circ \theta$ respectively. Then there exists an operator $\tilde{X}$ such that $\tilde{X}\tilde{\pi}_1 = \tilde{\pi}_2\tilde{X}$ and $\tilde{X}P_{\mathcal{K}_1} = P_{\mathcal{K}_2}\tilde{X}$.*

PROOF: We get for $Y \in C^*(\underline{T})$

$$\begin{pmatrix} \pi_1(Y) & 0 \\ 0 & \pi_2(Y) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix} \begin{pmatrix} \pi_1(Y) & 0 \\ 0 & \pi_2(Y) \end{pmatrix}.$$

Now the proof follows from Arveson's commutator lifting Theorem (Theorem 1.3.1 of [Ar1]). □

COROLLARY 3.3.3 *Let $\underline{T}$ be q-commuting contractive n-tuple on $\mathcal{H}$ and $\underline{\tilde{S}}$ be its standard q-commuting dilation on space $\mathcal{H}_1$. If $X \in C^*(\underline{T})'$ then there exists a unique $\tilde{X} \in C^*(\underline{\tilde{S}})' \cap \{P_{\mathcal{H}}\}'$ such that $P_{\mathcal{H}}\tilde{X}|_{\mathcal{H}} = X$. Also the map $X \mapsto \tilde{X}$ is a $*$-isomorphism.*

PROOF: The proof follows from direct application of Arveson's commutator lifting Theorem (Theorem 1.3.1 of [Ar1]). □

3.4 Distribution of $S_i + S_i^*$ and Related Operator Spaces

Let $\mathcal{R}$ be the von Neumann algebra generated by $G_i = S_i + S_i^*$ for all $1 \leq i \leq n$. We are interested in calculating the moments of $S_i + S_i^*$ with respect to the vaccum state and inferring about the distribution. The vacuum expectation is given by $\epsilon(T) = \langle \omega, T\omega \rangle$ where $T \in \mathcal{R}$. So,

$$\epsilon((S_i + S_i^*)^n) = \langle \omega, (S_i + S_i^*)^n \omega \rangle = \begin{cases} 0 & \text{if n is odd} \\ C_{\frac{n}{2}} = \frac{1}{\frac{n}{2}+1}\binom{n}{\frac{n}{2}} & \text{otherwise} \end{cases}$$

where C_n is the catalan number (refer [Com]). This shows that $S_i + S_i^*$ has semicircular distribution similar to the ones considered by Voiculescu ([Vo]). Further this vaccum expectation is not tracial on $\mathcal{R}$ for $n \geq 2$ as

$$\begin{aligned} \epsilon(G_2G_2G_1G_1) &= \langle \omega, (S_2 + S_2^*)(S_2 + S_2^*)(S_1 + S_1^*)(S_1 + S_1^*)\omega \rangle \\ &= \langle \omega, (S_2^*S_2^*S_1S_1 + S_2^*S_2S_1^*S_1)\omega \rangle = 1 \\ \epsilon(G_2G_1G_1G_2) &= \langle \omega, (S_2 + S_2^*)(S_1 + S_1^*)(S_1 + S_1^*)(S_2 + S_2^*)\omega \rangle \\ &= \langle \omega, (S_2^*S_1^*S_1S_2 + S_2^*S_1S_1^*S_2)\omega \rangle = \frac{1}{2} \end{aligned}$$

We would now investigate the operator space generated by G_i's, using notions of the theory of operator spaces introduced by Effros and Ruan [ER]. Here we follow the ideas of [BS2] and [HP]. For some Hilbert space $\tilde{\mathcal{H}}$ and $a_i \in B(\tilde{\mathcal{H}}), 1 \le i \le n$ define

$$\|(a_1, \cdots, a_n)\|_{max} = \max(\|\sum_{i=1}^n a_i a_i^*\|^{\frac{1}{2}}, \|\sum_{i=1}^n a_i^* a_i\|^{\frac{1}{2}}).$$

Let us denote the operator space

$$\left\{ \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ r_n & 0 & \cdots & 0 \end{pmatrix} \oplus \begin{pmatrix} r_1 & \cdots & r_n \\ 0 & & 0 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \cdots & 0 \end{pmatrix} : r_1, \cdots, r_n \in \mathbb{C} \right\} \subset M_n \oplus M_n$$

by E_n. Let $\{e_{ij} : 1 \le i, j \le n\}$ denote the standard basis of M_n and $\delta_i = e_{i1} \oplus e_{1i}$. Then one has

$$\|\sum_{i=1}^n a_i \otimes \delta_i\|_{B(\tilde{\mathcal{H}}) \otimes M_n} = \|(a_1, \cdots, a_n)\|_{max}.$$

THEOREM 3.4.1 *The operator space generated by G_i, $1 \le i \le n$ is completely isomorphic to E_n.*

PROOF: Its enough to show that for $a_i \in B(\tilde{\mathcal{H}}), 1 \le i \le n$ we have

$$\|(a_1, \cdots, a_n)\|_{max} \le \|\sum_{i=1}^n a_i \otimes G_i\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} \le 2\|(a_1, \cdots, a_n)\|_{max}$$

$$\begin{aligned} \|\sum_{i=1}^n a_i \otimes S_i^*\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} &= \|\sum_{i=1}^n (a_i \otimes 1)(1 \otimes S_i^*)\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} \\ &\le \|\sum_{i=1}^n a_i a_i^*\|_{\tilde{\mathcal{H}}}^{\frac{1}{2}} \|\sum_{i=1}^n S_i S_i^*\|_{\Gamma_q(\mathbb{C}^n)}^{\frac{1}{2}} \le \|\sum_{i=1}^n a_i a_i^*\|_{\tilde{\mathcal{H}}}^{\frac{1}{2}}. \end{aligned}$$

Similarly

$$\begin{aligned} \|\sum_{i=1}^n a_i \otimes S_i\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} &= \|\sum_{i=1}^n (1 \otimes S_i)(a_i \otimes 1)\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} \\ &\le \|\sum_{i=1}^n a_i^* a_i\|_{\tilde{\mathcal{H}}}^{\frac{1}{2}}. \end{aligned}$$

So

$$\|\sum_{i=1}^n a_i \otimes G_i\|_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} \le 2\|(a_1, \cdots, a_n)\|_{max}.$$

Let $\mathcal{S}$ denote the set of all states on $B(\tilde{\mathcal{H}})$. Now using the fact that $\epsilon(G_iG_j) = \langle \omega, S_i^* S_j \omega \rangle = \delta_{ij}$ we get

$$\begin{aligned} \| \sum_{i=1}^n a_i \otimes G_i \|^2_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} &\geq \sup_{\tau \in \mathcal{S}} (\tau \otimes \epsilon)[(\sum_{i=1}^n a_i \otimes G_i)^* \sum_{j=1}^n a_j \otimes G_j] \\ &= \sup_{\tau \in \mathcal{S}} \tau(\sum_{i=1}^n a_i^* a_i) = \| \sum_{i=1}^n a_i^* a_i \|_{\tilde{\mathcal{H}}}. \end{aligned}$$

Using similar arguements

$$\| \sum_{i=1}^n a_i \otimes G_i \|^2_{\tilde{\mathcal{H}} \otimes \Gamma_q(\mathbb{C}^n)} \geq \| \sum_{i=1}^n a_i a_i^* \|_{\tilde{\mathcal{H}}}.$$

□

Chapter 4

Minimal Cuntz-Krieger Dilation

In this Chapter we realize the generators of Cuntz-Krieger algebras through dilation and this helps us in understanding these algebras and their representations.

DEFINITION 4.0.1 Let $\mathcal{K}$ and $\mathcal{L}$ be two Hilbert spaces such that $\mathcal{K}$ is a closed subspace of $\mathcal{L}$ and $\underline{T}$, $\underline{R}$ be n-tuples of operators on $\mathcal{K}$, $\mathcal{L}$ respectively. Let A be a $0-1$-matrix. When $\underline{T}$ satisfies A-relations (see Definition 1.0.4), a minimal dilation $\underline{R}$ of $\underline{T}$ is said to be *minimal Cuntz-Krieger dilation* if $\underline{R}$ consists of partial isometries with orthogonal ranges satisfying A-relations and

$$R_i^* R_i = I - \sum_{j=1}^{n} (1 - a_{ij}) R_j R_j^*. \tag{4.1}$$

Each dilation generates a Cuntz-Krieger algebra $\mathcal{O}_A$ extended by compact operators.

4.1 Maximal A-relation Piece and A-Fock Space

DEFINITION 4.1.1 Assume matrix A to be $0-1$-matrix. For a tuple $\underline{R}$ on a Hilbert space $\mathcal{L}$, when polynomials are $p_{(l,m)} = z_l z_m - a_{lm} z_l z_m, (l,m) \in \{1,\cdots,n\} \times \{1,\cdots,n\} = \mathcal{I}$ we call the maximal piece with respect to $\{p_{(l,m)}\}_{(l,m)\in\mathcal{I}}$ as the *maximal A-relation piece* and $\mathcal{L}^p(\underline{R}), \underline{R}^p$ is denoted by $\mathcal{L}_A(\underline{R}), \underline{R}^A$. *The space* $\mathcal{L}_A(\underline{R})$ *is called the* maximal A-relation subspace.

LEMMA 4.1.2 *For a given* $0-1$*-matrix* A *let* $\underline{R} = (R_1, \ldots, R_n)$ *be a n-tuple of bounded operators on a Hilbert space* $\mathcal{M}$*,* $\mathcal{K}_{ij} = \overline{span}\{\underline{R}^\alpha (a_{ij} R_i R_j - R_i R_j) h : h \in \mathcal{M}, \alpha \in \tilde{\Lambda}\}$ *for all* $1 \le i,j \le n$*, and* $\mathcal{K} = \overline{span} \cup_{i,j=1}^n \mathcal{K}_{ij}$*. Then* $\mathcal{M}_A(\underline{R}) = \mathcal{K}^\perp$ *and* $\mathcal{M}_A(\underline{R}) = \{h \in \mathcal{M} : (a_{ij} R_j^* R_i^* - R_j^* R_i^*)(\underline{R}^\alpha)^* h = 0, \forall 1 \le i,j \le n, \alpha \in \tilde{\Lambda}\}$.

COROLLARY 4.1.3 *Let* $\underline{R}$ *and* $\underline{T}$ *be two n-tuples of bounded operators on* $\mathcal{M}$ *and* $\mathcal{H}$ *respectively.*

1. *The maximal A-relation piece of* $(R_1 \oplus T_1, \cdots, R_n \oplus T_n)$ *is* $(R_1^A \oplus T_1^A, \cdots, R_n^A \oplus T_n^A)$ *acting on* $\mathcal{M}_A \oplus \mathcal{H}_A$ *and the maximal A-relation piece of* $(R_1 \otimes I, \cdots, R_n \otimes I)$ *acting on* $\mathcal{M} \otimes \mathcal{H}$ *is* $(R_1^A \otimes I, \cdots, R_n^A \otimes I)$ *on* $\mathcal{M}_A \otimes \mathcal{H}$*.*

2. *Suppose* $\mathcal{H} \subseteq \mathcal{M}$ *and* $\underline{R}$ *is the dilation of* $\underline{T}$ *then* $\underline{R}^A$ *is the dilation of* $\underline{T}^A$ *with* $\mathcal{H}_A(\underline{T}) = \mathcal{M}_A(\underline{R}) \cap \mathcal{H}$.

The above Lemma and Corollary follows from Lemma 2.1.3, Corollary 2.1.4 and Proposition 2.1.5.

We need to introduce the following new type of Fock space for our further study of representations Cuntz-Krieger algebras and dilations.

DEFINITION 4.1.4 For a given $A = (a_{ij})_{n\times n}$ as above, the *A-Fock space* is defined as the subspace of $\Gamma(\mathbb{C}^n)$ which is the maximal A-relation subspace $(\Gamma(\mathbb{C}^n))_A(\underline{V})$ with respect to the left creation operators. It is denoted by $\Gamma_A(\mathbb{C}^n)$. We also define n-tuple $\underline{S} = (S_1, \cdots, S_n)$ where S_i's are the compressions of left creation operators V_i's on to $\Gamma_A(\mathbb{C}^n)$.

Similar Fock spaces was also studied by Muhly [Mu], Solel and others.

Given A-relations we denote $\{\alpha \in \tilde{\Lambda} :$ either $|\alpha| = m > 1$ and $a_{\alpha_i\alpha_{i+1}} = 1$ for $1 \leq i \leq m-1$, or $|\alpha| \leq 1\}$ by C_A. Here we give another description of the A-Fock space.

