University of Waterloo

Waterloo, Ontario

Mathematics 237

Mid-Term Test – Winter Term 2001

Duration: 1 hour 30 minutes

Date:

SCIENTIFIC CALCULATORS ALLOWED ONLY.
GIVE ANSWERS TO 2 SIGNIFICANT DIGITS
NO OTHER AIDS PERMITTED

Family Name: ____________________ Initials: ___ Id. Number: __________

Signature: ______________________________

Instructor: I. VanderBurgh (01)
K. Morris (02)

Instructions:

1. Complete the information above.
2. Attempt all questions, in the space provided. If you require more space, use the reverse of the **preceding** page.
3. The marks for each question are indicated. Marks will be deducted for poorly presented work. Your grade will be influenced by how clearly you express your ideas, and by how well you organize your solutions. Justification should be provided by referring to definitions and theorems where appropriate.
4. This examination has **eleven** pages. The last two pages are for rough work. These pages will not be marked.

FOR EXAMINERS' USE ONLY		
Question	Maximum	Mark
1	8	
2	6	
3	12	
4	12	
5	12	
Total	50	

[8] 1. $f(x, y, z) = xe^{1-zy}$

(a) Calculate the gradient vector of f at (1, 1, 1).

(b) What is the rate of change of f at (1, 1, 1) in the direction $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$?

(c) Find the equation of the tangent plane to the surface

$$xe^{1-zy} = 1$$

at (1, 1, 1).

[6] 2. The temperature of a metal sheet as a function of position (x, y) and time t is

$$T(x, y, t) = (40 + x^2 \sin^2 y)e^{-t}.$$

An ant is walking across the sheet. Its position is given by

$$(x, y) = (\cos t, 2t).$$

Find $T'(\frac{\pi}{2})$, the rate of change of temperature seen by the ant at time $\frac{\pi}{2}$.

[12] 3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{for} \quad (x,y) \neq (0,0) \quad \text{and} \quad f(0,0) = 0.$$

(a) Show that f is continuous at $(0,0)$.

(b) Find the linearization $L_{\mathbf{a}}(x,y)$ of the function f at the point $\mathbf{a} = (0,0)$.

(c) Show that f is not differentiable at $(0,0)$.

(d) Calculate the directional derivative of f at the point $(0,0)$ in an arbitrary direction $\mathbf{u} = (u_1, u_2)$.

[12] 4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = (xy)^{\frac{2}{3}}.$$

(a) Define differentiability at $\mathbf{a}$ for $f : \mathbb{R}^2 \to \mathbb{R}$.

(b) Show that f is differentiable at $(0,0)$.

c) Based on your answer to part (i), what, if anything, can you conclude about the continuity of f at $(0,0)$? Justify your answer briefly.

d) Based on your answer to part (a), what, if anything, can you conclude about the continuity of the partial derivatives f_x and f_y at $(0,0)$? Provide a brief explanation.

[12] 5. $f(x,y) = \sqrt{1+x^2+y^2}$.

(a) Calculate the linear approximation at (0,1).

(b) Prove that f is differentiable at (0,1).

(c) Use the linear approximation to estimate $f(0.01, 0.9)$.

(d) Show that the error in the linear approximation from part (a) never exceeds $x^2 + (y-1)^2$. What theorem are you using?