PROPOSITION 4.1.5

$$\Gamma_A(\mathbb{C}^n) = \overline{span}\{e^\alpha : \alpha \in C_A\}.$$

PROOF: Let $\alpha \in \Lambda^m$ be such that there exists $1 \leq k \leq m-1$ for which $a_{\alpha_k\alpha_{k+1}} = 0$. Denoting α_k, α_{k+1} by s, t, it is clear that

$$e^\alpha \in \overline{\text{span}}\{\underline{V}^\gamma(V_sV_t - a_{st}V_sV_t)h : h \in \Gamma(\mathbb{C}^n), \gamma \in \tilde{\Lambda}\},$$

which implies that such e^α are orthogonal to $\Gamma_A(\mathbb{C}^n)$. Where as if for all $1 \leq k \leq m-1$, $a_{\alpha_k\alpha_{k+1}} = 1$ then for all $1 \leq i, j \leq m-1, \beta \in \tilde{\Lambda}, h \in \Gamma(\mathbb{C}^n)$

$$\langle e^\alpha, \underline{V}^\beta(V_iV_j - a_{ij}V_iV_j)h\rangle = 0,$$

and so such $e^\alpha \in \Gamma_A(\mathbb{C}^n)$. Hence by taking completions the Proposition follows. □

Suppose $e^\alpha \in \Gamma_A(\mathbb{C}^n)$, and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \cdots, \alpha_m)$. Then

$$S_ie^\alpha = P_{\Gamma_A(\mathbb{C}^n)}V_ie^\alpha = \begin{cases} e_i & \text{if } |\alpha| = 0 \\ a_{i\alpha_1}e_i \otimes e^\alpha & \text{if } |\alpha| \geq 1 \end{cases}$$

$$S_i^*e^\alpha = V_i^*e^\alpha = \begin{cases} 0 & \text{if } |\alpha| = 0 \\ \delta_{i\alpha_1}\omega & \text{if } |\alpha| = 1 \\ \delta_{i\alpha_1}e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_m} & \text{if } |\alpha| > 1, \end{cases}$$

$$S_i^*S_ie^\alpha = \begin{cases} \omega & \text{if } |\alpha| = 0 \\ a_{i\alpha_1}e^\alpha & \text{if } |\alpha| \geq 1 \end{cases} \text{ and } S_iS_i^*e^\alpha = \begin{cases} 0 & \text{if } |\alpha| = 0 \\ \delta_{i\alpha_1}e^\alpha & \text{if } |\alpha| = 1. \end{cases}$$

PROPOSITION 4.1.6 *The maximal A-relation piece of a n-tuple of isometries with orthogonal ranges is a n-tuple of partial isometries with orthogonal ranges.*

Proof: Let $\underline{\hat{V}} = (\hat{V}_1, \cdots, \hat{V}_n)$ be a n-tuple of isometries with orthogonal ranges on a Hilbert space $\mathcal{K}$. Fix matrix $A = (a_{ij})_{n\times n}$ as above and denote the projection onto $\mathcal{K}_A(\underline{\hat{V}})$ by P. Any k_A in $\mathcal{K}_A(\underline{\hat{V}})$ can be written as $k_A = \oplus_{p=1}^n \hat{V}_p k_p \oplus k_0$ for some $k_p \in \mathcal{K},\ \ 1 \le p \le n$ and some $k_0 \in (I - \sum_{p=1}^n \hat{V}_p \hat{V}_p^*)\mathcal{K}$. Clearly k_0 is in $\mathcal{K}_A(\underline{\hat{V}})$ using Lemma 4.1.2. By the same Lemma one observes that other k_p's also belong to $\mathcal{K}_A(\underline{\hat{V}})$ as for $k \in \mathcal{K}, \alpha \in \tilde{\Lambda}$

$$\begin{aligned}
\langle k_p, \underline{\hat{V}}^{\alpha}(\hat{V}_i\hat{V}_j - a_{ij}\hat{V}_i\hat{V}_j)k\rangle &= \langle \hat{V}_p k_p, \hat{V}_p\underline{\hat{V}}^{\alpha}(\hat{V}_i\hat{V}_j - a_{ij}\hat{V}_i\hat{V}_j)k\rangle \\
&= \langle \oplus_{q=1}^n \hat{V}_q k_q \oplus k_0, \hat{V}_p\underline{\hat{V}}^{\alpha}(\hat{V}_i\hat{V}_j - a_{ij}\hat{V}_i\hat{V}_j)k\rangle \\
&= \langle k_A, \hat{V}_p\underline{\hat{V}}^{\alpha}(\hat{V}_i\hat{V}_j - a_{ij}\hat{V}_i\hat{V}_j)k\rangle = 0.
\end{aligned}$$

The above calculation uses the fact that ranges of $\hat{V}_q$'s and $I - \sum_i \hat{V}_i\hat{V}_i^*$ are all mutually orthogonal. Next we show that

$$P\hat{V}_i k_0 = \hat{V}_i k_0, \tag{4.2}$$

$$P\hat{V}_i\hat{V}_p k_p = a_{ip}\hat{V}_i\hat{V}_p k_p \tag{4.3}$$

Equation (4.2) follows from $\langle \hat{V}_i k_0, \underline{\hat{V}}^{\beta}(\hat{V}_s\hat{V}_t - a_{st}\hat{V}_s\hat{V}_t)k\rangle = 0$, for all $\beta \in \tilde{\Lambda}, 1 \le s, t \le n,\ \ k \in \mathcal{K}$ (since k_0 is orthogonal to range of $\hat{V}_t, 1 \le t \le n$). When $a_{ip} = 0$, we have $P\hat{V}_i\hat{V}_p k_p = P(\hat{V}_i\hat{V}_p - a_{ip}\hat{V}_i\hat{V}_p)k_p = 0 = a_{ip}\hat{V}_i\hat{V}_p k_p$. So it is enough to show for $a_{ip} = 1$ that $\hat{V}_i\hat{V}_p k_p \in \mathcal{K}_A(\underline{\hat{V}})$. When $|\alpha| > 1$ or $|\alpha| = 0$, it easy to see that for $1 \le s, t \le n,\ \ k \in \mathcal{K}$

$$\langle \hat{V}_i\hat{V}_p k_p, \underline{\hat{V}}^{\alpha}(\hat{V}_s\hat{V}_t - a_{st}\hat{V}_s\hat{V}_t)k\rangle = 0 \tag{4.4}$$

as $\hat{V}_i$'s are isometries with orthogonal ranges and $k_p \in \mathcal{K}_A(\underline{\hat{V}})$. When $|\alpha| = 1$,

$$\begin{aligned}
\langle \hat{V}_i\hat{V}_p k_p, \hat{V}_i(\hat{V}_p\hat{V}_t - a_{pt}\hat{V}_p\hat{V}_t)k\rangle &= \langle \hat{V}_p k_p, (\hat{V}_p\hat{V}_t - a_{pt}\hat{V}_p\hat{V}_t)k\rangle \\
&= \langle \oplus_{s=1}^n \hat{V}_s k_s \oplus k_0, (\hat{V}_p\hat{V}_t - a_{pt}\hat{V}_p\hat{V}_t)k\rangle = 0.
\end{aligned}$$

And also clearly equation (4.4) holds in all other cases when $\alpha = 1$. So equation (4.3) holds and we have

$$\begin{aligned}
\hat{V}_i^A(\hat{V}^A)_i^*\hat{V}_i^A k_A &= P\hat{V}_i\hat{V}_i^* P\hat{V}_i k_A \\
&= P\hat{V}_i\hat{V}_i^* P(\oplus_p \hat{V}_i\hat{V}_p k_p \oplus \hat{V}_i k_0) \\
&= P\hat{V}_i\hat{V}_i^*(\oplus_{p=1}^n a_{ip}\hat{V}_i\hat{V}_p k_p \oplus \hat{V}_i k_0) \\
&= \oplus_{p=1}^n a_{ip} P\hat{V}_i\hat{V}_p k_p \oplus P\hat{V}_i k_0 \\
&= P\hat{V}_i k_A = \hat{V}_i^A k_A.
\end{aligned}$$

So $\hat{V}_i^A$'s are partial isometries. The next assertion of the Proposition that for $1 \le i \ne j \le n$, range of $\hat{V}_i^A$ is orthogonal to range of $\hat{V}_j^A$ can be proved in the following way:

$$\begin{aligned}
(\hat{V}_j^A)^*\hat{V}_i^A k_A &= \hat{V}_j^* P\hat{V}_i k_A \\
&= \hat{V}_j^* P\hat{V}_i(\oplus_{p=1}^n \hat{V}_p k_p \oplus k_0) \\
&= \hat{V}_j^*(\oplus_p a_{ip}\hat{V}_i\hat{V}_p k_p \oplus \hat{V}_i k_0) = 0
\end{aligned}$$

□

Corollary 4.1.7 *The following holds for* $\underline{S}$:

1. $I - \sum_{i=1}^{n} S_i S_i^* = P_0'$ *where* P_0' *is the projection on to the vacuum space.*
2. S_i*'s are partial isometries with orthogonal ranges.*
3. $S_i^* S_i = I - \sum_{j=1}^{n} (1 - a_{ij}) S_j S_j^*$.

Proof:

1. $I - \sum S_i S_i^* = P_{\Gamma_A(\mathbb{C}^n)}(I - \sum V_i V_i^*) P_{\Gamma_A(\mathbb{C}^n)} = P_0'$.
2. Follows from Proposition 4.1.6.
3. Suppose $e^\alpha \in \Gamma_A(\mathbb{C}^n)$, and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \cdots, \alpha_m)$. Then

$$[I - \sum_j (1 - a_{ij}) S_j S_j^*] e^\alpha = \begin{cases} \omega & \text{if } |\alpha| = 0 \\ a_{i\alpha_1} e^\alpha & \text{if } |\alpha| \geq 1. \end{cases}$$

□

4.2 Minimal Cuntz-Krieger Dilation and Standard Non-commuting Dilation

Here we will consider the dilations of contractive n-tuple satisfying A-relations. One of the dilation is the standard noncommuting dilation and the other one is motivated from the relations satisfied by the generators of Cuntz-Krieger algebras.

Let $C^*(\underline{V})$ and $C^*(\underline{S})$ be the unital C^*-algebras generated by $V_1, \cdots, V_n$ and $S_1, \cdots, S_n$, respectively of Fock spaces $\Gamma(\mathbb{C}^n)$ and $\Gamma_A(\mathbb{C}^n)$. As $\underline{V}$ consists of isometries with orthogonal ranges, it easily follows that $C^*(\underline{V}) = \overline{\text{span}}\{\underline{V}^\alpha (\underline{V}^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}$. Even for $\underline{S}$, we have $C^*(\underline{S}) = \overline{\text{span}}\{\underline{S}^\alpha (\underline{S}^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}$ because of orthogonal ranges of S_i's and Corollary 4.1.7-3. Hence if $\underline{R}$ is a minimal Cuntz-Krieger dilation for some contractive tuple satisfying A-relations then there is a unique $*$-homomorphisms τ satisfying $\underline{S}^\alpha (\underline{S}^\beta)^* \mapsto \underline{R}^\alpha (\underline{R}^\beta)^*$ for $\alpha, \beta \in \tilde{\Lambda}$ (using Corollary 4.1.7-3). This is so because S_i's and R_i's have orthogonal ranges and for $1 \leq i \leq n$

$$\begin{aligned} \tau(S_i^* S_i) &= \tau(I - \sum_j (1 - a_{ij}) S_j S_j^*) = I - \sum_j (1 - a_{ij}) R_j R_j^* \\ &= R_i^* R_i = \tau(S_i^*) \tau(S_i). \end{aligned}$$

For a pure contractive tuple $\underline{T}$ on $\mathcal{H}$, there exists an isometry $K : \mathcal{H} \to \tilde{\mathcal{H}}$ where $\tilde{\mathcal{H}} = \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$, as given in equation (2.3). If $\underline{T}$ satisfies A-relations, clearly the range of K is contained in $\Gamma_A(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$. In this case $(S_1 \otimes I, \cdots, S_n \otimes I)$ is the dilation of $\underline{T}$ which implies $S_i^* \otimes I$ leaves $K(\mathcal{H})$ invariant. Now as $((I - \sum S_i S_i^*) \otimes I) K h = \omega \otimes \Delta_{\underline{T}} h$ and $\overline{\text{span}}\{\underline{S}^\alpha \omega : \alpha \in \tilde{\Lambda}\} = \Gamma_A(\mathbb{C}^n)$, we get $(S_1 \otimes I, \cdots, S_n \otimes I)$ to be the minimal dilation of $\underline{T}$.

Starting with any contractive tuple $\underline{T}$ satisfying A-relations on a Hilbert space $\mathcal{H}$, we obtain for $0 < r < 1$ the tuple $r\underline{T} = (rT_1, \cdots, rT_n)$ to be pure. Here range of K_r (as defined in chapter 2) is contained in $\Gamma_A(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$. So using arguement of chapter 2 (for ψ) and taking limit as r increases to 1 for the linear map $X \mapsto K_r^*(X \otimes I)K_r$ (refer page: 16) we get a unique unital completely positive map $\theta : C^*(\underline{S}) \to B(\mathcal{H})$ such that

$$\theta(\underline{S}^\alpha(\underline{S}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^* \quad \text{for } \alpha, \beta \in \tilde{\Lambda}. \tag{4.5}$$

For some Hilbert space $\hat{\mathcal{H}}$ containing $\mathcal{H}$, let a $*$-homomorphism $\pi_1 : C^*(\underline{S}) \to B(\hat{\mathcal{H}})$ be the minimal Stinespring dilation of θ such that

$$\theta(X) = P_{\mathcal{H}}\pi_1(X)|_{\mathcal{H}} \quad \forall X \in C^*(\underline{S})$$

and $\overline{\text{span}}\{\pi_1(X)h : X \in C^*(\underline{S}), h \in \mathcal{H}\} = \hat{\mathcal{H}}$. Then the tuple $\underline{\tilde{S}} = (\tilde{S}_1, \cdots, \tilde{S}_n)$ where $\tilde{S}_i = \pi_1(S_i)$, is the minimal Cuntz-Krieger dilation of $\underline{T}$ and this is unique upto unitary equivalence.

One sees that $\underline{\tilde{S}}$ consists of partial isometries with orthogonal ranges and satisfy A-relation. Also by applying π_1 to both sides of equation in Corollary 4.1.7-3, we see that

$$\tilde{S}_i^*\tilde{S}_i = I - \sum_j (1 - a_{ij})\tilde{S}_j\tilde{S}_j^*. \tag{4.6}$$

Now we will give another method of constructing the minimal Cuntz-Krieger dilation of a contractive n-tuple $\underline{T} = (T_1, \cdots, T_n)$ on Hilbert space $\mathcal{H}$ satisfying A-relations. Importance of this method is that it helps in getting a better understanding of the structure of this dilation. Here we would be using positive definite kernels. Define a set

$$\mathcal{M}_0 = \{(\alpha, u) : \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\}.$$

For some $\alpha, \beta \in \tilde{\Lambda}$ if $\alpha = 0$, or $|\alpha| < |\beta|$ and $\alpha_i = \beta_i$ for $1 \leq i \leq |\alpha|$ then we write $\alpha \subsetneq \beta$. Define

$$\tilde{a}_{\alpha,\beta} = \begin{cases} 1 & \text{if } |\alpha| \text{ or } |\beta| \leq 1 \\ a_{\beta_1\beta_2} \cdots a_{\beta_{|\beta|-1}\beta_{|\beta|}} & \text{if } 1 < |\alpha| \leq |\beta| \\ a_{\alpha_1\alpha_2} \cdots a_{\alpha_{|\alpha|-1}\alpha_{|\alpha|}} & \text{if } 1 < |\beta| < |\alpha|. \end{cases}$$

If $\alpha \subsetneq \beta$ or $\beta \subsetneq \alpha$ define

$$\gamma = \begin{cases} (\beta_{|\alpha|+1}, \cdots, \beta_{|\beta|}) & \text{if } |\alpha| < |\beta| \\ (\alpha_{|\beta|+1}, \cdots, \alpha_{|\alpha|}) & \text{if } |\beta| < |\alpha|. \end{cases}$$

Let $u, v \in \mathcal{H}$ be arbitrary. Consider a map $\tilde{K} : \mathcal{M}_0 \times \mathcal{M}_0 \to \mathbb{C}$ defined as follows:

$$\tilde{K}((\alpha, u), (\beta, v)) = \begin{cases} \langle u, v\rangle & \text{if } \alpha = \beta = 0 \\ \langle u, \tilde{a}_{\alpha,\beta}[I - \sum_k (1 - a_{\alpha_{|\alpha|}k})T_kT_k^*]v\rangle & \alpha = \beta \neq 0 \\ \langle u, \tilde{a}_{\alpha,\beta}\underline{T}^\gamma v\rangle & \text{if } \alpha \subsetneq \beta \\ \langle u, \tilde{a}_{\alpha,\beta}(\underline{T}^\gamma)^* v\rangle & \text{if } \alpha \supsetneq \beta \\ 0 & \text{otherwise.} \end{cases}$$

We would show that $\tilde{K}$ is a positive definite kernel. For this we consider the operator $N^{(m)} = (N^{(m)}_{\alpha,\beta})$ where $N^{(m)}$ is written as block matrix in terms of $N^{(m)}_{\alpha,\beta}$, and the row and column for the block matrix are indexed by $\alpha, \beta \in \tilde{\Lambda}$ and $|\alpha|, |\beta| \leq m$. (For all the matrices denoted by notations of the type $A^{(m)}$ below are in the form of block matrics indexed by $\alpha, \beta \in \tilde{\Lambda}$ and $|\alpha|, |\beta| \leq m$). Here

$$N^{(m)}_{\alpha,\beta} := \begin{cases} I & \text{if } \alpha = \beta = 0 \\ \tilde{a}_{\alpha,\beta}[I - \sum_k (1 - a_{\alpha_{|\alpha|}k}) T_k T_k^*] & \alpha = \beta \neq 0 \\ \tilde{a}_{\alpha,\beta} \underline{T}^{\gamma} & \text{if } \alpha \subsetneq \beta \\ \tilde{a}_{\alpha,\beta} (\underline{T}^{\gamma})^* & \text{if } \alpha \supsetneq \beta \\ 0 & \text{otherwise.} \end{cases}$$

We would show that $N^{(m)}$ is positive which would clearly imply that $\tilde{K}$ is positive definite kernel. Here we use induction to show this. First we define the matrices $L^{(m)}, F^{(m)}$ and $M^{(m)}$ as

$$L^{(m)}_{\alpha,\beta} := \begin{cases} T_{\beta_1} & \text{if } \alpha = 0, |\beta| = 1 \\ I & \text{if } \alpha = \beta \neq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad F^{(m)}_{\alpha,\beta} := \begin{cases} I - \sum_i T_i T_i^* & \text{if } \alpha = \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and } M^{(m)}_{\alpha,\beta} := \begin{cases} 0 & \text{if } \alpha = 0 \text{ or } \beta = 0 \\ N^{(m)}_{\alpha,\beta} & \text{otherwise.} \end{cases}$$

We further denote $I - \sum_k (1 - a_{ik}) T_k T_k^*$ by $G_i, 1 \leq i \leq n$. Notice that $N^{(0)}, M^{(0)}$ are clearly positive. Moreover as $T_i G_i = T_i$,

$$N^{(1)} = L^{(1)} M^{(1)} (L^{(1)})^* + F^{(1)}$$

where $M^{(1)}$ consists of only diagonal block and the diagonal entries are $0, G_1, \cdots, G_n$ in order. So $N^{(1)}, M^{(1)}$ are also positive. If $\alpha = (\alpha_1, \cdots, \alpha_m)$ then let us denote $(i, \alpha_1, \cdots, \alpha_m)$ by (i, α). When $\alpha = 0$, (i, α) is taken to be same as (i). Also for $m \geq 2$

$$N^{(m)} = L^{(m)} M^{(m)} (L^{(m)})^* + F^{(m)}$$

where $M^{(m)}$ also consists of only diagonal block and the diagonal entries are $0, E_1, \cdots, E_n$. These E_i are

$$(E_i)_{\alpha,\beta} = M^{(m)}_{(i,\alpha),(i,\beta)}.$$

Also

$$E_i = D_i M^{(m-1)} (D_i)^*$$

where

$$(D_i)_{\alpha,\beta} := \begin{cases} a_{i,\beta_1} T_{\beta_1} & \text{if } \alpha = 0, |\beta| = 1 \\ a_{i,\alpha_1} I & \text{if } \alpha = \beta \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and $M^{(m-1)}$ is positive by hypothesis. So we get $M^{(m)}$ and hence $N^{(m)}$ to be positive. From this we can say that there exists a Hilbert space $\mathcal{K}$ and a injective map $\lambda : \mathcal{M}_0 \to \mathcal{K}$ such that $\overline{\text{span}}\{\lambda(\alpha, u) : 1 \leq i \leq n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \mathcal{K}$ and

$$\langle \lambda(\alpha, u), \lambda(\beta, v) \rangle = \tilde{K}((\alpha, u), (\beta, v)).$$

Now the claim is that the $\underline{\tilde{S}} = (\tilde{S}_1, \cdots, \tilde{S}_n)$ consisting of maps $\tilde{S}_i : \mathcal{K} \to \mathcal{K}$ defined as

$$\tilde{S}_i \lambda((\alpha_1, \cdots, \alpha_m), u) = \lambda((i, \alpha_1, \cdots, \alpha_m), u),$$

constitute a tuple $\underline{\tilde{S}}$ which is the minimal Cuntz-Krieger dilation of $\underline{T}$. That these have orthogonal ranges is clear from the following equations and the definition of kernel $\tilde{K}$: For $i \neq j$

$$\begin{aligned} & \langle \tilde{S}_i \lambda((\alpha_1, \cdots, \alpha_m), u), \tilde{S}_j \lambda((\beta_1, \cdots, \beta_k), v)\rangle \\ = & \langle \lambda((i, \alpha_1, \cdots, \alpha_m), u), \lambda((j, \beta_1, \cdots, \beta_k), v)\rangle \\ = & \tilde{K}((i, \alpha_1, \cdots, \alpha_m), u), (j, \beta_1, \cdots, \beta_k), v)) = 0. \end{aligned}$$

As required for dilations we have $\tilde{S}_i^* \lambda(0, u) = \lambda(0, T_i^* u)$ as seen below:

$$\begin{aligned} \langle \tilde{S}_i^* \lambda(0, u), \lambda(\beta, v)\rangle &= \langle \lambda(0, u), \tilde{S}_i \lambda(\beta, v)\rangle \\ &= \langle \lambda(0, u), \lambda((i, \beta), v)\rangle \\ &= \tilde{K}((0, u), ((i, \beta), v)) \\ &= \tilde{K}((0, T_i^* u), (\beta, v)) \\ &= \langle \lambda(0, T_i^* u), \lambda(\beta, v)\rangle \end{aligned}$$

Now we would evaluate $\tilde{S}_i^* \tilde{S}_i \lambda(\alpha, u)$.

$$\begin{aligned} & \langle \tilde{S}_i^* \tilde{S}_i \lambda(0, u), \lambda(\beta, v)\rangle = \langle \lambda((i), u), \lambda((i, \beta), v)\rangle \\ = & \tilde{K}(((i), u), ((i, \beta), v)) \\ = & \begin{cases} \langle u, [I - \sum_k (1 - a_{ik}) T_k T_k^*] v\rangle & \text{if } \beta = 0 \\ \langle u, a_{i\beta_1} \underline{T}^\beta v\rangle & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \langle \lambda(0, u) - \sum_k (1 - a_{ik}) \lambda((k), T_k^* u), \lambda(\beta, v)\rangle \\ = & \tilde{K}((0, u), (\beta, v)) - \sum_k (1 - a_{ik}) \tilde{K}(((k), T_k^* u), (\beta, v)) \\ = & \begin{cases} \langle u, [I - \sum_k (1 - a_{ik}) T_k T_k^*] v\rangle & \text{if } \beta = 0 \\ \langle u, [\underline{T}^\beta - \sum_k (1 - a_{ik}) T_k T_k^* \underline{T}^\beta] v\rangle & \text{otherwise} \end{cases} \\ = & \begin{cases} \langle u, [I - \sum_k (1 - a_{ik}) T_k T_k^*] v\rangle & \text{if } \beta = 0 \\ \langle u, [\underline{T}^\beta - (1 - a_{i\beta_1}) \underline{T}^\beta] v\rangle = \langle u, a_{i\beta_1} \underline{T}^\beta v\rangle & \text{otherwise.} \end{cases} \end{aligned}$$

Hence

$$\tilde{S}_i^* \tilde{S}_i \lambda(\alpha, u) = \begin{cases} \lambda(0, u) - \sum_k (1 - a_{ik}) \lambda((k), T_k^* u) & \text{if } \alpha = 0 \\ a_{i\alpha_1} \lambda(\alpha, u) & \text{otherwise.} \end{cases}$$

From one gets $\tilde{S}_i \tilde{S}_i^* \tilde{S}_i = \tilde{S}_i$, i.e., $\tilde{S}_i, 1 \leq i \leq n$ are partial isometries. Further

$$\begin{aligned} & [I - \sum_k (1 - a_{ik}) \tilde{S}_k \tilde{S}_k^*] \lambda(\alpha, u) \\ = & \lambda(\alpha, u) - \sum_k (1 - a_{ik}) \tilde{S}_k \tilde{S}_k^* \underline{\tilde{S}}^\alpha \lambda(0, u) \\ = & \begin{cases} \lambda(0, u) - \sum_k (1 - a_{ik}) \lambda((k), T_k^* u) & \text{if } \alpha = 0 \\ a_{i\alpha_1} \lambda(\alpha, u) & \text{otherwise.} \end{cases} \end{aligned}$$

In the previous equation we have made use of the fact that $\tilde{S}_i, 1 \leq i \leq n$ are partial isometry with orthogonal range. We conclude that $\underline{\tilde{S}}$ satisfy equation (4.1). Minimality holds as

$$\begin{aligned} & \overline{\operatorname{span}}\{\underline{\tilde{S}}^{\beta}\lambda(0,u) : \beta \in \tilde{\Lambda}, u \in \mathcal{H}\} \\ = & \overline{\operatorname{span}}\{\lambda(\alpha,u) : 1 \leq i \leq n, \alpha \in \tilde{\Lambda}, u \in \mathcal{H}\} = \mathcal{K}. \end{aligned}$$

Next we would study the decomposition of the minimal Cuntz-Krieger dilation in terms of nondegenerate part of the associated Stinespring dilation. As for any $\alpha, \beta \in \tilde{\Lambda}$ the rank one operator $\eta \to \langle \underline{S}^{\beta}\omega, \eta\rangle \underline{S}^{\alpha}\omega$ on $\Gamma_A(\mathbb{C}^n)$ can be written as $\underline{S}^{\alpha}(I - \sum S_i S_i^*)(\underline{S}^{\beta})^*$ and they span the subalgebra of compact operators in $C^*(\underline{S})$, we conclude that $C^*(\underline{S})$ also contains all compact operators. So $\hat{\mathcal{H}}$ can be decomposed as $\hat{\mathcal{H}} = \hat{\mathcal{H}}_C \oplus \hat{\mathcal{H}}_N$ where

$$\hat{\mathcal{H}}_C := \overline{\operatorname{span}}\{\pi_1(X)h : h \in \hat{\mathcal{H}}, X \in C^*(\underline{\tilde{S}}) \text{ and compact}\}$$

and $\hat{\mathcal{H}}_C$ is bi-invariant with respect to $\tilde{S}_i$'s, that is, invariant with respect to $\tilde{S}_i$'s and $\tilde{S}_i^*$'s. Also π_1 can be decomposed as $\pi_{1C} \oplus \pi_{1N}$ where $\pi_{1C}(X) = P_{\hat{H}_C}\pi_1(X)P_{\hat{H}_C}$ and $\pi_{1N}(X) = P_{\hat{H}_N}\pi_1(X)P_{\hat{H}_N}$ Here as π_{1N} kills compacts, $\pi_{1N}(I - \sum S_i S_i^*) = \pi_{1N}(P_0') = 0$. Hence $(\pi_{1N}(S_1), \cdots, \pi_{1N}(S_n))$ satisfy Cuntz-Krieger relations (as well as A-relations). So $\pi_{1N}(S_i), 1 \leq i \leq n$ generate a Cuntz-Krieger algebra. Also by standard C^*-algebra theory $\hat{\mathcal{H}}_C = \Gamma_A(\mathbb{C}^n) \otimes \mathcal{K}$ for some Hilbert space $\mathcal{K}$ such that $\pi_{1C}(S_i) = S_i \otimes I$.

From equation (2.3) we get for $\alpha, \beta \in \tilde{\Lambda}$

$$\begin{aligned} & K^*[\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^* \otimes I]Kh \\ = & K^*[\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^* \otimes I](\sum_{\gamma} \underline{S}^{\gamma}\omega \otimes \Delta_{\underline{T}}(T^{\gamma})^* h) \\ = & K^*(\sum_{\gamma} \underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^* \underline{S}^{\gamma}\omega \otimes \Delta_{\underline{T}}(T^{\gamma})^* h) \\ = & K^*(\underline{S}^{\alpha}\omega \otimes \Delta_{\underline{T}}(T^{\beta})^* h) = \underline{T}^{\alpha}\Delta_{\underline{T}}^2(\underline{T}^{\beta})^* h. \end{aligned}$$

As $\pi_1 = \pi_{1C} \oplus \pi_{1N}$ is a $*$-homomorphism, by Stinespring Theorem there exists a isometry $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$ such that L_1 maps $\mathcal{H}$ to $\Gamma_A(\mathbb{C}^n) \otimes \mathcal{K}$ and L_2 maps $\mathcal{H}$ to some Hilbert space $\mathcal{K}_1$. Now for $\alpha, \beta \in \tilde{\Lambda}$

$$\begin{aligned} \underline{T}^{\alpha}\Delta_{\underline{T}}^2(\underline{T}^{\beta})^* h &= \theta(\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^*)(h) \\ &= L_1[\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^* \otimes I]L_1^*(h) \\ &\quad + L_2[\pi_{1N}(\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^*)]L_2^*(h) \\ &= L_1[\underline{S}^{\alpha}(I - \sum_i S_i S_i^*)(\underline{S}^{\beta})^* \otimes I]L_1^*(h) \end{aligned}$$

as π_{1N} kills compacts. Hence the map L_1 can be chosen to be K and $\mathcal{K} := \overline{\Delta_{\underline{T}}(\mathcal{H})}$. So $\mathbb{C} \otimes \mathcal{K}$ is a wandering subspace which generates $\hat{\mathcal{H}}_C$. In fact given just a n-tuple of partial isometries satisfying equation 4.1 and satisfying A-relations its clear that we will get such decomposition, as minimal Cuntz-Krieger dilation of such tuple is itself and such decomposition is called *Wold decomposition.* Further using arguements similar to Theorem 1.3 in [Po1] we get that

$$\hat{\mathcal{H}}_N = \cap_{m=0}^{\infty}\overline{\text{span}}\{\underline{\tilde{S}}^{\alpha}h : h \in \hat{\mathcal{H}}, |\alpha| = m\}.$$

COROLLARY 4.2.1 *1. rank* $(I - \sum_i \tilde{V}_i\tilde{V}_i^*) = rank(I - \sum_i \tilde{S}_i\tilde{S}_i^*)$ $= rank(I - \sum_i T_iT_i^*)$.

2. $\lim_{k\to\infty} \sum_{|\alpha|=k} \underline{\tilde{S}}^{\alpha}(\underline{\tilde{S}}^{\alpha})^* = P_{\hat{\mathcal{H}}_N}$.

PROOF: Clear. □

Now we would see how the minimal Cuntz-Krieger dilation and minimal isometric dilation are related.

THEOREM 4.2.2 *Let $\underline{T}$ be a contractive n-tuple on Hilbert space $\mathcal{H}$ satisfying A-relations. Then the maximal A-relation piece of the minimal isometric dilation of $\underline{T}$ is a realization of the minimal Cuntz-Krieger dilation of $\underline{T}$.*

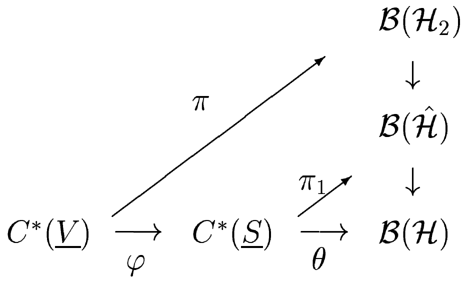
PROOF: Let $\theta : C^*(\underline{S}) \to B(\mathcal{H})$ be the unital completely positive map as in equation (4.5) and let π_1 be the corresponding minimal Stinespring dilation. Also $\tilde{S}_i = \pi_1(S_i)$ as before. As standard tuple $\underline{S}$ on $\Gamma_A(\mathbb{C}^n)$ is also a contractive tuple, there is a completely positive map φ from the C^*-algebra $C^*(\underline{V})$ generated by the left creation operators to $C^*(\underline{S})$, satisfying

$$\varphi(\underline{V}^{\alpha}(\underline{V}^{\beta})^*) = \underline{S}^{\alpha}(\underline{S}^{\beta})^* \quad \text{for} \quad \alpha, \beta \in \tilde{\Lambda}.$$

So, the completely positive map ψ from $C^*(\underline{V})$ to $B(\mathcal{H})$ defined in page number 24, satisfies $\psi = \theta \circ \varphi$. Let the minimal Stinespring dilation of $\pi_1 \circ \varphi$ be the $*$-homomorphism $\pi : C^*(\underline{V}) \to B(\mathcal{H}_2)$ for some Hilbert space $\mathcal{H}_2 = \overline{\text{span}}\{\pi(X)h : X \in C^*(\underline{V}), h \in \hat{\mathcal{H}}\}$. This satisfies

$$\pi_1 \circ \varphi(X) = P_{\hat{\mathcal{H}}}\pi(X)|_{\hat{\mathcal{H}}} \quad \forall X \in C^*(\underline{V}).$$

In the following commuting diagram



all the horizontal arrows are unital completely positive maps, down arrows are compressions and diagonal arrows are minimal Stinespring dilations. Let $\hat{V}_i = \pi(V_i)$ and $\underline{\hat{V}} = (\hat{V}_1, \cdots, \hat{V}_n)$. We would first show that $\underline{\tilde{S}}$ is the maximal A-relation piece of $\underline{\hat{V}}$ and then show that $\underline{\hat{V}}$ is the standard noncommuting dilation of $\underline{T}$.

Here we would use the presentation of minimal isometric dilation $\underline{\hat{V}}$ given by Popescu as in equation (2.2). Let for $1 \le i \le n, h \in \hat{\mathcal{H}}$ and $d_\alpha \in \mathcal{D}$

$$\hat{V}_i(h \oplus \sum_{\alpha\in\tilde{\Lambda}} e^{\alpha} \otimes d_{\alpha}) := \tilde{S}_i h \oplus D(e_i \otimes h) \oplus e_i \otimes \sum_{\alpha\in\tilde{\Lambda}} e^{\alpha} \otimes d_{\alpha} \tag{4.7}$$

on the dilation space $\mathcal{H}_2 = \hat{\mathcal{H}} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathcal{D})$ where $D : \underbrace{\hat{\mathcal{H}} \oplus \cdots \oplus \hat{\mathcal{H}}}_{n-\text{copies}} \to \underbrace{\hat{\mathcal{H}} \oplus \cdots \oplus \hat{\mathcal{H}}}_{n-\text{copies}}$ is

$$\begin{aligned} D^2 &= [\delta_{ij} I - \tilde{S}_i^* \tilde{S}_j]_{n\times n} \\ &= [\delta_{ij}(I - \tilde{S}_i^* \tilde{S}_i)]_{n\times n}. \end{aligned}$$

Note that D^2 is a projection as $\tilde{S}_i$'s are partial isometries and so $D^2 = D$. Let $\mathcal{D}$ denote the range of D. We would identify $\underbrace{\hat{\mathcal{H}} \oplus \cdots \oplus \hat{\mathcal{H}}}_{n-\text{copies}}$ by $\mathbb{C}^n \otimes \hat{\mathcal{H}}$ at some places and hence $(h_1, \cdots, h_n)$ by $\sum_{i=1}^n e_i \otimes (I - \tilde{S}_i^* \tilde{S}_i) h_i$ and $\mathbb{C}^n \omega \otimes \mathcal{D}$ by $\mathcal{D}$.

$$D(h_1, \cdots, h_n) = D(\sum_{i=1}^n e_i \otimes h_i) = \sum_{i=1}^n e_i \otimes (I - \tilde{S}_i^* \tilde{S}_i) h_i.$$

As $\underline{\tilde{S}}$ satisfying A-relation and $\hat{V}_i^*$ keeps $\hat{\mathcal{H}}$ invariant, clearly $\hat{\mathcal{H}} \subseteq (\mathcal{H}_2)_A(\hat{\underline{V}})$. Lets begin with arbitrary $z \in \hat{\mathcal{H}}^\perp \cap (\mathcal{H}_2)_A(\hat{\underline{V}})$ and then we would show $z = 0$. This z can be written as $0 \oplus \sum_{\alpha \in \tilde{\Lambda}} e^\alpha \otimes z_\alpha$ such that $z_\alpha \in \mathcal{D}$. If possible, let $z \neq 0$, then $\langle \omega \otimes z_\alpha, (\hat{\underline{V}}^\alpha)^* z \rangle = \langle e^\alpha \otimes z_\alpha, z \rangle = \langle z_\alpha, z_\alpha \rangle \neq 0$. As $(\hat{\underline{V}}^\alpha)^* z \in (\mathcal{H}_2)_A(\hat{\underline{V}})$, we can assume $\|z_0\| = 1$ without loss of generality. Also $z_0 = D(h_1, \cdots, h_n)$ for some $h_i \in \hat{\mathcal{H}}$ as projection have closed range. Let $x = \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{S}_j^* h_i)$. Here $x \in \mathbb{C}^n \otimes \mathcal{D}$.

$$\begin{aligned} & \sum_{i,j}^n (\hat{V}_i \hat{V}_j - a_{ij} \hat{V}_i \hat{V}_j) \tilde{S}_j^* h_i \\ = & \sum_{i,j=1}^n (\tilde{S}_i \tilde{S}_j - a_{ij} \tilde{S}_i \tilde{S}_j) \tilde{S}_j^* h_i + \sum_{i=1}^n D(e_i \otimes \sum_{j=1}^n (1 - a_{ij}) \tilde{S}_j \tilde{S}_j^* h_i) \\ & + \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{S}_j^* h_i) = 0 + \sum_{i=1}^n D[e_i \otimes (I - \tilde{S}_i^* \tilde{S}_i) h_i] + x \\ = & D^2(h_1, \cdots, h_n) + x = \tilde{z}_0 + x \end{aligned}$$

So, $\langle z, \tilde{z}_0 + x \rangle = 0$ by Lemma 4.1.2.

$$\begin{aligned} \|x\|^2 &= \| \sum_{i,j=1}^n (1 - a_{ij}) e_i \otimes D(e_j \otimes \tilde{S}_j^* h_i) \|^2 \\ &= \sum_{i=1}^n \langle \sum_{j=1}^n (1 - a_{ij}) D(e_j \otimes \tilde{S}_j^* h_i), \sum_{j'=1}^n (1 - a_{ij'}) e_{j'} \otimes \tilde{S}_{j'}^* h_i \rangle \\ &= \sum_{i=1}^n \langle \sum_{j=1}^n (1 - a_{ij}) (I - \tilde{S}_j^* \tilde{S}_j) \tilde{S}_j^* h_i, \sum_{j=1}^n (1 - a_{ij}) \tilde{S}_j^* h_i \rangle \\ &= \sum_{i=1}^n \langle \sum_{j=1}^n (1 - a_{ij}) (\tilde{S}_j^* - \tilde{S}_j^*) h_i, \sum_{j=1}^n (1 - a_{ij}) \tilde{S}_j^* h_i \rangle = 0 \end{aligned}$$

So, $x = 0$. Thus $\|\tilde{z}_0\|^2 = \langle z, \tilde{z}_0 \rangle = 0$ which is a contradiction. Hence $z = 0$ which implies $\mathcal{H} = (\mathcal{H}_2)_A(\hat{\underline{V}})$.

Finally it can be shown that $\underline{\tilde{V}}$ is the minimal isometric dilation of $\underline{T}$ by using arguments similar to the proof of Theorem 2.2.1 and with this the proof is complete. □

In same way one can show that similar result holds even if commuting tuples are replaced by q-commuting case. To keep the presentation simpler we have worked with the above special case.

4.3 Representations of Cuntz-Krieger Algebras

Cuntz-Krieger algebra $\mathcal{O}_A$ admits inequivalent representations. When a tuple $\underline{T} = (T_1, \cdots, T_n)$ on Hilbert space $\mathcal{H}$ satisfying A-relations and $\sum_{i=1}^n T_iT_i^* = I$, the minimal Cuntz-Krieger dilation $\underline{\tilde{S}} = (\tilde{S}_1, \cdots, \tilde{S}_n)$ is such that $C^*(\underline{\tilde{S}})$ is a Cuntz-Krieger algebra and the generators $\tilde{S}_1, \cdots, \tilde{S}_n$ satisfy Cuntz-Krieger relations. When $\underline{\tilde{S}}$ is the minimal Cuntz-Krieger dilation of tuple $\underline{T}$ satisfying A-relation and $\sum T_iT_i^* = I$, the unital completely positive map $\rho_{\underline{T}} : \mathcal{O}_A \to C^*(\underline{\tilde{S}})$ given by $\rho_{\underline{T}}(s_i) = \tilde{S}_i$ is a representation of $\mathcal{O}_A$. We would classify such representations when the tuples under consideration are commuting.

For a tuple $\underline{R} = (R_1, \cdots, R_n)$ on a Hilbert space $\mathcal{K}$, we would use the concept of *maximal commuting piece* and the space $\mathcal{K}^c(\underline{R})$ as defined in Section 2.1. We refer to $\mathcal{K}^c(\underline{R})$ as *maximal commuting subspace.*

DEFINITION 4.3.1

1. A commuting tuple $\underline{T} = (T_1, \cdots, T_n)$ is called *spherical unitary* if $\sum T_iT_i^* = I$ and T_i's are normal.
2. A representation ρ of $\mathcal{O}_A$ on $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$, is said to be *spherical* if $\mathcal{K} = \{\underline{R}^\alpha(k) : k \in \mathcal{K}^c(\underline{R})$ and $\alpha \in \tilde{\Lambda}\}$ where $R_i = \rho(s_i), 1 \leq i \leq n$.

DEFINITION 4.3.2 The *maximal commuting A-subspace* of a n-tuple of isometries $\underline{\hat{V}}$ with orthogonal ranges is defined as the intersection of its maximal commuting subspace and maximal A-relation subspace. The n-tuple obtained by compressing each $\hat{V}_i$ to the maximal commuting A-subspace is called *maximal commuting A-piece.*

REMARK 4.3.3 Making use of Lemma 4.1.2, it is clearly seen that the maximal commuting A-subspace of a n-tuple is infact the maximal commuting subspace of the the maximal A-relation piece. It is also seen that the maximal commuting A-subspace is the maximal A-relation subspace of the maximal commuting piece.

Let $P_0 = 1$ on $\mathbb{C}$ and P_m acting on $(\mathbb{C}^n)^{\otimes m}$ be the projection $\frac{1}{m!}\sum_{\sigma \in \mathcal{S}_m} U_\sigma^m$ where

$$U_\sigma^m(y_1 \otimes \cdots \otimes y_m) = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(m)}$$

where $y_i \in \mathbb{C}^n$. Also we denote $\oplus_{m=0}^\infty P_m$ by P'. Given A-relations we denote $\{\alpha \in \tilde{\Lambda}$: either $|\alpha| = m > 1$ and $a_{\alpha_i\alpha_j} = 1$ for $1 \leq i \neq j \leq m,$ or $|\alpha| \leq 1\}$ by $\tilde{C}_A$. It may be noted that this is different from C_A. It may be noted that this is different from C_A as defined just before Proposition 4.1.5.

DEFINITION 4.3.4 The subspace of $\Gamma_A(\mathbb{C}^n)$ defined by

$$\overline{\text{span}}\{P'e^\alpha : \alpha \in \tilde{C}_A\}.$$

is called *commuting A-Fock space* and denoted by $\Gamma_{sA}(\mathbb{C}^n)$.

To see that $\Gamma_{sA}(\mathbb{C}^n)$ is the maximal commuting A-subspace of $\underline{V}$ we first note that the maximal commuting A-subspace of $\underline{V}$ is the intersection of symmetric Fock space $\Gamma_s(\mathbb{C}^n)$ (refer [BBD]) and the maximal A-relation subspace of $\underline{V}$. Also

$$\Gamma_s(\mathbb{C}^n) = \overline{\text{span}}\{P'e^\alpha : \alpha \in \tilde{\Lambda}\}.$$

Suppose $\alpha \in \Lambda^m$ and for all $1 \leq k \neq l \leq m$, $a_{\alpha_k \alpha_l} = 1$ then for $h \in \Gamma(\mathbb{C}^n)$ and all i, j,

$$\langle P'e^\alpha, \underline{V}^\beta (V_i V_j - a_{ij} V_i V_j) h\rangle = 0.$$

So, from the definition it is clear that

$$\Gamma_{sA}(\mathbb{C}^n) \subseteq \Gamma_s(\mathbb{C}^n) \bigcap \Gamma_A(\mathbb{C}^n).$$

Let $\hat{P}$ denote the projection onto $\Gamma_s(\mathbb{C}^n) \bigcap \Gamma_A(\mathbb{C}^n)$ and let $z \in \Gamma_s(\mathbb{C}^n) \bigcap \Gamma_A(\mathbb{C}^n)$ be arbitrary. Suppose $\alpha \in \Lambda^m$ be such that $\langle e^\alpha, z\rangle$ is not equal to 0. As $z \in \Gamma_A(\mathbb{C}^n)$, it follows that $\alpha \in C_A$. Further for any $\sigma \in S_m$

$$\begin{aligned} \langle U^m_\sigma e^\alpha, z\rangle &= \langle U^m_\sigma e^\alpha, \hat{P} z\rangle = \langle \hat{P} U^m_\sigma e^\alpha, z\rangle \\ &= \langle \hat{P} e^\alpha, z\rangle = \langle e^\alpha, z\rangle. \end{aligned}$$

Thus $\langle U^m_\sigma e^\alpha, z\rangle$ is not equal to 0. This implies that $\alpha \in \tilde{C_A}$ and hence $z \in \Gamma_{sA}(\mathbb{C}^n)$. We conclude that $\Gamma_{sA}(\mathbb{C}^n)$ is the maximal commuting A-subspace of $\underline{V}$.

Here V_i^* leaves $\Gamma_{sA}(\mathbb{C}^n)$ invariant and S_i^* leaves $\Gamma_{sA}(\mathbb{C}^n)$ invariant. Let the compression of V_i on $\Gamma_{sA}(\mathbb{C}^n)$ be denoted by W_i. Suppose $\alpha \in \tilde{C_A}$,and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \cdots, \alpha_m)$ where $m = |\alpha|$. The operator W_i turns out to be

$$W_i P'e^\alpha = \begin{cases} e_i & \text{if } |\alpha| = 0 \\ P'e_i \otimes e^\alpha & \text{if } a_{i\alpha_j} a_{\alpha_j i} = 1, \forall 1 \leq j \leq m \\ 0 & \text{otherwise.} \end{cases}$$

Form this it is observed that $\underline{W} = (W_1, \cdots, W_n)$ is the maximal commuting A-piece of $\underline{V}$. Let us denote the maximal commuting piece of $\underline{V}$ on $\Gamma(\mathbb{C}^n)$ by $\hat{\underline{S}} = (\hat{S}_1, \cdots, \hat{S}_n)$. Then for $\alpha \in \tilde{C}_A, \alpha = (\alpha_1, \cdots, \alpha_m), m > 1$ the commutators

$$[W_i, W_i^*] P'e^\alpha = \begin{cases} [\hat{S}_i, \hat{S}_i^*] P'e^\alpha & \text{if } a_{i\alpha_j} a_{\alpha_j i} = 1, \forall 1 \leq j \leq m \text{ or if } \alpha = 0 \\ \frac{1}{m!} P'e^\alpha & \text{if } \alpha_j = i \text{ for some } 1 \leq j \leq m \text{ and } a_{ii} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is known that $[\hat{S}_i, \hat{S}_i^*]$'s are compact (refer [BBD], [Ar4] or [BB]). Hence clearly $[W_i, W_i^*]$'s are compact. From this it follows that

$$C^*(\underline{W}) = \overline{\text{span}}\{\underline{W}^\alpha (\underline{W}^\beta)^* : \alpha, \beta \in \tilde{\Lambda}\}.$$

Clearly the vacuum vector is contained in $\Gamma_{sA}(\mathbb{C}^n)$ and $I - \sum W_i W_i^*$ is the projection on to the vacuum space. $C^*(\underline{W})$ contains all the rank one operators of the type $\mu \to \langle \underline{W}^\alpha \omega, \mu\rangle \underline{W}^\beta \omega$ on $\Gamma_{sA}(\mathbb{C}^n)$ as those can be written as $\underline{W}^\alpha (\underline{W}^\beta)^*$. As these rank one operators span the subalgebra of compact operators, we conclude that $C^*(\underline{W})$ contains all compacts.

For a commuting pure tuple $\underline{T}$ satisfy A-relations, with easy computation it can be seen that the range of the isometry $K_r : \mathcal{H} \to \Gamma(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}\mathcal{H}}, 1 \leq r \leq 1$, defined in equation (2.3) is contained in $\Gamma_{sA}(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}\mathcal{H}}$ and we get a unital completely positive map $\phi : C^*(\underline{W}) \to B(\mathcal{H})$ defined as strong operator topology limit of $X \mapsto K_r^*(. \otimes I)K_r$ as r increases to 1. Let $\pi_0 : C^*(\underline{W}) \to B(\mathcal{H}_0)$ be the minimal Stinespring dilation of ϕ for some Hilbert space $\mathcal{H}_0$ and $\tilde{W}_i = \pi_0(W_i)$ where $\mathcal{H}_0 = \overline{\text{span}}\{\underline{\tilde{W}}^{\alpha} h : \alpha \in \tilde{\Lambda}, h \in \mathcal{H}\}$.

DEFINITION 4.3.5 The above defined tuple $\underline{\tilde{W}} = (\tilde{W}_1, \cdots, \tilde{W}_n)$ is said to be the *standard commuting A-dilation of $\underline{T}$.*

REMARK 4.3.6 It follows from Theorem 2.2.1 that for spherical unitary $\underline{T}$ satisfying A-relation the maximal commuting piece of the minimal isometric dilation is $\underline{T}$. As $\underline{T}$ satisfy A-relations, it clear that $\underline{T}$ is also the maximal commuting A-piece.

Next Lemma would be crucial for classifying certain types of representations of Cuntz-Krieger algebras.

LEMMA 4.3.7 *The maximal commuting piece of the minimal Cuntz-Krieger dilation of a commuting tuple $\underline{T}$ satisfying A-relations is the standard commuting A-dilation.*

PROOF: Let the unital completely positive map $\phi : C^*(\underline{W}) \to B(\mathcal{H})$, π_0 and $\mathcal{H}_0$ be as above. We denote the operator $\pi_0(W_i)$ by $\tilde{W}_i$ and denote the n-tuple $(\tilde{W}_1, \cdots, \tilde{W}_n)$ by $\underline{\tilde{W}}$. As $\underline{W}$ is a contractive tuple satisfying A-relation, there is a unital completely positive map $\eta : C^*(\underline{S}) \to C^*(\underline{W})$ such that $\eta(\underline{S}^{\alpha}(\underline{S}^{\beta})^*) = \underline{W}^{\alpha}(\underline{W}^{\beta})^*$. The completely positive map θ as in equation (4.5) is equal to $\phi \circ \eta$. Let $\tilde{\pi}_1$ be the minimal Stinespring dilation of $\pi_0 \circ \eta$ and $\tilde{S}_i = \tilde{\pi}_1(S_i)$. We have the following commuting diagram.

$$\begin{array}{ccccc} & & & & \mathcal{B}(\tilde{\mathcal{H}}_1) \\ & & \nearrow^{\tilde{\pi}_1} & & \downarrow \\ & & & & \mathcal{B}(\mathcal{H}_0) \\ & & & \nearrow^{\pi_0} & \downarrow \\ C^*(\underline{S}) & \underset{\eta}{\longrightarrow} & C^*(\underline{W}) & \underset{\phi}{\longrightarrow} & \mathcal{B}(\mathcal{H}) \end{array}$$

Here the horizontal arrows are completely positive maps, diagonal arrows are $*$-homomorphism and down arrows are compressions.

Its easy to see that $C^*(\underline{W})$ contains all compact operators and so $\mathcal{H}_0$ can be decomposed as $\mathcal{H}_{0C} \oplus \mathcal{H}_{0N}$ where $\mathcal{H}_{0C} = \overline{\text{span}}\{\pi_0(X)h : h \in \mathcal{H}, X \in C^*(\underline{W}), X \text{ compact }\}$ and $\mathcal{H}_{0N} = \mathcal{H}_0 \ominus \mathcal{H}_{0C}$.

$$\pi_0(X) = \begin{pmatrix} \pi_{0C}(X) & \\ & \pi_{0N}(X) \end{pmatrix}$$

where $\pi_{0C}(X)$ and $\pi_{0N}(X)$ are compressions of $\pi_0(X)$ to $\mathcal{H}_{0C}$ and $\mathcal{H}_{0N}$ respectively. Further $\mathcal{H}_{0C} = \Gamma_{sA}(\mathbb{C}^n) \otimes \overline{\Delta_{\underline{T}}(\mathcal{H})}$ and $\pi_{0C}(X) = X \otimes I$. Let $E_i = \pi_{0N}(W_i)$ and $\underline{E} = (E_1, \cdots, E_n)$. As $[W_i, W_i^*]$ and $I - \sum W_i W_i^*$ are compacts, clearly $\underline{E}$ is a spherical unitary satisfying A-relations.

From the properties of Popescu's Poisson transform and $\Gamma_{sA}(\mathbb{C}^n)$, it follows that $(W_1 \otimes I, \cdots, W_n \otimes I)$ is the maximal commuting A-piece of its minimal isometric dilations $(V_1 \otimes$

$I, \cdots, V_n \otimes I)$. Also from Remark 4.3.6 we get $\underline{E}$ to be the maximal commuting A-piece of its minimal isometric dilations. So from Remark 4.3.3 and Theorem 4.2.2 we observe that each of them is the maximal commuting piece of their minimal Cuntz-Krieger dilation. Hence by Corollary 4.1.3, $\tilde{\underline{W}}$ is the maximal commuting piece of $\tilde{\underline{S}}$. From this using arguments similar to Theorem 2.2.1 it can be shown that $\tilde{\underline{S}}$ is the minimal Cuntz-Krieger dilation of $\underline{W}$. Hence the Lemma follows. □

If a commuting contractive tuple $\underline{T}$ also satisfy A-relations for $A = (a_{ij})_{n \times n}$, then without loss of generality we can take A to be symmetric, i.e., $A = A^*$. Then A is the adjacency matrix of the graph G with set of vertices $\{1, 2, \cdots, n\}$ and set of edges $E = \{(i,j) : a_{ij} = 1, 1 \leq i < j \leq n\}$. We call all the vertices i to be *zero vertices* if $a_{ii} = 0$. Let us associate for this graph a subset M of $\{(z_1, \cdots, z_n) : \sum_{i=1}^n |z_i|^2 = 1\}$ defined as the set of elements satisfying A-relations, that is

$$M = \{(z_1, \cdots, z_n) : \sum_{i=1}^n |z_i|^2 = 1, z_i z_j = a_{ij} z_i z_j, 1 \leq i, j \leq n\}.$$

The set M can be described in the following way: For a zero vertex i, the corresponding z_i of any element of M will always taken to be zero. For any element $(z_1, \cdots, z_n)$ of M, some elements $z_{i_1}, \cdots, z_{i_k}$ for different $1 \leq i_k \leq n$ can be simultaneously choosen to be nonzero if and only if $i_1, \cdots, i_k$ are nonzero vertices and form vertices of an induced subgraph of G which is also complete.

Let C_n^M be the C^*-algebra of continuous complex valued functions on M. Consider the tuple $\underline{z} = (z_1, \cdots, z_n)$ of co-ordinate functions z_i in C_n^M. To any spherical unitary $\underline{R} = (R_1, \cdots, R_n)$ satisfying A-relations, there is a unique representation of C_n^M mapping z_i to R_i. As for any commuting tuple $\underline{T}$ satisfying A-relations with $\sum T_i T_i^* = I$, then standard commuting dilation $\tilde{\underline{S}} = (\tilde{S}_1, \cdots, \tilde{S}_n)$ is a spherical unitary (refer Section 2.2) and we have a representation $\eta_{\underline{T}}$ of C_n^M such that $\eta_{\underline{T}}(z_i) = \tilde{S}_i$. From Lemma 4.3.7, it is easy to see that if $\underline{D}$ and $\underline{E}$ are two commuting n-tuple of operators satisfying same A-relations (on not necessarily same Hilbert space), the corresponding representations $\rho_{\underline{D}}$ and $\rho_{\underline{E}}$ of $\mathcal{O}_A$ are unitarily equivalent if and only if the representations $\eta_{\underline{D}}$ and $\eta_{\underline{E}}$ of C_n^M are unitarily equivalent.

Any $z = (z_1, \cdots, z_n) \in M$ satisfy A-relations as operator tuple on $\mathbb{C}$ and is a spherical unitary. We can get a one dimensional representation η_z of C_n^M which maps f to $f(z)$. Let $(V_1^z, \cdots, V_n^z)$ and $(S_1^z, \cdots, S_n^z)$ be the minimal isometric dilation and the minimal Cuntz-Krieger dilation respectively of this operator tuple $z = (z_1, \cdots, z_n)$. The dilation space of minimal isometric dilation is

$$\mathcal{H}^z = \mathbb{C} \oplus (\Gamma(\mathbb{C}^n) \otimes \mathbb{C}_z^n)$$

where $\mathbb{C}_z^n$ is the $(n-1)$-dimensional subspace of $\mathbb{C}^n$ orthogonal to $(\overline{z}_1, \cdots, \overline{z}_n)$ and

$$V_i^z(h \oplus \sum_\alpha e^\alpha \otimes d_\alpha) = a_i \oplus D(e_i \otimes h) \oplus e_i \otimes (\sum_\alpha e^\alpha \otimes d_\alpha).$$

Using the minimal Cuntz-Krieger dilation $\underline{S}^z$ we get a representation $\vartheta : \mathcal{O}_A \to C^*(\underline{S}^z)$ mapping s_i to S_i^z. This is the GNS representation of Cuntz state

$$\underline{s}^\alpha (\underline{s}^\beta)^* \to \underline{z}^\alpha \overline{(\underline{z}^\beta)}.$$

We call such states coming out of $(z_1, \cdots, z_n)$ as *Cuntz-Krieger states.*

THEOREM 4.3.8 *Any spherical representation of $\mathcal{O}_A$ can be written as direct integral of GNS representations of Cuntz-Krieger states.*

PROOF: An arbitrary representation of C_n^M is a countable direct sum of multiplicity free representations. Also any multiplicity free representation of C_n^M can be seen as a map which send $g \in C_n^M$ to a operator which acts as multiplication by g on $L^2(M, \mu)$ for some finite Borel measure μ on M and the associated representation ϑ of $\mathcal{O}_A$ can be expressed as direct integral of representations ϑ_z with respect to measure μ acting on $\oint \mathcal{H}^z \mu(dz)$. Thus the Theorem follows. □

EXAMPLE 4.3.9 Let $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Then for any commuting contractive n-tuple satisfying A-relations also saisfy A'-relations where A' is the symmetric matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1. \end{pmatrix}$ Further the set of vertices of the graph is $\{1, 2, 3, 4\}$, the set of edges is $E = \{(1, 2), (1, 3)\}$ and zero vertex is 3. Hence $M = [(\mathbb{C}^2 \times \{0\}^2) \cup (\{0\}^3 \times \mathbb{C})] \cap \partial B_n$.

COROLLARY 4.3.10 *Any representation of $\mathcal{O}_A$ can be decomposed as $\pi_s \oplus \pi_t$ where π_s is spherical representation and $(\pi_t(s_1), \cdots, \pi_t(s_n))$ has trivial maximal commuting piece.*

PROOF: Similar to proof of Theorem 2.3.4. □

It also follows that for irreducible representation of $\mathcal{O}_A$, the maximal commuting piece of $(\pi(s_1), \cdots, \pi(s_n))$ is either one dimensional or trivial.

4.4 Universal Properties and WOT-closed Algebras Related to Minimal Cuntz-Krieger Dilation

Assume $\underline{\tilde{S}}$ be the minimal Cuntz-Krieger dilation of a contractive tuple $\underline{T}$ satisfying A-relations. Define $C^*(\underline{\tilde{S}})$ to be the unital C^*-algebra generated by $\underline{\tilde{S}}$. Clearly the linear map from $C^*(\underline{\tilde{S}})$ to $B(\mathcal{H})$ such that $\underline{\tilde{S}}^\alpha(\underline{\tilde{S}}^\beta)^* \mapsto P_{\mathcal{H}}\underline{\tilde{S}}^\alpha(\underline{\tilde{S}}^\beta)^*|_{\mathcal{H}} = \underline{T}^\alpha(\underline{T}^\beta)^*$ is a unital completely positive map. We would investigate some universal properties of minimal Cuntz-Krieger dilations using methods employed by Popescu (for minimal isometric dilations) [Po4]. We skip that proofs of Proposition 4.4.1 and Theorem 4.4.2 as they are similar to those appearing in Section 2 of [Po4] and to Proposition 3.3.1, Theorem 3.3.2 and Corollary 3.3.3 of this thesis.

PROPOSITION 4.4.1 *Suppose $\underline{\tilde{S}}$ is the minimal Cuntz-Krieger dilation of a contractive tuple $\underline{T}$ on Hilbert space $\mathcal{H}$ satisfying A-relations.*

1. *Consider a unital C^*-algebra $C^*(\underline{d})$ generated by the entries of the tuple $\underline{d} = (d_1, \cdots, d_n)$ where $\underline{d}$ satisfy equation (4.1) with respect to some matrix A. Assume that $\underline{d}$ also satisfy $d_i^* d_j = 0$ for $1 \leq i \neq j \leq n$. Let there be a completely positive map $\varrho : C^*(\underline{d}) \to B(\mathcal{H})$ such that $\varrho(\underline{d}^\alpha(\underline{d}^\beta)^*) = \underline{T}^\alpha(\underline{T}^\beta)^*$. Then there is a $*$-homomorphism form $C^*(\underline{d})$ to $C^*(\underline{\tilde{S}})$ such that $d_i \mapsto \tilde{S}_i$ for all $1 \leq i \leq n$.*

2. *Suppose* $\pi : C^*(\underline{T}) \to B(\tilde{\mathcal{K}})$ *is a* $*$*-homomorphism and* $\theta : C^*(\underline{\tilde{S}}) \to C^*(\underline{T})$ *be the completely positive map obtained by restricting compression map (to* $B(\mathcal{H})$*) for* $B(\hat{\mathcal{H}})$ *to* $C^*(\underline{\tilde{S}})$*. Assume the minimal Stinespring dilation of* $\pi \circ \theta$ *to be* $\tilde{\pi}$ *such that* $\pi \circ \theta(X) = P_{\tilde{\mathcal{K}}}\tilde{\pi}(X)|_{\tilde{\mathcal{K}}}$*. Then* $(\tilde{\pi}(\tilde{S}_1), \cdots, \tilde{\pi}(\tilde{S}_n))$ *is the minimal Cuntz-Krieger dilation of* $(\pi(T_1), \cdots, \pi(T_n))$*.*

THEOREM 4.4.2 *Let* $\underline{T}$ *be contractive n-tuple on* $\mathcal{H}$ *satisfying A-relations and* $\underline{\tilde{S}}$ *be its minimal Cuntz-Krieger dilation.*

1. *Suppose* π_1 *and* π_2 *are two* $*$*-homomorphism from* $C^*(\underline{T})$ *to* $B(\mathcal{K}_1)$ *and* $B(\mathcal{K}_2)$ *respectively, for some Hilbert space* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *respectively. Let* θ *be as defined in the previous Proposition. If* X *be an operator such that* $X\pi_1(Y) = \pi_2(Y)X$ *for all* $Y \in C^*(\underline{T})$*, and* $\tilde{\pi}_1$ *and* $\tilde{\pi}_2$ *are the minimal Stinespring dilations of* $\pi_1 \circ \theta$ *and* $\pi_2 \circ \theta$ *respectively. Then there exists an operator* $\tilde{X}$ *such that* $\tilde{X}\tilde{\pi}_1 = \tilde{\pi}_2\tilde{X}$ *and* $\tilde{X}P_{\mathcal{K}_1} = P_{\mathcal{K}_2}\tilde{X}$*.*

2. *If* $X \in C^*(\underline{T})'$ *then there exists a unique* $\tilde{X} \in C^*(\underline{\tilde{S}})' \cap \{P_{\mathcal{H}}\}'$ *such that* $P_{\mathcal{H}}\tilde{X}|_{\mathcal{H}} = X$*. Also the map* $X \mapsto \tilde{X}$ *is a* $*$*-isomorphism.*

Though many of the arguements in the following part of this Section are similar to the proofs in [DKS] and [DP2] we have given the complete arguement for the sake of making this thesis self-contained. Using equation (4.6) we observe that

$$\begin{aligned}\tilde{S}_j^*\tilde{S}_i^*\tilde{S}_i\tilde{S}_j &= \tilde{S}_j^*[I - \sum_k (1 - a_{ik})\tilde{S}_k\tilde{S}_k^*]\tilde{S}_j \\ &= a_{ij}\tilde{S}_j^*\tilde{S}_j\tilde{S}_j^*\tilde{S}_j = a_{ij}\tilde{S}_j^*\tilde{S}_j.\end{aligned}$$

From this it follows that for $\alpha = (\alpha_1, \cdots, \alpha_m)$

$$\begin{aligned}(\underline{\tilde{S}}^{\alpha})^*\underline{\tilde{S}}^{\alpha} &= a_{\alpha_1\alpha_2}\cdots a_{\alpha_{m-1}\alpha_m}\tilde{S}_{\alpha_m}^*\tilde{S}_{\alpha_m} \\ &= a_{\alpha_1\alpha_2}\cdots a_{\alpha_{m-1}\alpha_m}[I - \textstyle\sum_k (1 - a_{\alpha_m k})\tilde{S}_k\tilde{S}_k^*].\end{aligned} \tag{4.8}$$

and

$$\underline{\tilde{S}}^{\alpha}(\underline{\tilde{S}}^{\alpha})^*\underline{\tilde{S}}^{\alpha} = \underline{\tilde{S}}^{\alpha}.$$

We notice that each $\underline{\tilde{S}}^{\alpha}$ is partial isometry. Let $\mathcal{H}_2$ and $\hat{\mathcal{H}}$ be the dilation spaces associated with $\underline{\tilde{V}}$ and $\underline{\tilde{S}}$ respectively as before and let us denote $\hat{\mathcal{H}} \ominus \mathcal{H}$ by $\mathcal{E}$. We know that $\tilde{V}_i$ and $\tilde{S}_i$ leaves $\mathcal{E}$ and $\mathcal{H}^{\perp}$ respectively invariant. Let $\Phi : B(\mathcal{H}_2) \to B(\mathcal{H}_2)$ be the completely positive map defined by

$$\Phi(X) = \sum_{i=1}^{n} \tilde{V}_i P_{\mathcal{H}^{\perp}} X P_{\mathcal{H}^{\perp}} \tilde{V}_i^*.$$

So, $\Phi(P_{\mathcal{E}}) \leq \Phi(I)$. Also let $Q_i := P_{\mathcal{E}}\tilde{S}_iP_{\mathcal{E}}$. Then for $h \in \mathcal{E}$

$$\begin{aligned}\langle \sum_{|\alpha|=m} \underline{Q}^{\alpha}(\underline{Q}^{\alpha})^*h, h\rangle &= \langle \sum_{|\alpha|=m} \underline{\tilde{S}}^{\alpha}P_{\mathcal{E}}(\underline{\tilde{S}}^{\alpha})^*h, h\rangle \\ &= \langle \sum_{|\alpha|=m} \underline{\tilde{V}}^{\alpha}P_{\mathcal{H}^{\perp}}P_{\mathcal{E}}P_{\mathcal{H}^{\perp}}(\underline{\tilde{V}}^{\alpha})^*h, h\rangle \\ &= \langle \Phi^m(P_{\mathcal{E}})h, h\rangle \\ &\leq \langle \Phi^m(I)h, h\rangle.\end{aligned}$$

But as $\lim_{m\to\infty}\langle\Phi^m(I)h,h\rangle = 0$ we have $\lim_{m\to\infty}\sum_{|\alpha|=m}\langle \underline{Q}^\alpha(\underline{Q}^\alpha)^*h,h\rangle = 0$ which implies that $\underline{Q}$ is pure. In the above computation we used $\tilde{V}_i^*$ invariance of $\hat{\mathcal{H}}$ for $1 \le i \le n$. Here we are interested in understanding the structure of weak operator topology (WOT)-closed algebra generated by the minimal Cuntz-Krieger dilation $\underline{\tilde{S}}$ of some contractive tuple $\underline{T} = (T_1,\cdots,T_n)$ satisfying A-relations where $T_i \in B(\mathcal{H})$. Let $\mathcal{A}$ denote the unital WOT-closed algebra generated by all $\tilde{S}_i, 1 \le i \le n$.

LEMMA 4.4.3 *1. If $\mathcal{A}$ has no wandering vector then every non-trivial invariant subspace reduces $\mathcal{A}$.*

2. $\mathcal{K} := \mathcal{E} \ominus [\sum_i^n \tilde{S}_i\mathcal{E}]$ is a wandering subspace for $\underline{\tilde{S}}$.

PROOF: Let there be no wandering vector for $\mathcal{A}$ and let if possible $\mathcal{N}$ be a non-trivial invariant subspace for $\mathcal{A}$. If $\sum_{i=1}^n \tilde{S}_i\mathcal{N}$ is not equal to $\mathcal{N}$ then $\mathcal{N}\ominus\sum_{i=1}^n \tilde{S}_i\mathcal{N}$ would be wandering as seen using orthogonal of ranges of $\tilde{S}_i$'s, equation (4.8) and the following: For $n_1, n_2 \in \mathcal{N}\ominus\sum_{i=1}^n \tilde{S}_i\mathcal{N}$

$$\langle \tilde{S}_i^*\tilde{S}_i\tilde{S}_{\alpha_1}\cdots\tilde{S}_{\alpha_m}n_1,n_2\rangle = \langle a_{i\alpha_1}\tilde{S}_{\alpha_1}\cdots\tilde{S}_{\alpha_m}n_1,n_2\rangle = 0.$$

But this is not possible by our assumption. So

$$\mathcal{N} = \sum_{i=1}^n \tilde{S}_i\mathcal{N}. \tag{4.9}$$

Now let $h \in \mathcal{N}$ be arbitrary. From the above equation it follows that one can write h as $\sum_{i,j=1}^n \tilde{S}_i\tilde{S}_j n_{ij}$ for some $n_{ij} \in \mathcal{N}$. From this and equation (4.8) its clear that $\tilde{S}_k^*h \in \mathcal{N}$ for all $1 \le k \le n$. So $\mathcal{N}$ is reducing for $\mathcal{A}$. Hence part 1 follows.

$\mathcal{E}$ is also invariant subspace for $\mathcal{A}$. The nontrivial case is when $\mathcal{E}$ is non-zero. $\mathcal{E} \ne \sum_{i=1}^n \tilde{S}_i\mathcal{E}$ as otherwise $\mathcal{E}$ would be reducing which is not possible as $\mathcal{H}$ spans $\hat{\mathcal{H}}$. It follows from above that $\mathcal{K}$ is a wandering subspace of $\mathcal{A}$. □

So we can write $\hat{\mathcal{H}} = \mathcal{H}\oplus\mathcal{H}'\oplus(\Gamma_A(\mathbb{C}^n)\otimes\mathcal{K})$ for some Hilbert space $\mathcal{H}'$. So $\sum_{\alpha\in\tilde{\Lambda}}\underline{\tilde{S}}^\alpha\mathcal{K} = \Gamma_A(\mathbb{C}^n)\otimes\mathcal{K}$ and this would be left invariant by $\tilde{S}_i$'s. Also $\tilde{S}_iP_{\Gamma_A(\mathbb{C}^n)\otimes\mathcal{K}}$ is $S_i\otimes I$ for $1 \le i \le n$. Let us denote by $\mathcal{B}$ the WOT-closed algebra generated by $T_1,\cdots,T_n$. In order to get reducing subspaces for $\mathcal{A}$ its sufficient to demand for $\mathcal{B}^*$-invariant subspace as seen in the next Lemma.

LEMMA 4.4.4 *Let $\mathcal{L}$ be a $\mathcal{B}^*$-invariant subspace of $\mathcal{H}$. Then $\mathcal{A}[\mathcal{L}]$ reduces $\mathcal{A}$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthogonal $\mathcal{B}^*$-invariant subspace of $\mathcal{H}$ then $\mathcal{A}[\mathcal{L}_1]$ and $\mathcal{A}[\mathcal{L}_2]$ are also mutually orthogonal.*

PROOF: $\tilde{S}_i^*$ leaves $\mathcal{L}$ invariant as $\tilde{S}_i^*$ and T_i^* leaves $\mathcal{H}$ and $\mathcal{L}$ respectively invariant. Also

$$\mathcal{A}[\mathcal{L}] = \overline{\text{span}}\{\underline{\tilde{S}}^\alpha h : \alpha \in \tilde{\Lambda}, h \in \mathcal{L}\}.$$

Now for any $x \in \mathcal{L}$ and $\alpha = (\alpha_1,\cdots,\alpha_m)$, using equation (4.8)

$$\tilde{S}_i^*\underline{\tilde{S}}^\alpha x = \begin{cases} [I-\sum_k(1-a_{ik})\tilde{S}_k\tilde{S}_k^*]x & \text{if } \alpha_1 = i, |\alpha| = 1 \\ a_{\alpha_1\alpha_2}\tilde{S}_{\alpha_2}\cdots\tilde{S}_{\alpha_m}x & \text{if } \alpha_1 = i, |\alpha| > 1 \\ 0 & \text{if } \alpha_1 \ne i \\ \tilde{S}_i^*x & \text{if } |\alpha| = 0 \end{cases}$$

As $\mathcal{L}$ is invariant for $\mathcal{A}^*$, $\tilde{S}_i^* \underline{\tilde{S}}^\alpha x \in \mathcal{A}[\mathcal{L}]$ and hence reduce $\mathcal{A}$.

Further when $\mathcal{L}_1$ and $\mathcal{L}_2$ are orthogonal $\mathcal{B}^*$-invariant subspaces, to establish that $\mathcal{A}[\mathcal{L}_1]$ is $\mathcal{A}[\mathcal{L}_2]$ are orthogonal it is sufficient to check if for $|\alpha| \leq |\beta|$ and $l_1 \in \mathcal{L}_1, l_2 \in \mathcal{L}_2$, $\langle \underline{\tilde{S}}^\alpha l_1, \underline{\tilde{S}}^\beta l_2 \rangle = 0$. This holds easily for all cases because of the orthogonality of ranges of different $\tilde{S}_i$'s, $\mathcal{B}^*$-invariance of $\mathcal{L}_i$ and the equation (4.8) except for the following case we need more careful verification: For $\alpha = (\alpha_1, \cdots, \alpha_m)$

$$\begin{aligned} \langle (\underline{\tilde{S}}^\alpha)^* (\underline{\tilde{S}}^\alpha) l_1, l_2 \rangle &= \langle a_{\alpha_1\alpha_2} \cdots a_{\alpha_{m-1}\alpha_m} [I - \sum_k (1 - a_{\alpha_m k}) \tilde{S}_k \tilde{S}_k^*] l_1, l_2 \rangle \\ &= a_{\alpha_1\alpha_2} \cdots a_{\alpha_{m-1}\alpha_m} \{ \langle l_1, l_2 \rangle - \sum_k (1 - a_{\alpha_m k}) \langle \tilde{S}_k^* l_1, \tilde{S}_k^* l_2 \rangle \} \\ &= 0. \end{aligned}$$

Hence the Lemma follows. □

Let us define $\mathcal{H}_N := \hat{\mathcal{H}}_N \cap \mathcal{H}$. In the next Proposition we would take $\mathcal{H}$ to be finite dimensional. Proof is similar to the proof of Lemma 4.1, Corollary 4.2 and Corollary 4.3 in [DPS].

PROPOSITION 4.4.5 *Let $\underline{T}$ be a contractive tuple satisfying A-relations of operators on a finite dimensional Hilbert space $\mathcal{H}$.*

1. *Let $\mathcal{K}$ be a reducing subspace of $\hat{\mathcal{H}}_N$ with respect to $\mathcal{A}$ and let there exists $h \in \hat{\mathcal{H}}$ such that $P_\mathcal{K} h$ is non-zero. Then there exists $k \in \mathcal{A}^*[h] \cap \mathcal{H}_N$ such that $P_\mathcal{K} k$ is non-zero.*

2. *Any non-zero subspace of $\hat{\mathcal{H}}_N$ which is co-invariant with respect to $\tilde{S}_i, 1 \leq i \leq n$ has a non-trivial intersection with $\mathcal{H}_N$.*

3. *$\hat{\mathcal{H}}_N = \mathcal{A}[\mathcal{H}_N]$. When $\sum T_i T_i^* = I$ and $\mathcal{B} = B(\mathcal{H})$ every co-invariant subspace of $\tilde{\mathcal{H}}$ with respect to all $\tilde{S}_i$'s contains $\mathcal{H}$.*

PROOF:

1. Let $h' := P_\mathcal{K} h$ and choose $0 < \epsilon < \min\{\frac{\|h'\|}{2}, (\frac{\|h'\|}{2})^{\frac{1}{2}}\}$. As the tuples obtained by compression of $\underline{\tilde{S}}$ to $\mathcal{H}^\perp$ and $\mathcal{H}_N$ are pure and $\tilde{S}_i$'s leaves these spaces invariant, we can choose m sufficiently large that

$$\sum_{|\alpha|=m} \|P_{\mathcal{H}^\perp} (\underline{\tilde{S}}^\alpha)^* h\|^2 < \epsilon^2$$

$$\sum_{|\alpha|=m} \|P_{\mathcal{H}^\perp} (\underline{\tilde{S}}^\alpha)^* h'\|^2 < \epsilon^2$$

and

$$\sum_{|\alpha|=m} \|P_{\hat{\mathcal{H}}_N} (\underline{\tilde{S}}^\alpha)^* h'\|^2 < \epsilon^2.$$

Also as $\tilde{\mathcal{H}}_N$ is bi-invariant with respect to $\tilde{S}_i$, the properties of $\hat{\mathcal{H}}_N$ clearly yields $\sum_{|\alpha|=m} \underline{\tilde{S}}^\alpha (\underline{\tilde{S}}^\alpha)^* P_\mathcal{K} = P_\mathcal{K}$ so one gets

$$\sum_{|\alpha|=m} \|P_\mathcal{H} (\underline{\tilde{S}}^\alpha)^* h'\|^2 = \sum_{|\alpha|=m} (\|(\underline{\tilde{S}}^\alpha)^* h'\|^2 - \|P_{\mathcal{H}^\perp} (\underline{\tilde{S}}^\alpha)^* h'\|^2) > \|h'\|^2 - \epsilon^2.$$

Let A_1 be the set of $\alpha \in \Lambda^m$ such that

$$\|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\|^2 > \epsilon^{-1}\|P_{\mathcal{H}^{\perp}}(\underline{\tilde{S}}^{\alpha})^* h\|^2$$

and let A_2 be the set of $\alpha \in \Lambda^m$ such that

$$\|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\|^2 > \epsilon^{-1}\|P_{\hat{\mathcal{H}}_N}(\underline{\tilde{S}}^{\alpha})^* h\|^2$$

The set $A_1 \cap A_2$ is non-empty as

$$\begin{aligned}
&\sum_{\alpha \in A_1 \cap A_2} \|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\|^2 \\
> \ & \|h'\|^2 - \epsilon^2 - \sum_{\alpha \in A_1} \|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\|^2 - \sum_{\alpha \in A_2} \|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\|^2 \\
> \ & \|h'\|^2 - \epsilon^2 - \sum_{\alpha \in A_1} \epsilon^{-1}\|P_{\mathcal{H}^{\perp}}(\underline{\tilde{S}}^{\alpha})^* h\|^2 - \sum_{\alpha \in A_2} \epsilon^{-1}\|P_{\hat{\mathcal{H}}_N}(\underline{\tilde{S}}^{\alpha})^* h\|^2 \\
> \ & \|h'\|^2 - \epsilon^2 - \epsilon - \epsilon > \frac{\|h'\|^2}{4}.
\end{aligned}$$

Also as $\underline{T}$ is a contractive tuple, $\sum_{\alpha \in A_1 \cap A_2} \|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h\|^2 \leq \|h\|^2$. Form that it follows easily that

$$\|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h'\| > \frac{\|h'\|}{2\|h\|}\|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha})^* h\|.$$

Now with respect to $\epsilon_m = 1/m$ one can obtain sequence of $\alpha^m \in \tilde{\Lambda}$. The unit vectors $l_m = (\tilde{S}^{\alpha^m})^* h / \|(\tilde{S}^{\alpha^m})^* h\|$ satisfy

$$\begin{aligned}
\lim_{m \to \infty} \|P_{\mathcal{H}^{\perp}} l_m\| \ &\leq \ \lim_{m \to \infty} \frac{1}{\sqrt{m}} \frac{\|P_{\mathcal{H}}(\underline{\tilde{S}}^{\alpha^m})^* P_{\mathcal{K}} h\|}{\|(\underline{\tilde{S}}^{\alpha^m})^* h\|} \\
&= \ \lim_{m \to \infty} \frac{1}{\sqrt{m}} \frac{\|P_{\mathcal{H}} P_{\mathcal{K}}(\underline{\tilde{S}}^{\alpha^m})^* h\|}{\|(\underline{\tilde{S}}^{\alpha^m})^* h\|} = 0
\end{aligned}$$

and similarly $\lim_{m \to \infty} \|P_{\mathcal{H}_N} l_m\| = 0$. Further

$$\begin{aligned}
\|P_{\mathcal{K}} l_m\| \ &= \ \|(\tilde{S}^{\alpha^m})^* h'\| / \|(\tilde{S}^{\alpha^m})^* h\| \geq \|P_{\mathcal{H}}(\tilde{S}^{\alpha^m})^* h'\| / \|(\tilde{S}^{\alpha^m})^* h\| \\
&> \ (\|h'\| \|P_{\mathcal{H}} l_m\|) / (2\|h\|).
\end{aligned}$$

As the unit ball of finite dimensional $\mathcal{H}$ is compact, there is a subsequence converging to a unit vector k in $\mathcal{H}$. Its clear that

$$P_{\mathcal{H}_N} k = 0, k \in \mathcal{A}^*[h] \cap \mathcal{H}_N \text{ and } \|P_{\mathcal{K}} k\| \geq \|h'\| / 2\|h\|.$$

Hence part 1 is proved.

2. The proof follows from part 1 of this Proposition by choosing h from the non-zero subspace invariant with respect to $\tilde{S}_i^*$ and taking $\mathcal{K} = \hat{\mathcal{H}}_N$.

3. Clearly T_i^*'s leave $\mathcal{H}_N$ invariant and from Lemma (4.4.2) we get that $\mathcal{A}[\mathcal{H}_N] \subseteq \hat{\mathcal{H}}_N$ is a reducing subspace for $\mathcal{A}$. If we assume that $\hat{\mathcal{H}}_N \ominus \mathcal{A}[\mathcal{H}_N]$ is not zero, part 2 of this Proposition gives a contradiction. Hence $\mathcal{A}[\mathcal{H}_N] = \hat{\mathcal{H}}_N$. When $\sum T_i T_i^* = I$ and $\mathcal{B} = B(\mathcal{H})$ one has $\hat{\mathcal{H}} = \hat{\mathcal{H}}_N$. By above arguement any non-trivial co-invariant subspace with respect to $\tilde{S}_i$'s has a non-trivial intersection with $\mathcal{H}$. But as $\mathcal{B} = B(\mathcal{H})$ it is infact true that this invariant subspace contains $\mathcal{H}$.

□

Now we consider the tuple $\underline{Y}$ consisting of right creation operators. One can easily notice using methods similar to proof of Lemma 4.1.2 that for polynomials $p_{(l,m)} = z_l z_m - a_{ml} z_l z_m$, $(l,m) \in \{1,\cdots,n\} \times \{1,\cdots,n\} = \mathcal{I}$, we get $(\Gamma(\mathbb{C}^n))^p(\underline{Y}) = \Gamma_A(\mathbb{C}^n)$. Let X_i denote the compression of Y_i to $\Gamma_A(\mathbb{C}^n)$, i.e., $X_i = P_{\Gamma_A(\mathbb{C}^n)} Y_i|_{\Gamma_A(\mathbb{C}^n)}$. Suppose $e^\alpha \in \Gamma_A(\mathbb{C}^n)$, and when $|\alpha| > 0$ let $\alpha = (\alpha_1, \cdots, \alpha_m)$. Then

$$X_i e^\alpha = P_{\Gamma_A(\mathbb{C}^n)} Y_i e^\alpha = \begin{cases} e_i & \text{if } |\alpha| = 0 \\ a_{\alpha_m i} e^\alpha \otimes e_i & \text{if } |\alpha| \geq 1 \end{cases}$$

Moreover from Proposition 4.1.6 it follows that $\underline{X}$ consists of partial isometries with orthogonal ranges satisfying A^T-relations. Let $\mathcal{L}$ and $\mathcal{X}$ denote the WOT-closed algebras generated by $S_1, \cdots, S_n$ and $X_1, \cdots, X_n$ respectively. Now we would analyze the structure of these WOT-closed algebras. Let Q_k denote the projection onto $\text{span}\{e^\alpha : \alpha \in C_A, |\alpha| = k\}$.

PROPOSITION 4.4.6 *1. $\mathcal{L}$ coincides with the commutant of $\mathcal{X}$ in $B(\Gamma(\mathbb{C}^n))$, that is $\mathcal{L} = \mathcal{X}'$. Also $\mathcal{X} = \mathcal{L}'$ and hence $\mathcal{L}$ and $\mathcal{X}$ are double commutants of themselves.*

2. $\mathcal{L}$ and $\mathcal{X}$ are inverse closed. Also the only normal elements in $\mathcal{L}$ and $\mathcal{X}$ are scalars.

PROOF: Any element in $\mathcal{L}$ can be written as $\sum_{|\alpha|\geq 1} b_\alpha \underline{S}^\alpha$ for some $b_\alpha \in \mathbb{C}$. Let for $\beta = (\beta_1, \cdots, \beta_m)$, β' denote $(\beta_m, \cdots, \beta_1)$.

$$\underline{S}^\alpha \underline{X}^{\beta'} e^\gamma = \begin{cases} a_{\alpha_{|\alpha|}\gamma_1} a_{\gamma_{|\gamma|}\beta_1} e^\alpha \otimes e^\gamma \otimes e^\beta & \text{if } |\gamma| > 0 \\ a_{\alpha_{|\alpha|}\beta_1} e^\alpha \otimes e^\beta & \text{if } |\gamma| = 0 \end{cases} = \underline{X}^{\beta'} \underline{S}^\alpha e^\gamma.$$

So, $\mathcal{L} \subseteq \mathcal{X}'$. The proof of $\mathcal{L} \supseteq \mathcal{X}'$ is similar to proof of Theorem 1.2 in [DP2] as seen below:

Noticing that $X_i Q_k = Q_{k+1} X_i$. Let us define $\theta_j : B(\Gamma_A(\mathbb{C}^n)) \to B(\Gamma_A(\mathbb{C}^n))$ by

$$\theta_j(L) = \sum_{k \geq \max\{0,-j\}} Q_k L Q_{k+j}.$$

Then for $L \in \mathcal{X}'$

$$X_i \theta_j(L) = \theta_j(L) X_i.$$

Consider the Cesáro operators for $L \in \mathcal{X}'$ defined for $k \geq 1$ by

$$\Sigma_k(L) = \sum_{|j|<k} (1 - \frac{|j|}{k}) \theta_j(L)$$

By the properties of Cesáro sum one gets $\Sigma_k(L) \in \mathcal{X}'$ and $\Sigma_k(L)$ converges in SOT to L as $k \to \infty$. Moreover taking $L\omega = \sum_\alpha d_\alpha \underline{S}^\alpha \omega$ we observe that

$$\Sigma_k(L)\omega = \sum_{|\alpha|<k} (1 - \frac{|\alpha|}{k}) d_\alpha \underline{S}^\alpha \omega$$

and the Cesáro sums $\sum_{|\alpha|<k}(1-|\alpha|/k)d_\alpha \underline{S}^\alpha \in \mathcal{L}$. Hence $\mathcal{L} \supseteq \mathcal{X}'$.

To prove part 2 we take the approach of proof of Corollary 1.4, 1.5 in [DP2]. $\mathcal{L}$ is inverse closed as for any invertible $L \in \mathcal{X}'$ and any $M \in \mathcal{X}$

$$L^{-1}M = L^{-1}MLL^{-1} = L^{-1}LML^{-1} = ML^{-1}.$$

Similarly $\mathcal{X}$ is also inverse closed and one observes that this happens for any algebra which is also commutant of some algebra. Assume K be a normal element in $\mathcal{L}$ with $c = \langle K\omega, \omega\rangle$. Clearly ω is a eigenvector of $\mathcal{L}^*$. So $K^*\omega = \overline{c}\omega$ and hence $K\omega = c\omega$. Moreover as K is in the commutator of $\mathcal{X}$ we observe that

$$K\underline{S}^\alpha\omega = K\underline{X}^\alpha\omega = \underline{X}^\alpha K\omega = c\underline{S}^\alpha\omega.$$

Thus K is scalar operator cI. □

Proposition 4.4.7 *Any element $A \in \mathcal{L}$ leaves range of $\underline{X}^\alpha(\underline{X}^\alpha)^*$ invariant.*

Proof: Note that one can argue as we did for $\underline{S}^\alpha$ and show that $\underline{X}^\alpha$ are also partial isometries. Further as $\mathcal{L} = \mathcal{X}'$

$$\begin{aligned} \underline{X}^\alpha(\underline{X}^\alpha)^* A \underline{X}^\alpha(\underline{X}^\alpha)^* &= \underline{X}^\alpha(\underline{X}^\alpha)^*\underline{X}^\alpha A(\underline{X}^\alpha)^* \\ &= \underline{X}^\alpha A(\underline{X}^\alpha)^* = A\underline{X}^\alpha(\underline{X}^\alpha)^*. \end{aligned}$$

the Proposition follows. □

In these algebras the wandering subspace description is much simpler than the general case as can be seen from the next result.

Proposition 4.4.8
1. *If $\mathcal{N}$ is a invariant subspace of $\mathcal{L}$ then $\mathcal{M} = \mathcal{N} \ominus \sum_{i=1}^n S_i\mathcal{N}$ is a wandering subspace and $\mathcal{L}[\mathcal{M}] = \mathcal{N}$.*
2. *A subspace is cyclic and invariant with respect to $\mathcal{L}$ if and only if it is the range of some element in $\mathcal{X}$.*

Proof: From the proof of part 1 of Lemma (4.4.3) its clear that $\mathcal{M}$ is a wandering subspace. Rest is similar to proof of Theorem 2.1 in [DP2]. Apart from $\mathcal{L}[\mathcal{M}]$ being subset of $\mathcal{N}$, we get from Wold decomposition that $\mathcal{K} := \mathcal{M} \ominus \mathcal{L}[\mathcal{M}]$ is bi-invariant for S_i's and satisfy $\mathcal{K} = \sum_i S_i\mathcal{K}$. If we assume $\mathcal{K}$ is not zero then there exists smallest integer m_0 such that $Q_{m_0}\mathcal{K} \neq 0$ which yields a contradiction as

$$Q_{m_0}\mathcal{K} \subseteq \sum_i Q_{m_0}S_i\mathcal{K} = \sum_i S_iQ_{m_0-1}\mathcal{K} = 0.$$

So part 1 holds.

Cyclic subspaces of $\mathcal{N}$ are of the type $\mathcal{L}[\eta]$. For this vector, define operator L by $L\underline{S}^\alpha\omega = \underline{S}^\alpha\eta$. Clearly L commutes with S_i and hence is in $\mathcal{X}$. Also range of L is $\mathcal{L}[\eta]$. For the converse if K be in $\mathcal{X}$, one can show that $\eta = K\omega$ generates the range of K. □

Chapter 5

Examples

Ample interesting examples of q-commuting tuples exists one of which is the following:

EXAMPLE 5.0.1 Suppose $\mathcal{W}$ is a finite abelian group with $o(\mathcal{W}) = N$ and $\hat{\mathcal{W}}$ is its dual group. with $o(\mathcal{W}) = N$ Let the binary operation be denoted by $+$. Define unitary operators $U_g, g \in \mathcal{W}$ and $V_\alpha, \alpha \in \hat{\mathcal{W}}$ on $L^2(\mathcal{W})$

$$U_g f(x) = f(x+g), \quad V_\alpha f(x) = \alpha(x) f(x) \text{ for } x \in \mathcal{W}.$$

Then we have for $g, h \in \mathcal{W}$ and $\alpha, \beta \in \hat{\mathcal{W}}$

$$U_g U_h = U_{g+h}, \quad V_\alpha V_\beta = V_{\alpha\beta}, \quad U_g V_\alpha = \alpha(g) V_\alpha U_g.$$

These are called *Weyl commutation* relations. If we consider a tuple $(U_{g_1}, \cdots, U_{g_m}, V_{\alpha_1}, \cdots, V_{\alpha_l})$ for $g_i \in \mathcal{W}$ and $\alpha_i \in \hat{\mathcal{W}}$ then this tuple is a q-commuting tuple. Here for $k := m + l, q := (q_{ij})_{k \times k}$ and

$$q_{ij} := \begin{cases} 1 & \text{if } 1 \leq i, j \leq m \\ 1 & \text{if } m+1 \leq i, j \leq k \\ \alpha_{j-m}(g_i) & \text{if } 1 \leq i \leq m, m+1 \leq j \leq k \\ (\alpha_{i-m}(g_j))^{-1} & \text{if } m+1 \leq i \leq k, 1 \leq j \leq m. \end{cases}$$

The next example shows that the fermionic Fock space can also be realized through "maximal piece" concept. Minimal Cuntz-Krieger dilation is illustrated in the last example.

EXAMPLE 5.0.2 Also we would like to remark that the Fermionic Fock space $\Gamma_a(\mathbb{C}^n)$ is the intersection of the maximal q-commuting space and maximal A-relation subspace with respect to the following $q = (q_{ij})_{n \times n}$ and $A = (a_{ij})_{n \times n}$:

$$q_{ij} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{otherwise} \end{cases} \text{ and } a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise.} \end{cases}$$

This can be shown using arguements similar to that used to show $\Gamma_{sA}(\mathbb{C}^n)$ is maximal commuting A-subspace with respect to $\underline{V}$ given in page number 58. In other words, Fermionic Fock space $\Gamma_a(\mathbb{C}^n)$ is the maximal piece for the set of polynomials:

$$p_{1ij}(\underline{z}) = z_j z_i - q_{ij} z_i z_j \text{ and } p_{2ij}(\underline{z}) = z_i z_j - a_{ij} z_i z_j \forall 1 \leq i, j \leq n.$$

EXAMPLE 5.0.3 For $\mathcal{H} = \mathbb{C}^2$, let $T_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $T_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then one observes that $\underline{T}$ satisfy A-relation, $T_1T_1^* + T_2T_2^* = I$ and T_i's are partial isometries. Further the D used in the Theorem 4.2.2 turns out to be

$$D = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{pmatrix}.$$

Let us denote the two basis vectors in the range of D corresponding to the entries 1 appearing in D by f_1 and f_2 such that

$$D(e_1 \otimes (a_1, a_2) + e_2 \otimes (b_1, b_2)) = a_1 f_1 + b_2 f_2$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{C}$. The dilation space for the minimal isometric dilation $\underline{V} = (\tilde{V}_1, \tilde{V}_2)$ of $\underline{T}$ is $\mathcal{H} \oplus \Gamma(\mathbb{C}^n) \otimes \mathcal{D}$ where $\mathcal{D}$ is the range of D.

$$\tilde{V}_1\tilde{V}_1(a_1, a_2) = (0, 0) + \omega \otimes (a_2, 0) + e_1 \otimes (a_1, 0),$$

$$\tilde{V}_2\tilde{V}_2(a_1, a_2) = (0, 0) + \omega \otimes (0, a_1) + e_2 \otimes (0, a_2).$$

As a_1, a_2 are arbitrary using the above equations with the description of maximal A-relation space $\hat{\mathcal{H}}$ from Lemma 4.1.2 we get $\hat{\mathcal{H}} = \mathcal{H}$.

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Addendum

Motivation: Sz. Nagy dilation of contractions to isometries (or unitaries) is a very important basic construction in operator theory. There have been several attempts to extend it to n-tuples of operators. One such extension is the notion of *minimal isometric dilation.* Another such extension is *standard commuting dilation* of tuples, and this works only for commuting tuples. So there are two standard dilations for commuting tuples. Then it is a very natural question as to how are they related. This question has been answered completely in this Thesis making use of the new concept of *maximal commuting piece.*

Cuntz algebras are among the most important non-commutative C^*-algebras. They are simple, separable, nuclear and purely infinite. They have rich classes of representations. Their representations appear in a wide variety of fields, such as quantum theory, wavelet theory etc. They also appear quite naturally in dilation theory of n-tuples of operators. We give a complete classification of representations of Cuntz algebras arising from commuting tuples. This should be useful in other fields.

Tuples with a slightly twisted commutativity called q-commutativity appear in quantum theory and a study of their dilations by Bhat and Bhattacharyya in [BB] gave interesting results. We tried to extend our theory to this type of tuples and we could successfully do so. Cuntz-Krieger algebras were introduced by Cuntz and Krieger while studying topological Markov chains. One obtains them by making simple modifications to the defining relations of Cuntz algebras. Here we have defined minimal Cuntz-Krieger dilation with two purposes in mind- to look at some kind of partial isometric dilation and to extend our classification results to Cuntz-Krieger algebras.

Interrelations between three main themes:

Our three main themes are basically relationship of (i) commuting dilations, (ii) q-commuting dilations, and (iii) Cuntz-Krieger dilations with the minimal isometric dilation. So it is fairly obvious that they are very closely related.

In Chapter 2 we see how the standard commuting dilation sits inside the minimal isometric dilation. Chapter 3 extends this main result of Chapter 2 by replacing standard commuting dilations by more general standard q-commuting dilations. When the set of polynomials of maximal piece of Chapter 2 for minimal isometric dilations is replaced by another special set, we obtain minimal Cuntz-Krieger dilation and we discuss about this dilation in Chapter 4. In fact results of Chapter 2 can be derived from those of the later chapters as special cases but it is worth proving them separately as this case is particularly important and many of the proofs shortens due to special properties available in this case.

Moreover in Chapter 2 we get a complete classification of representations of Cuntz algebras coming from dilation of commuting tuples. We couldn't extend this result to q-commuting tuples. Chapter 3 also has some results on operator spaces and universal properties of tuples we have. In Chapter 4 we have an alternative derivation of minimal

Cuntz-Krieger dilation through positive definite kernels.

Problems for the future: We still don't know much about the following interesting questions:

1. Let $\underline{T}$ be a contractive n-tuple of operators on a Hilbert space $\mathcal{H}$. Does the maximal commuting piece of the minimal isometric dilation of $\underline{T}$ coincides with the standard commuting dilation of maximal commuting piece of $\underline{T}$ under conditions like $\overline{\Delta_{\underline{T}}(\mathcal{H})} = \overline{\Delta_{\underline{T}}(\mathcal{H}^c)}$?

2. Whether the results we have on minimal Cuntz-Krieger dilations and the related classification results help in understanding the topological Markov chains (refer [CK]) better?

3. Are all the unital C^*-algebras generated by standard q-commuting dilations for different values of q non-isomorphic?

4. Let $\underline{S}$ denote the n-tuple obtained by compressing $\underline{V}$ to $\Gamma_q(\mathbb{C}^n)$. Can we classify the von Neumann algebras generated by all $G_i = S_i + S_i^*, 1 \leq i \leq n$ for different values of q?